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n-SHAPE FIBRATIONS

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1. Introduction

In [11] and [12] the authors introduced and studied a class of maps between compact metric spaces called shape fibrations. A shape fibration is a map $p: E \rightarrow B$ which is induced by a level map of ANR-sequences $\underline{p}: \underline{E} \rightarrow \underline{B}$, $p = \lim \underline{p}$, such that \underline{p} has a certain homotopy lifting property with respect to the class of all topological spaces. It is shown in [11] that the approximate fibrations of Coram and Duvall [1] are precisely shape fibrations between ANR's. Also, it is shown in [11] that cell-like maps between finite-dimensional metric compacta are shape fibrations. However, the Taylor map [16] is an example of a cell-like map between infinite-dimensional continua which fails to be a shape fibration.

The purpose of this paper is to consider the homotopy lifting property with respect to some other classes \mathcal{X} of spaces. In particular, if \mathcal{X} consists of all topological spaces of dimension $\leq n$, we obtain the notion of an n -shape fibration. Maps which are n -fibrations for all n are also interesting. We call such maps weak shape fibrations. Most of the results for shape fibrations obtained in [11] and [12]

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remain true in appropriate formulations for n -shape fibrations and for weak shape fibrations. For maps between ANR's, weak shape fibrations coincide with approximate fibrations and therefore also with shape fibrations. Furthermore, every cell-like map between metric compacta (regardless of dimension) is a weak shape fibration. Consequently, the Taylor map provides an example of a weak shape fibration which fails to be a shape fibration.

The authors would like to point out that several of the results of this paper are modeled on results of Coram and Duvall in [2]. Also, the authors would like to thank Jerzy Dydak for bringing to their attention Lemma 8.3 of [4] which is referred to in Section 7 of this paper.

2. Shape Fibrations with Respect to a Class of Spaces

We shall consider inverse sequences of metric compacta $\underline{E} = (E_i, q_{ii'})$, $\underline{B} = (B_i, r_{ii'})$. If all E_i and B_i are ANR's, we speak of ANR-sequences. Polyhedral sequences and Q -manifold sequences are especially useful since polyhedra and Q -manifolds are convenient ANR's. A level preserving map of sequences $p: \underline{E} \rightarrow \underline{B}$ (abbreviated as level map) is a sequence of maps $p_i: E_i \rightarrow B_i$ such that

$$(1) p_i q_{ii'} = r_{ii'} p_{i'}$$

Definition 1. Let \mathcal{X} be a class of spaces. A level map $p: \underline{E} \rightarrow \underline{B}$ is said to have the *homotopy lifting property with respect to \mathcal{X}* (\mathcal{X} -HLP) provided each i admits a $j \geq i$ such that for any $X \in \mathcal{X}$ and any maps $h: X \rightarrow E_j$, $H: X \times I \rightarrow B_j$ with

$$(2) p_j h = H_0.$$

There is a homotopy $\tilde{H}: X \times I \rightarrow E_i$ satisfying

$$(3) \tilde{H}_0 = q_{ij}h, \text{ and}$$

$$(4) p_i\tilde{H} = r_{ij}H.$$

Every such j is called a *lifting index* for i .

Definition 2. We say that $\underline{p}: \underline{E} \rightarrow \underline{B}$ has the *approximate homotopy lifting property with respect to \mathcal{X} (\mathcal{X} -AHLP)* provided each i and each $\epsilon > 0$ admit a $j \geq i$ and a $\delta > 0$ such that for any $X \in \mathcal{X}$ and any $h: X \rightarrow E_j, H: X \times I \rightarrow B_j$ with the distance

$$(5) d(p_jh, H_0) \leq \delta,$$

there is a homotopy $\tilde{H}: X \times I \rightarrow E_i$ satisfying

$$(6) d(\tilde{H}_0, q_{ij}h) < \epsilon$$

$$(7) d(p_i\tilde{H}, r_{ij}H) < \epsilon.$$

We call δ a *lifting mesh* for (i, ϵ) .

Remark 1. If each E_i is an ANR and if $\underline{p}: \underline{E} \rightarrow \underline{B}$ has the \mathcal{X} -AHLP where all members of \mathcal{X} are paracompact Hausdorff spaces, then \underline{p} also has the formally stronger property obtained by replacing (6) by (3). This follows from the proof of Proposition 1 of [11]. (Paracompactness is needed in order to construct the function $\phi: X \rightarrow (0,1]$ in that proof.)

Remark 2. If each B_i is an ANR and all the members of \mathcal{X} are paracompact Hausdorff spaces, then $\underline{p}: \underline{E} \rightarrow \underline{B}$ has the \mathcal{X} -AHLP provided it has the formally weaker property obtained by replacing (5) by (2). This follows from the proof of Proposition 2 of [11].

Definition 3. Let $\underline{p}: \underline{E} \rightarrow \underline{B}$ be a level map of inverse sequences and let $(E, q_i) = \lim \underline{E}, (B, r_i) = \lim \underline{B}$. The unique map $p: E \rightarrow B$ such that $p_i q_i = r_i p$ for all i , is said to be

induced by \underline{p} or to be the limit of \underline{p} , denoted by $p = \lim \underline{p}$.

Definition 4. Let \mathcal{X} be a class of spaces. A map $p: E \rightarrow B$ between metric compacta is called a *shape fibration with respect to \mathcal{X}* (\mathcal{X} -*shape fibration* for short) provided there exists a level map of ANR-sequences $\underline{p}: \underline{E} \rightarrow \underline{B}$ such that $p = \lim \underline{p}$ and \underline{p} has the \mathcal{X} -AHLP.

Remark 3. Notice that whenever p is an \mathcal{X} -shape fibration and $\mathcal{X}' \subseteq \mathcal{X}$, then p is a fortiori an \mathcal{X}' -shape fibration.

The proof of Theorem 1 of [11] applies in the present situation and yields the following result.

Theorem 1. If $\underline{p}: \underline{E} \rightarrow \underline{B}$ and $\underline{p}': \underline{E}' \rightarrow \underline{B}'$ are two level maps of ANR-sequences with the same limit $p = \lim \underline{p} = \lim \underline{p}'$ and if for some class \mathcal{X} \underline{p} has the \mathcal{X} -AHLP, then so does \underline{p}' .

The next result follows from the proof of Theorem 2 of [11] together with Remark 1.

Theorem 2. Let all members of \mathcal{X} be paracompact Hausdorff spaces. If $\underline{p}: \underline{E} \rightarrow \underline{B}$ is a level map of ANR-sequences such that $\lim \underline{p} = p: E \rightarrow B$ and \underline{p} has the \mathcal{X} -AHLP, then there is a level map $\underline{p}': \underline{E}' \rightarrow \underline{B}$ of ANR-sequences such that $\lim \underline{p}' = p$ and \underline{p}' has the \mathcal{X} -HLP.

Remark 4. If all of the members of \mathcal{X} are paracompact Hausdorff spaces and if $p: E \rightarrow B$ is an \mathcal{X} -shape fibration, then for every closed subset $B' \subseteq B$ the restriction $p' = p|_{E'}: E' \rightarrow B'$, $E' = p^{-1}(B')$, is also an \mathcal{X} -shape fibration. This is obtained by following the proof of the analogous

statement for shape fibrations ([11], Proposition 4) and applying Theorem 2 at the outset.

3. Shape Fibrations

Definition 5. A map $p: E \rightarrow B$ between metric compacta is called a *shape fibration* if it is an \mathcal{X} -shape fibration where \mathcal{X} is the class of all topological spaces.

Remark 5. The following theorem (Theorem 3) shows that shape fibrations as defined above (Definition 5) coincide with the shape fibrations as defined in [11]. Therefore, for ANR's, shape fibrations (in the sense of Definition 5) agree with approximate fibrations (see [11], Corollary 1).

Theorem 3. *Shape fibrations coincide with \mathcal{X} -shape fibrations where \mathcal{X} is the class of all separable metric spaces.*

Proof. Let $p: \underline{E} \rightarrow \underline{B}$ be a level map of ANR-sequences such that $p = \lim p$ and p has the AHLPL with respect to separable metric spaces. For given i and ϵ , let $j \geq i$ and δ be the lifting index and mesh respectively. Let X be an arbitrary topological space and let $h: X \rightarrow E_j$ and $H: X \times I \rightarrow B_j$ be maps satisfying $d(p_j h, H_0) \leq \delta$. Consider the set

$$(1) \Delta = \{(e, \omega) \in E_j \times B_j^I : d(p_j(e), \omega(0)) \leq \delta\},$$

where B_j^I is given the compact-open topology. Notice that Δ is a separable metric space because B_j^I is one ([3], Theorem 8.2(3), p. 270 and Theorem 5.2, p. 265).

We shall now define maps $f: X \rightarrow \Delta$, $g: \Delta \rightarrow E_j$ and $G: \Delta \times I \rightarrow B_j$ such that

$$(2) gf = h,$$

$$(3) G(f \times 1) = H.$$

(4) The diagram

$$\begin{array}{ccc}
 \Delta & \xleftarrow{f} & X \\
 \downarrow i_0 & & \downarrow i_0 \\
 \Delta \times I & \xleftarrow{f \times I} & X \times I
 \end{array}$$

commutes strictly (where $i_0(y) = (y, 0)$), and

(5) the diagram

$$\begin{array}{ccc}
 E_j & \xleftarrow{g} & \Delta \\
 \downarrow p_j & & \downarrow i_0 \\
 B_j & \xleftarrow{G} & \Delta \times I
 \end{array}$$

commutes up to δ , i.e.,

(6) $d(p_j g(y), G(y, 0)) \leq \delta$ for $y \in \Delta$.

We define f by

(7) $f(x) = (h(x), \hat{H}(x))$, where $[\hat{H}(x)](t) = H(x, t)$.

The continuity of f follows from ([3], Theorem 3.1(1), p. 261). We define g and G by

(8) $g(e, \omega) = e$, and

(9) $G((e, \omega), t) = \omega(t)$.

The map G is continuous because the evaluation map $(\omega, t) \mapsto \omega(t)$ is continuous ([3], Theorem 2.4(2), p. 260).

By the choice of j and δ there is a map $\tilde{G}: \Delta \times I \rightarrow E_i$ such that

(10) $d(\tilde{G}_0, q_{ij} g) < \varepsilon$, and

(11) $d(p_i \tilde{G}, r_{ij} G) < \varepsilon$.

Now the map $\tilde{H}: X \times I \rightarrow E_i$ given by

$$(12) \tilde{H} = \tilde{G} \circ (f \times 1)$$

has the desired properties, i.e., it satisfies 2.(6) and 2.(7).

Theorem 4. Shape fibrations coincide with \mathcal{X} -shape fibrations where \mathcal{X} is the class of all separable, locally compact polyhedra.

We shall now prove the theorem by using the following lemma the proof of which we postpone until the next section.

Lemma 1. Let E and B be ANR's, let X be a separable metric space and let the following diagram commute

$$(13) \quad \begin{array}{ccc} E & \xleftarrow{h} & X \\ P \downarrow & & \downarrow i_0 \\ B & \xleftarrow{H} & X \times I \end{array} .$$

Then for any $\eta > 0$ and $\delta > 0$ there exists a separable, locally compact polyhedron P and there exist maps $f: X \rightarrow P$, $g: P \rightarrow E$, $G: P \times I \rightarrow B$ such that image of f is a dense subset of P and

$$(14) \quad d(gf, h) < \eta$$

$$(15) \quad d(G(f \times 1), H) < \delta.$$

If $\dim X \leq n$, then one can achieve that $\dim P \leq n$ also.

Proof of Theorem 4. Let j and δ be the lifting index and the lifting mesh for i and $\epsilon/2$ with respect to separable, locally compact polyhedra. We also assume that δ is so small that δ -close points in E_j and B_j map by q_{ij} and r_{ij} respectively to $\epsilon/2$ -close points in E_i and B_i . Let X be a separable

metric space and let $h: X \rightarrow E_j$, $H: X \times I \rightarrow B_j$ be maps such that $p_j h = H_0$. Apply Lemma 1 to $h, H, p_j, \delta/2$ and η , where $\eta < \delta$ is so small that η -close points map under p_j onto $\delta/2$ -close points. We obtain a separable, locally compact polyhedron P and maps $f: X \rightarrow P, g: P \rightarrow E_j, G: P \times I \rightarrow B_j$ such that $f(X)$ is a dense subset of P and (14) and (15) hold. (Moreover, if $\dim X \leq n$, then $\dim P \leq n$. We shall use this fact in the proof of Theorem 5.)

Since $p_j h = H_0$, (14) and (15) imply

$$(16) \quad d(p_j g f, G i_0 f) < \delta.$$

Since $f(X)$ is dense in P , (16) implies

$$(17) \quad d(p_j g, G i_0) \leq \delta.$$

In other words, the diagram

$$(18) \quad \begin{array}{ccc} E_j & \xleftarrow{g} & P \\ P_j \downarrow & & \downarrow i_0 \\ B_j & \xleftarrow{G} & P \times I \end{array}$$

commutes up to δ . By the choice of δ and j there is a homotopy

$\tilde{G}: P \times I \rightarrow E_i$ such that

$$(19) \quad d(\tilde{G}_0, q_{ij} g) < \epsilon/2 \text{ and}$$

$$(20) \quad d(p_i \tilde{G}, r_{ij} G) < \epsilon/2.$$

We now put

$$(21) \quad \tilde{H} = \tilde{G} \circ (f \times 1): X \times I \rightarrow E_i.$$

Clearly,

$$(22) \quad d(\tilde{H}_0, q_{ij} h) < \epsilon \text{ and}$$

$$(23) \quad d(p_i \tilde{H}, r_{ij} H) < \epsilon.$$

4. The Proof of Lemma 1

Let us first recall that a map f of a space X into the nerve $|N(\mathcal{V})|$ of a locally finite open covering \mathcal{V} of X is called a *canonical map* for \mathcal{V} provided

$$(1) f^{-1}(\text{St}(V, N(\mathcal{V}))) \subseteq V$$

for all $V \in \mathcal{V}$. Equivalently, if $x \in V_0 \cap \dots \cap V_n$, then $f(x)$ is contained in the closed simplex spanned by the vertices V_0, \dots, V_n .

Remark 6. For every map $f: X \rightarrow |K|$ of X onto a dense subset of the carrier of a locally finite simplicial complex K there is a star-finite open covering \mathcal{V} of X such that $N(\mathcal{V}) = K$ and f is a canonical map for \mathcal{V} . It suffices to consider the star-finite covering ω formed by all $\text{St}(q, K)$ where $q \in K^0$. With every q associate the open set $V = f^{-1}(\text{St}(q, K))$. These sets form a star-finite open covering of X . Clearly, $V_0 \cap \dots \cap V_n \neq \emptyset$ if and only if $\text{St}(q_0, K) \cap \dots \cap \text{St}(q_n, K) \neq \emptyset$, i.e., if and only if the vertices q_0, \dots, q_n span a simplex in K . In other words $N(\mathcal{V}) = K$. Furthermore, for each vertex q and for the corresponding $V \in \mathcal{V}$ one has

$$(2) f^{-1}(\text{St}(q, K)) = V$$

which shows that f is indeed a canonical map $f: X \rightarrow |N(\mathcal{V})| = |K|$ for \mathcal{V} .

The next lemma will be used in the proof of Lemma 1.

Lemma 2. Let X be a separable metric space. Then every open covering of X admits a countable star-finite refinement \mathcal{V} and a canonical map $f: X \rightarrow |N(\mathcal{V})|$ such that $f(X)$ is dense in $|N(\mathcal{V})|$. If $\dim X \leq n$ one can also achieve that $\dim N(\mathcal{V}) \leq n$.

Proof. Separable metric spaces are regular and Lindelöf and therefore strongly paracompact ([15], V. 4.B, p. 172), i.e., every open covering \mathcal{U}' admits a star-finite refinement \mathcal{U} . If $\dim X \leq n$, then every open covering admits a refinement \mathcal{U} of order $\leq n + 1$ (which is a fortiori star-finite). Notice that every star-finite covering \mathcal{U} of X must be countable. This is true since \mathcal{U} admits a countable refinement $\mathcal{W} = (W_1, W_2, \dots)$ and each W_i is contained in at most finitely many members of \mathcal{U} .

Let $g: X \rightarrow |N(\mathcal{U})|$ be a canonical map and let $\mathcal{U} = (U_1, U_2, \dots)$. For a highest dimensional simplex S in $St(U_1, N(\mathcal{U}))$, whose interior is not entirely contained in $Cl(g(X))$, one composes g with a projection of $S \setminus y$ into ∂S , where $y \in \text{Int } S \setminus Cl(g(X))$. By repeating this procedure finitely many times, we obtain a modified map f_1 such that every simplex S in $St(U_1, N(\mathcal{U}))$ is either entirely contained in $Cl(f_1(X))$ or its interior is disjoint with $f_1(X)$. We now modify f_1 over $St(U_2, N(\mathcal{U}))$ to obtain f_2 , etc. Because of local finiteness, $f = \lim f_n$ exists and $Cl(f(X)) = |K|$ for some subcomplex K of $N(\mathcal{U})$.

Notice that if for some $x \in X$ the point $f(x)$ belongs to $St(U_i, N(\mathcal{U}))$, then also $g(x) \in St(U_i, N(\mathcal{U}))$. By Remark 6, $f: X \rightarrow |K|$ is a canonical map for the star-finite covering \mathcal{V} which consists of sets $V_i = f^{-1}(St(U_i, K))$ where U_i is a vertex of K . If $x \in V_i$, then $f(x) \in St(U_i, K) \subseteq St(U_i, N(\mathcal{U}))$ and therefore $g(x) \in St(U_i, N(\mathcal{U}))$, i.e., $x \in g^{-1}(St(U_i, N(\mathcal{U})))$. Since g is a canonical map for \mathcal{U} , the set $g^{-1}(St(U_i, N(\mathcal{U})))$ is contained in the member U_i of \mathcal{U} . Hence, $V_i \subseteq U_i$ and we see that \mathcal{V} refines \mathcal{U} and thus also \mathcal{U}' .

If $\dim X \leq n$, then $N(\mathcal{V}) = K \subseteq N(\mathcal{U})$ and $\dim N(\mathcal{U}) \leq n$ imply that $\dim N(\mathcal{V}) \leq n$.

Proof of Lemma 1. It follows from ([6], Theorem 8.1, p. 146) that there exists an open covering \mathcal{U}' of $X \times I$ such that for any locally finite refinement \mathcal{U} of \mathcal{U}' there exist maps $G: |N(\mathcal{U})| \rightarrow B$ and $g: |N(\mathcal{U}|X \times 0)| \rightarrow E$ such that for any canonical map $\phi: X \times I \rightarrow |N(\mathcal{U})|$ one has

$$(3) \ d(G\phi, H) < \delta \text{ and}$$

$$(4) \ d(g(\phi|X \times 0), i_{0,h}) < \eta.$$

Notice that the restriction $\mathcal{U}|X \times 0$ of \mathcal{U} to $X \times 0$ consists of the sets $U \cap (X \times 0)$ where $U \in \mathcal{U}$. The nerve $N(\mathcal{U}|X \times 0)$ can be viewed as a subcomplex of $N(\mathcal{U})$ if one identifies the vertex $U \cap (X \times 0)$ of $N(\mathcal{U}|X \times 0)$ with the vertex U of $N(\mathcal{U})$, $U \in \mathcal{U}$. Consequently, if $(x, 0) \in U_0 \cap \dots \cap U_r$, then $\phi(x, 0)$ is contained in the closed simplex spanned by the vertices U_0, \dots, U_r and therefore $\phi(x, 0) \subseteq |N(\mathcal{U}|X \times 0)|$. The composition $g(\phi|X \times 0)$ is thus well-defined.

It is well-known (e.g. [5], IX Theorem 5.6, p. 241) that one can refine \mathcal{U}' by a stacked covering of $X \times I$ over a covering \mathcal{V} of X . By Lemma 2 one can assume that \mathcal{V} is a countable star-finite covering which admits a canonical map $f: X \rightarrow |N(\mathcal{V})| = P$ such that $f(X)$ is dense in P . (Moreover, if $\dim X \leq n$, one can achieve that $\dim P \leq n$.) By the definition of stacked coverings, for each $V \in \mathcal{V}$ there is a finite collection of open intervals J_1^V, \dots, J_n^V which covers I and whose nerve is a triangulation of I . Furthermore, each $V \times J_i^V$ is contained in a member of \mathcal{U}' .

Now consider all of the sets $St(V, N(\mathcal{V})) \times J_i^V$. They form

an open covering ω of $P \times I$. Let (L, L') be a countable locally finite triangulation of $(P \times I, P \times 0)$ such that the sets $\text{St}(q, L)$, $q \in L^0$, refine ω . Then we put

$$(5) \mathcal{U} = \{(f \times 1)^{-1}(\text{St}(q, L)) : q \in L^0\}.$$

This is a countable star-finite covering of $X \times I$. For each vertex $q \in L^0$ there is a $V \in \mathcal{V}$ and an $i \leq n(V)$ such that

$$(6) \text{St}(q, L) \subseteq \text{St}(V, K) \times J_i^V.$$

Therefore,

$$(7) (f \times 1)^{-1}(\text{St}(q, L)) \subseteq (f \times 1)^{-1} \left[\text{St}(V, N(V)) \times J_i^V \right] \\ = f^{-1} \left[\text{St}(V, N(V)) \times J_i^V \right].$$

Since $f: X \rightarrow |N(V)|$ is a canonical mapping for V , one has $f^{-1}(\text{St}(V, N(V))) \subseteq V$ and therefore

$$(8) (f \times 1)^{-1}(\text{St}(q, L)) \subseteq V \times J_i^V,$$

which proves that \mathcal{U} refines \mathcal{U}' .

We now identify $q \in L^0$ with $U = (f \times 1)^{-1}(\text{St}(q, L)) \in \mathcal{U}$ and conclude by Remark 6 that $N(\mathcal{U}) = L$ and that $f \times 1: X \times I \rightarrow P \times I = |N(\mathcal{U})|$ is a canonical map for the covering \mathcal{U} .

The proof will be complete if we show that

$$(9) |N(\mathcal{U}|X \times 0)| = P \times 0.$$

Indeed, by the choice of \mathcal{U}' , we then obtain maps $G: P \times I \rightarrow B$ and $g: P \times 0 \rightarrow E$ such that conditions (14) and (15) of Lemma 1 hold.

Notice that (9) is equivalent to the following assertions:

- (i) $U \cap (X \times 0) \neq \emptyset$ implies that $q \in L'$,
- (ii) $q \in L'$ implies that $U \cap (X \times 0) \neq \emptyset$,
- (iii) $U_0 \cap \dots \cap U_n \cap (X \times 0) \neq \emptyset$ implies that $\langle q_0, \dots, q_n \rangle \in L'$, and

(iv) $\langle q_0, \dots, q_n \rangle \in L'$ implies that $U_0 \cap \dots \cap U_n \cap (X \times 0) \neq \phi$.

(i) If $q \notin L'$, then $St(q, L) \cap |L'| = \phi$. Thus $(f \times 1)^{-1}(St(q, L)) \cap (f \times 1)^{-1}(|L'|) = \phi$ or $U \cap (X \times 0) = \phi$.

(ii) If $q \in L'$, then $St(q, L) \cap |L'|$ is a nonempty, open subset of $|L'|$. Since $f(X)$ is dense in $|L'|$, there exists an $x \in X$ such that $f(x) \in St(q, L) \cap |L'|$. Thus $(x, 0) \in (f \times 1)^{-1}(St(q, L)) \cap (f \times 1)^{-1}(|L'|)$ or $(x, 0) \in U \cap (X \times 0)$.

(iii) If $(x, 0) \in U_0 \cap \dots \cap U_n$, then $(f \times 1)(x, 0) \in (f \times 1)(U_0) \cap \dots \cap (f \times 1)(U_n) \cap |L'| \subseteq St(q_0, L) \cap \dots \cap St(q_n, L) \cap |L'| = St(q_0, L') \cap \dots \cap St(q_n, L')$. Therefore, q_0, \dots, q_n span a simplex in L' .

(iv) If q_0, \dots, q_n span a simplex in L' , then $St(q_0, L') \cap \dots \cap St(q_n, L')$ is a nonempty open subset of $|L'|$. By denseness of $f(X)$ in $|L'|$, there exists an $x \in X$ such that $(f \times 1)(x, 0) = (f(x), 0) \in St(q_0, L') \cap \dots \cap St(q_n, L')$. Thus $(x, 0) \in (f \times 1)^{-1}(St(q_0, L')) \cap \dots \cap (f \times 1)^{-1}(St(q_n, L')) \subseteq (f \times 1)^{-1}(St(q_0, L)) \cap \dots \cap (f \times 1)^{-1}(St(q_n, L)) \cap (X \times 0) = U_0 \cap \dots \cap U_n \cap (X \times 0)$.

5. n-Shape Fibrations

Definition 6. A map $p: E \rightarrow B$ between metric compacta is called an *n-shape fibration* if it is an χ -shape fibration where χ is the class of all topological spaces X of dimension $\leq n$.

Here we are using covering dimension based on numerable

coverings.

Theorem 5. A map $p: E \rightarrow B$ is an n -shape fibration if and only if it is an \mathcal{X} -shape fibration where \mathcal{X} is either of the following two classes:

- (a) \mathcal{X} is the class of separable metric space X with $\dim X \leq n$,
- (b) \mathcal{X} is the class of separable, locally compact polyhedra P with $\dim P \leq n$.

Proof. We shall first show that p is an n -shape fibration whenever (a) holds. Let j and δ be a lifting index and a lifting mesh for i and ϵ with respect to separable metric spaces of dimension $\leq n$. Let X be a topological space, $\dim X \leq n$, and let $h: X \rightarrow E_j, H: X \times I \rightarrow B_j$ satisfy

$$(1) \ d(p_j h, H i_0) \leq \delta.$$

We introduce the separable metric space Δ and the maps $f: X \rightarrow \Delta, g: \Delta \rightarrow E_j, G: \Delta \times I \rightarrow B_j$ as in the proof of Theorem 3. Then the diagram 3.(5) commutes up to δ . By a well-known factorization theorem (e.g., [14], Lemma 2.2, p. 34) the map f factors through a separable metric space Y where $\dim Y \leq \dim X \leq n$, i.e., there are maps $f': X \rightarrow Y, f'': Y \rightarrow \Delta$ such that

$$(2) \ f'' f' = f.$$

One can also achieve that f' be surjective.

Now consider the diagram

$$(3) \quad \begin{array}{ccc} E_j & \xleftarrow{gf''} & Y \\ P_j \downarrow & & \downarrow i_0 \\ B_j & \xleftarrow{G(f'' \times 1)} & Y \times I \end{array}$$

which commutes up to δ since $f': X \rightarrow Y$ is onto. By the choice of j and δ there exists a map $\tilde{G}: Y \times I \rightarrow E_i$ such that

$$(4) \ d(\tilde{G}_0, q_{ij}gf'') < \epsilon, \text{ and}$$

$$(5) \ d(p_i \tilde{G}, r_{ij}G(f'' \times 1)) < \epsilon.$$

We now define $\tilde{H}: X \times I \rightarrow E_i$ by

$$(6) \ \tilde{H} = \tilde{G}(f' \times 1).$$

It is readily seen that

$$(7) \ d(\tilde{H}_0, q_{ij}h) < \epsilon, \text{ and}$$

$$(8) \ d(p_i \tilde{H}, r_{ij}H) < \epsilon.$$

The proof that (b) implies (a) follows the proof of Theorem 4 with the single modification of using the case $\dim X \leq n$ in Lemma 1.

We now further reduce the defining class λ for n -shape fibrations.

Theorem 6. A map $p: E \rightarrow B$ is an n -shape fibration if and only if it is an λ -shape fibration where λ is either of the following two classes:

(a) λ is the collection of all k -cells with $k \leq n$, i.e.,

$$\lambda = \{I^0, I^1, \dots, I^n\},$$

(b) λ is the class of compact polyhedra of dimension $\leq n$.

We precede the proof of Theorem 6 by a lemma.

Lemma 3. Let $p: \underline{E} \rightarrow \underline{B}$ be a level map which has the λ -AHLP where $\lambda = \{I^0, I^1, \dots, I^n\}$. Then for every i and $\epsilon > 0$ there is a $j \geq 1$ and a $\delta > 0$ such that the following approximate partial homotopy lifting property holds: Let X be a compact polyhedron with $\dim X \leq n$ and let $A \subseteq X$ be a sub-polyhedron. Let $h: (X \times 0) \cup (A \times I) \rightarrow E_j$, and $H: X \times I \rightarrow B_j$

be maps such that

$$(9) \quad d(p_j h, H| (X \times 0) \cup (A \times I)) < \delta.$$

Then, there is a homotopy $\tilde{H}: X \times I \rightarrow E_i$ such that

$$(10) \quad \tilde{H}| (X \times 0) \cup (A \times I) = q_{ij} h, \text{ and}$$

$$(11) \quad d(p_i \tilde{H}, r_{ij} H) < \epsilon.$$

Proof. The proof proceeds by induction on $\dim(X - A)$.

If $\dim(X - A) = 0$, let j and δ be a lifting index and a lifting mesh for i and ϵ . Then we extend $q_{ij} h$ to a map $\tilde{H}: (X \times 0) \cup ((A \cup X^0) \times I) \rightarrow E_i$ where for any vertex $x \in X^0 \setminus A$ the restriction $\tilde{H}_x = \tilde{H}|_{x \times I}$ is a homotopy satisfying

$$(12) \quad \tilde{H}_x(x, 0) = q_{ij} h(x) \text{ (see Remark 1), and}$$

$$(13) \quad d(p_i \tilde{H}_x, r_{ij} H|_{x \times I}) < \epsilon.$$

We now assume the assertion true for $\dim(X - A) < k \leq n$.

For a given i and ϵ choose a lifting index i' and a lifting mesh ϵ' for $\{I^0, I^1, \dots, I^n\}$. Now choose j and δ in accordance with the inductive hypothesis for the integer $k - 1$ and for i' and ϵ' . We also require that δ -close points map by $r_{i',j}$ into ϵ' -close points.

Now assume that $\dim(X - A) = k$ and that we are given h and H satisfying (9). By the inductive hypothesis there is a map $\tilde{H}': (X \times 0) \cup ((X^{k-1} \cup A) \times I) \rightarrow E_i$, such that

$$(14) \quad \tilde{H}'| (X \times 0) \cup (A \times I) = q_{i',j} h| (X \times 0) \cup (A \times I)$$

and

$$(15) \quad d(p_i \tilde{H}', r_{i',j} H| (X \times 0) \cup ((X^{k-1} \cup A) \times I)) < \epsilon'.$$

For each k -simplex C of $X \setminus A$ we consider the maps $\tilde{H}'| (C \times 0) \cup (\partial C \times I)$ and $r_{i',j} H| C \times I$. By the choice of i' and ϵ' , by (15) and by Remark 1, there is a map $\tilde{H}_C: C \times I \rightarrow E_i$ such that

$$(16) \tilde{H}_C | (C \times 0) \cup (\partial C \times I) = q_{ii}, \tilde{H}' | (C \times 0) \cup (\partial C \times I)$$

and

$$(17) d(p_i \tilde{H}_C, r_{ij} H | C \times I) < \epsilon.$$

Now the desired homotopy $\tilde{H}: X \times I \rightarrow E_i$ is given by

$$(18) \tilde{H} | (X \times 0) \cup ((X^{k-1} \cup A) \times I) = q_{ii}, \tilde{H}' \text{ and}$$

$$(19) \tilde{H} | C \times I = \tilde{H}_C.$$

This completes the proof of Lemma 3.

Proof of Theorem 6. It suffices (by Theorem 4) to show that (a) implies that p is an λ -shape fibration for the class of all locally compact separable polyhedra X with $\dim X \leq n$.

For a given i and ϵ choose $j \geq i$ and $\delta > 0$ in accordance with Lemma 3. Once again apply Lemma 3, this time to j and δ , to obtain j' and δ' . Let $X = |K|$, where K is a countable locally finite simplicial complex, $\dim X \leq n$, and let $h: X \rightarrow E_j$, and $H: X \times I \rightarrow B_j$, be such that $d(p_j, h, H_0) < \delta'$. By a standard construction, one can find two sequences of finite subpolyhedra $X_k, Y_k \subseteq X$, $k = 1, 2, \dots$, which cover X and are such that $X_k \cap X_{k'} = \phi$, $Y_k \cap Y_{k'} = \phi$ for $k \neq k'$ and $Y_k \cap (\cup X_i) = Y_k \cap (X_{k-1} \cup X_k)$. By the choice of j' and δ' there is a homotopy $\tilde{H}': (\cup X_k) \times I \rightarrow E_j$ such that

$$(20) \tilde{H}' | (\cup X_k) \times 0 = q_{jj}, h | (\cup X_k) \text{ and}$$

$$(21) d(p_j \tilde{H}', r_{jj}, H | (\cup X_k) \times I) < \delta.$$

Again we apply Lemma 3, this time to each pair $(Y_k, Y_k \cap (X_{k-1} \cup X_k))$ to obtain a homotopy $\tilde{H}_k: Y_k \times I \rightarrow E_i$ which extends $q_{ij} \tilde{H}' | (Y_k \cap (X_{k-1} \cup X_k)) \times I$ and $q_{ij} h | Y_k$ and is such that

$$(22) d(p_i \tilde{H}_k, r_{ij}, H | Y_k \times I) < \epsilon.$$

Clearly, the map $\tilde{H}: X \times I \rightarrow E_i$ given by $\tilde{H} | (\cup X_k) \times I = q_{ij} \tilde{H}'$

and $\tilde{H}|_{Y_k \times I} = \tilde{H}_k$ has all of the desired properties.

Example 1. For every n there is a map $p: E \rightarrow B$ which is an n -shape fibration, but which fails to be an $(n + 1)$ -shape fibration. Indeed, let $E = S^{n+1} \subseteq E^{n+2}$ be the unit sphere and let B be the segment $0 \times \cdots \times 0 \times [-1, 1] \subseteq E^{n+2}$ whose end-points are the south-pole e_0 and the north-pole e_1 of S^{n+1} . Let $p: E \rightarrow B$ be the projection defined by $p(x_1, \dots, x_{n+1}, x_{n+2}) = (0, \dots, 0, x_{n+2})$. Then p fails to have the AHLF for $X = S^{n+1}$ and is not an $(n + 1)$ -shape fibration. One can see this by considering the maps $h = \text{identity}$ and H defined by $H(x_1, \dots, x_{n+1}, x_{n+2}, t) = (0, \dots, 0, (1 - t)x_{n+2})$. However, the trivial level map \underline{p} (i.e., $\underline{p} = (p_i)$ and $p_i = p$) does have the AHLF with respect to compact polyhedra X where $\dim X \leq n$. This is an easy consequence of the following facts: For an arbitrary $\varepsilon > 0$ each map $h: X \rightarrow E$ is ε -homotopic to a map $X \rightarrow E \setminus \{e_0, e_1\}, 0 \times \cdots \times 0 \times [\varepsilon - 1, 1 - \varepsilon]$ is a strong deformation retract of B , and $p|_{E \setminus \{e_0, e_1\}}: E \setminus \{e_0, e_1\} \rightarrow B \setminus \{e_0, e_1\}$ is a product map and therefore has the homotopy lifting property.

Theorem 7. Let $p: E \rightarrow B$ be an n -shape fibration, let $e \in E$, $b = p(e)$, and $F = p^{-1}(b)$. Then, p induces an isomorphism of the homotopy pro-groups

$$(23) \quad \underline{p}_*: \text{pro} - \pi_k(E, F, e) \rightarrow \text{pro} - \pi_k(B, b) \text{ for all } k \leq n.$$

Theorem 2 of [12] is the analogue of Theorem 7 and its proof applies without change to establish Theorem 7. Notice that one can use Theorem 2 of the present paper to achieve that p be induced by a level map having the n -HLP.

Again by arguing as in [12], one obtains the following theorem.

Theorem 8. If $p: E \rightarrow B$ is an n -shape fibration, $e \in E$, $b = p(e)$, $F = p^{-1}(b)$, then the following (finite) sequence is exact:

$$(24) \quad \begin{array}{c} \text{pro} - \pi_n(F, e) \xrightarrow{\underline{i}_*} \text{pro} = \pi_n(E, e) \xrightarrow{\underline{p}_*} \text{pro} - \pi_n(B, b) \\ \xrightarrow{\underline{\delta}} \text{pro} - \pi_{n-1}(F, e) \rightarrow \dots \end{array}$$

The above sequence is obtained from the exact pro-homotopy sequence of the pointed pair (E, F, e) (see [10]) by replacing $\text{pro} - \pi_k(E, F, b)$ by $\text{pro} - \pi_k(B, b)$, $k \leq n$ (using Theorem 6). This also explains the morphisms \underline{i}_* , \underline{p}_* and $\underline{\delta}$.

6. Weak Shape Fibrations

Definition 7. A map $p: E \rightarrow B$ is called a *weak shape fibration* if it is an n -shape fibration for every n .

Notice that for the corresponding level maps p and for given i and ϵ the lifting index and lifting mesh depend on n .

The next corollary follows from Theorems 7 and 8.

Corollary 2. For a weak shape fibration $p: E \rightarrow B$ the morphism

$$\underline{p}_*: \text{pro} - \pi_k(E, F, e) \longrightarrow \text{pro} - \pi_k(B, b)$$

is an isomorphism of pro-groups for all k and one has an (infinite) exact sequence corresponding to (24).

Theorem 9. Let $p: E \rightarrow B$ be a weak shape fibration and let the points $x, y \in B$ belong to the same path component. If $\dim E < \infty$ and $\dim B < \infty$, then the fibers $X = p^{-1}(x)$ and $Y = p^{-1}(y)$ have the same shape.

Proof. The proof of Theorem 9 will follow from the scheme of proof of Theorem 3 of [11] once we establish the following fact: There exist an integer r and a level map $p': \underline{E}' \rightarrow \underline{B}$ of polyhedral sequences such that

- (1) $\dim E'_i \leq r$ for each i ,
- (2) $p = \lim p'$, and
- (3) p' has the HLP with respect to metric compacta of dimension $\leq r + 1$.

In order to establish this fact, embed E and B in Euclidean spaces R^n and R^m , respectively. One now obtains $p: \underline{E} \rightarrow \underline{B}$ by extending p to a map $\tilde{p}: R^n \rightarrow R^m$ and by considering suitable decreasing sequences of polyhedral neighborhoods of E and B respectively. It follows from Theorem 1 that for each k , p has the χ -AHLP where χ is the class of metric compacta of dimension $\leq k$. The proof of Theorem 2 now yields $p': \underline{E}' \rightarrow \underline{B}$ such that for each k (in particular, for $k = r + 1$ defined below), p' has the χ -HLP where χ is the class of metric compacta of dimension $\leq k$. Furthermore, the proof of Theorem 2 allows one to assume that the members E'_i of \underline{E}' are subpolyhedra of $E_i \times B_i$ which implies that $\dim E'_i \leq n + m = r$.

Theorem 10. *If E and B are compact ANR's, then $p: E \rightarrow B$ is a weak shape fibration if and only if it is an approximate fibration, or equivalently a shape fibration.*

Proof. An approximate fibration between compact ANR's is a shape fibration and thus also a weak shape fibration.

Conversely, if p is a weak shape fibration, then for each n the trivial level map of ANR-sequences p has the AHLP with respect to $\{I^0, I^1, \dots, I^n\}$ by Theorems 1 and 6. This

implies that p has the AHLP for all I^k , $k = 0, 1, \dots$. Hence, by Theorem 2.6 of [2], p is an approximate fibration.

7. Cell-like Maps are Weak Shape Fibrations

Theorem 11. Every cell-like map $p: E \rightarrow B$ between metric compacta is a weak shape fibration.

By specializing Lemma 8.3 of [4] one obtains the following lemma. (This Lemma can also be proved by using the techniques of Lemma 2.3 of [7].)

Lemma 4. Let $p: Q \rightarrow Q$ be a map, let $B \subseteq Q$ be a compact subset of Q , let $E = p^{-1}(B)$ and let $\text{Sh}(p^{-1}(b)) = 0$ for each $b \in B$. Then for any two neighborhoods $U \supseteq E$, $V \supseteq B$ with $p^{-1}(V) \subseteq U$ and for any $n \in \mathbb{N}$ and $\epsilon > 0$ there exist neighborhoods $U' \supseteq E$, $V' \supseteq B$ with $U' \subseteq U$, $V' \subseteq V$, $p^{-1}(V') \subseteq U'$ having the following property: If X is a compact polyhedron, $\dim X \leq n$, and $A \subseteq X$ a compact subpolyhedron and if $h: A \rightarrow U'$, $g: X \rightarrow V'$ are maps such that $ph = g|_A$, then there exists a map $\tilde{g}: X \rightarrow U$ such that $\tilde{g}|_A = h$ and $d(p\tilde{g}, g) < \epsilon$.

Proof of Theorem 11. Let $p: E \rightarrow B$ be a cell-like map between metric compacta. We embed E in Q and consider the quotient space $\tilde{Q} = Q/\{p^{-1}(b) : b \in B\}$ and the quotient map π . One can identify B with $\pi(E)$ so that $\pi|_E = p$. Since \tilde{Q} is a compact metric space, it can be embedded in Q . Notice that for $\pi: Q \rightarrow \tilde{Q} \subseteq Q$ one has $\pi^{-1}(B) = E$. Applying Lemma 4 to π one can construct by induction compact ANR-neighborhoods U_i and V_i of E and B respectively such that $\pi^{-1}(V_i) \subseteq U_i$, the neighborhoods U_i and V_i , $i \in \mathbb{N}$, form inclusion ANR-sequences with limits E and B respectively and $U' = U_i$, $V' = V_i$ satisfy

Lemma 4 for $U = U_{i-1}$, $V = V_{i-1}$, $n = i$ and $\varepsilon = \frac{1}{i}$.

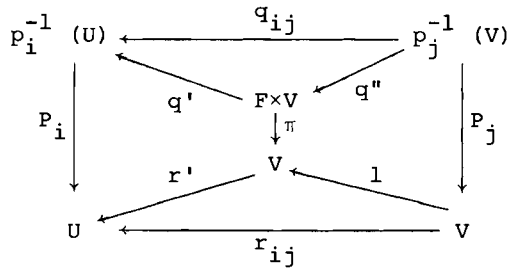
The maps $p_i = \pi|_{U_i}: U_i \rightarrow V_i$ form a level map \underline{p} , which induces p . Furthermore, for given i , $n \in \mathbb{N}$ and $\varepsilon > 0$ one can choose $j \in \mathbb{N}$ such that $j > \max\{i, n, \frac{1}{\varepsilon}\}$. If X is a compact polyhedron, $\dim X \leq n$, and if $h: X \rightarrow U_j$ and $H: X \times I \rightarrow V_j$ satisfy $H_0 = p_j h = \pi h$, then there is a homotopy $\tilde{H}: X \times I \rightarrow U_{j-1} \subseteq U_i$ such that $\tilde{H}_0 = h$ and $d(\pi\tilde{H}, H) < \frac{1}{j} < \varepsilon$. In view of Theorem 6 and Remark 2 this completes the proof.

Example 2. The Taylor map is a weak shape fibration (Theorem 11), which fails to be a shape fibration (Example 6 of [11]).

8. Shape Cell-bundles

In this section we exhibit for every cell-like map $p: E \rightarrow B$ a level map \underline{p} between ANR-sequences which induces p and which has a certain local factorization property. This yields an alternate proof of Theorem 11 and is of some independent interest.

Definition 8. A map $p: E \rightarrow B$ between metric compacta is called a *shape disk-bundle* (*shape Q-bundle*) if there is a level map of ANR-sequences $\underline{p}: \underline{E} \rightarrow \underline{B}$ which induces p and which has the following property: For each i there is a $j \geq i$ such that for each $b \in B_j$ there exists an open neighborhood V of b in B_j and an open set $U \subseteq B_i$ with $r_{ij}(V) \subseteq U$ such that for F the n -cell (the Hilbert cube Q) we have the following factorization diagram where $\pi: F \times V \rightarrow V$ is the second projection. We call shape disk-bundles and shape Q -bundles jointly *shape cell-bundles*.



Theorem 12. Every shape cell-bundle is a weak shape fibration.

Proof. Let $p: E \rightarrow B$ be a level map as in Definition 8. Notice that it will more than suffice to establish the following: given $i, n \in \mathbb{N}$ there exists $j \geq i$ such that for any compact polyhedron X , $\dim X \leq n$, and subpolyhedron $A \subseteq X$, and for any maps $H: X \rightarrow B_j$, $h: A \rightarrow E_j$ with $p_j h = H|_A$, there is a map $\tilde{H}: X \rightarrow E_i$ such that

- (1) $\tilde{H}|_A = q_{ij} h$, and
- (2) $p_i \tilde{H} = r_{ij} H$.

In order to prove this fact choose integers $i = i_0 \leq i_1 \leq \dots \leq i_{n+1} = j$ such that i_k satisfies Definition 8 with respect to i_{k-1} . Notice that we can find open covers \mathcal{V}^{i_k} of B_{i_k} , $k = 0, 1, \dots, n + 1$, such that any $V \in \mathcal{V}^{i_{k+1}}$ admits a $U \in \mathcal{V}^{i_k}$ satisfying Definition 8.

Suppose we are given the maps $H: X \rightarrow B_j$, $h: A \rightarrow E_j$ with $p_j h = H|_A$. Let K be a triangulation of X of fine enough mesh that simplices of K are mapped into elements of $\mathcal{V}^{i_{n+1}}$ under H . Let L^r denote the set of all simplices from K of dimension $\leq r$ which are not contained in A . One can easily find a map $\tilde{H}^0: A \cup |L^0| \rightarrow E_{i_n}$ such that

$$(3) \tilde{H}^0|A = q_{i_n j} h, \text{ and}$$

$$(4) P_{i_n} \tilde{H}^0 = r_{i_n j} H|A \cup |L^0|.$$

Inductively, suppose that $\tilde{H}^r: A \cup |L^r| \rightarrow E_{i_{n-r}}$ is such that

$$(5) \tilde{H}^r|A = q_{i_{n-r} j} h, \text{ and}$$

$$(6) P_{i_{n-r}} \tilde{H}^r = r_{i_{n-r} j} H|A \cup |L^r|.$$

In order to define $\tilde{H}^{r+1}: A \cup |L^{r+1}| \rightarrow E_{i_{n-r-1}}$ we extend $q_{i_{n-r-1} i_{n-r}} \tilde{H}^r$ to all $(r+1)$ -simplices Δ of L^{r+1} as follows. Since $\tilde{H}^r: \partial\Delta \rightarrow E_{i_{n-r}}$ is already defined, one can consider the map

$$(7) \pi' q'' \tilde{H}^r: \partial\Delta \rightarrow F,$$

where $\pi': F \times V \rightarrow F$ is the first projection. Since F is contractible, the map (7) can be extended to a map $g: \Delta \rightarrow F$.

Then we define $\tilde{H}^{r+1}|_{\Delta}: \Delta \rightarrow E_{i_{n-r-1}}$ by

$$(8) \tilde{H}^{r+1}|_{\Delta} = q'(g \times r_{i_{n-k} j} H|_{\Delta}).$$

It is easy to check that \tilde{H}^{r+1} so defined has all the properties needed in the inductive step. Finally, $\tilde{H} = \tilde{H}^n: X \rightarrow E_i$ is the required map satisfying (1) and (2).

Theorem 13. Every cell-like map $p: E \rightarrow B$ between finite-dimensional metric compacta is a shape disk-bundle.

We shall prove two preliminary lemmas.

Lemma 5. Let $p: E \rightarrow B$ be a cell-like map between finite dimensional metric compacta. Then, there exist $m, n \in \mathbb{N}$ and embeddings $j: B \rightarrow \mathbb{R}^n$, $i: E \rightarrow \mathbb{R}^m \times \mathbb{R}^n$ such that the projection $\pi: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the following conditions:

$$(9) \pi i = j p, \text{ and}$$

$$(10) \text{ for each } b \in B \text{ the set } i p^{-1}(b) \text{ is cellular in the}$$

hyperplane $\pi^{-1}j(b) = R^m \times j(b)$.

Proof. For $n \geq 2 \dim B + 1$ there is an embedding $j: B \rightarrow R^n$. For $r \geq 2 \dim E + 1$ there is an embedding $k: E \rightarrow R^r$. An easy general position argument enables one to verify that for each $b \in B$, $kp^{-1}(b) \subseteq R^r \times 0$ satisfies the cellularity criterion [13] in $R^{r+3} = R^r \times R^3$ provided $r \geq 2$ (e.g., p. 600 of [8]). Let $m = r + 3 \geq 5$ and let $i: E \rightarrow R^m \times R^n$ be given by

$$(11) \quad i(x) = (k(x), jp(x)).$$

Then, i is an embedding because k is an embedding. Furthermore, $\pi i = jp$ and for each $b \in B$ the set $ip^{-1}(b) = kp^{-1}(b) \times j(b)$ is cellular in $R^m \times j(b)$ by Theorem 3 of [13]. This completes the proof of Lemma 5.

Remark 7. From now on we shall identify $x \in E$ with $i(x)$ and $b \in B$ with $j(b)$. Therefore, $p: E \rightarrow B$ can be viewed as the restriction $\pi|_E$ of the projection $\pi: R^m \times R^n \rightarrow R^n$. Notice that $p^{-1}(b)$ is now cellular in $R^m \times b$ for each $b \in B$.

Definition 9. A finite collection \mathcal{C} of pairs (B_k, D_k) , $k = 1, 2, \dots, r$, will be called an *admissible collection* for p if it satisfies the following conditions:

$$(12) \quad B_1, \dots, B_r \text{ are PL } n\text{-cells in } R^n \text{ and } B \subseteq \bigcup_{k=1}^r \text{Int } B_k$$

$$(13) \quad B_k \cap B \neq \emptyset \text{ for } k = 1, \dots, r$$

$$(14) \quad D_1, \dots, D_r \text{ are PL } m\text{-cells in } R^m \text{ such that for}$$

$$k = 1, \dots, r,$$

$$p^{-1}(B \cap B_k) \subseteq (\text{Int } D_k) \times B_k.$$

With each admissible collection \mathcal{C} we associate two polyhedra

$$(15) \quad B_{\mathcal{C}} = \bigcup_{i=1}^r B_k \subseteq R^n, \text{ and}$$

$$(16) E_C = \bigcup_{i=1}^r (D_k \times B_k) \subseteq R^m \times R^n.$$

Notice that $\pi(E_C) = B_C$, where $\pi: R^m \times R^n \rightarrow R^n$ is the second projection. Also notice that B_C is a neighborhood of B by (12), and that E_C is a neighborhood of E since for $e \in E$ there exists by (12) a B_k such that $p(e) \in \text{Int } B_k$ and so by (14) $e \in \text{Int } D_k \times \text{Int } B_k \subseteq E_C$. However by the invariance of domain $\text{Int } D_k \times \text{Int } B_k$ is open in $R^m \times R^n$ and therefore $e \in \text{Int } E_C$.

Remark 8. An admissible collection always exists. In particular, if B_1 is a PL n -cell in R^n with $B \subseteq \text{Int } B_1$ and D_1 is a PL m -cell in R^m with $\pi'(E) \subseteq \text{Int } D_1$ (where π' is the first projection), then the single pair (B_1, D_1) forms an admissible collection.

Lemma 6. Let $C = \{(B_k, D_k)\}_{k=1}^r$ be an admissible collection for $p: E \rightarrow B$, let $X \subseteq E_C$ be an open neighborhood of E in $R^m \times R^n$ and let $Y \subseteq \bigcup \text{Int } B_k$ be a compact neighborhood of B in R^n . Let δ be the Lebesgue number for the open cover $\{\text{Int } B_k \cap Y\}_{k=1}^r$ of Y . Then, there exists an admissible collection $C' = \{(B'_k, D'_k)\}_{k'=1}^{r'}$ for p such that

$$(17) B_{C'} \subseteq Y,$$

$$(18) \text{diam } B'_{k'} < \delta/3, \quad k' = 1, \dots, r',$$

$$(19) D'_{k'} \times B'_{k'} \subseteq X, \quad k' = 1, \dots, r', \text{ hence } E_{C'} \subseteq X, \text{ and}$$

$$(20) D'_{k'} \times B'_{k'} \subseteq (\text{Int } D_k) \times B_k, \text{ for all } k \text{ such that}$$

$$B'_{k'} \subseteq \text{Int } B_k.$$

Proof. Let $b \in B$ and let $\Delta(b)$ denote the set of all indices $k \in \{1, \dots, r\}$ such that $b \in \text{Int } B_k$. Then, by (14)

$$(21) p^{-1}(b) \subseteq \bigcap_{k \in \Delta(b)} ((\text{Int } D_k) \times B_k).$$

Therefore,

$$(22) \pi'p^{-1}(b) \subseteq \bigcap_{k \in \Delta(b)} \text{Int } D_k.$$

By Lemma 5 we may choose a PL m -cell $D'_b \subseteq R^m$ such that

$$(23) \pi'p^{-1}(b) \subseteq \text{Int } D'_b \subseteq D'_b \subseteq \bigcap_{k \in \Delta(b)} \text{Int } D_k.$$

Since $p^{-1}(b) \subseteq X$, one can also achieve that

$$(24) D'_b \times b \subseteq X.$$

Now choose a PL n -cell neighborhood B'_b of b in Y so small that

$$(25) \text{diam } B'_b < \delta/3,$$

$$(26) p^{-1}(B \cap B'_b) \subseteq (\text{Int } D'_b) \times B'_b \subseteq D'_b \times B'_b \subseteq (\bigcap_{k \in \Delta(b)} \text{Int } D_k)$$

$$\times B'_b, \text{ and}$$

$$(27) D'_b \times B'_b \subseteq X.$$

Since B is compact, the covering $\{\text{Int } B'_b \mid b \in B\}$ of B admits a finite subcover. Thus, we obtain a finite collection B'_1, \dots, B'_r , of PL n -cells and a corresponding collection D'_1, \dots, D'_r , of PL m -cells. Then $C' = \{(B'_k, D'_k)\}_{k=1}^r$ is an admissible collection which satisfies (17)-(20). Notice in particular that (14) and (20) follow from (26).

Proof of Theorem 13. Let $p: E \rightarrow B$ be a cell-like map between finite-dimensional compacta. Then we identify p with $\pi|E$ as described in Remark 7.

We shall now define by induction a sequence of admissible collections C_i , $i = 1, 2, \dots$. We choose for C_1 an arbitrary admissible collection (Remark 8). Given $C_i = \{(B_k, D_k)\}_{k=1}^r$ we define $C_{i+1} = \{(B'_k, D'_k)\}_{k=1}^r$ by applying Lemma 6 to $C = C_i$, $X = X_i$ contained in the $1/i$ -neighborhood $N(E, 1/i)$, $Y = Y_i$ contained in $N(B, 1/i)$ and to $\delta = \delta_i$ the Lebesgue number for $\{\text{Int } B_k \cap Y_i\}_{k=1}^r$. We obtain in this way a level map $p = (p_i)$ of polyhedral inclusion sequences with $p_i = \pi|E C_i$

where $\lim \underline{p} = p = \pi|_E$:

$$\begin{array}{ccccc} E C_1 & \longleftarrow & E C_2 & \longleftarrow & \dots \\ \downarrow p_1 & & \downarrow p_2 & & \\ B C_1 & \longleftarrow & B C_2 & \longleftarrow & \dots \end{array}$$

We claim that \underline{p} satisfies all the conditions of Definition 8 with $j = i + 1$ and F an m -cell D_k (depending on $b \in B$).

Indeed, if $b \in B$, one can choose by (12) a k'_0 such that $b \in \text{Int } B'_{k'_0} = V$. Then by (16) we have

$$(28) \quad (p_{i+1})^{-1}(V) = \cup_{k'=1}^r D'_{k'} \times (V \cap B'_{k'}).$$

Let $J = \{k' : V \cap B'_{k'} \neq \emptyset\}$. Then by (18) there exists a k such that

$$(29) \quad \cup_{k' \in J} B'_{k'} \subseteq \text{Int } B_k = U.$$

Notice that $k'_0 \in J$ and therefore

$$(30) \quad V \subseteq U.$$

By (20) and (29)

$$(31) \quad \cup_{k' \in J} (D'_{k'} \times B'_{k'}) \subseteq (\text{Int } D_k) \times U$$

and so by definition of J and V

$$(32) \quad \cup_{k'=1}^r D'_{k'} \times (V \cap B'_{k'}) \subseteq (\text{Int } D_k) \times V.$$

By (28) and (32) we have

$$(33) \quad (p_{i+1})^{-1}(V) \subseteq D_k \times V,$$

and by (29), (30) and (16) we have

$$(34) \quad D_k \times V \subseteq \cup_{k=1}^r D_k \times (U \cap B_k) = (p_i)^{-1}(U).$$

Consequently, $V, U, F = D_k$ and the inclusions r', q'', q' given by (30), (33) and (34) respectively satisfy all the conditions of Definition 8.

Theorem 14. Every cell-like map $p: E \rightarrow B$ between metric compacta is a shape Q -bundle.

The proof of Theorem 14 is the same as the proof of Theorem 13 except that Lemma 7 plays the role of Lemma 5.

The roles of the PL-cells in R^m is taken by PL-Hilbert cubes in $Q = \prod_{j=1}^{\infty} [-1,1]$ defined as compacta of the form

$$(35) D \times \prod_{j>i} [-1,1] \subseteq Q,$$

where $D \subseteq \prod_{j=1}^i [-1,1] \subseteq R^i$ is a PL i -cell.

Notice that the union of a finite collection of PL-Hilbert cubes is an ANR. (In fact, it is a Q -manifold.)

The role of cellular sets in R^m is taken by cellular sets $X \subseteq Q$ defined as intersections of sequences

$$(36) Q_1 \supseteq \text{Int } Q_1 \supseteq Q_2 \supseteq \dots,$$

where each Q_i is a PL-Hilbert cube, i.e., is of the form (35).

Remark 9. The proof of Theorem 1 of [9] shows that every continuum X of trivial shape which is contained in $Q^+ = \prod_{i=1}^{\infty} [0,1]$ is a cellular set in Q .

Lemma 7. Let $p: E \rightarrow B$ be a cell-like map between metric compacta. Then, there exist embeddings $j: B \rightarrow Q$ and $i: E \rightarrow Q \times Q$ such that the second projection $\pi: Q \times Q \rightarrow Q$ satisfies the following conditions:

$$(37) \pi i = jp,$$

$$(38) \text{ For each } b \in B, \text{ the set } ip^{-1}(b) \text{ is cellular in } \pi^{-1}j(b) = Q \times j(b).$$

Proof. Let $j: B \rightarrow Q$ and $k: E \rightarrow Q^+ \subseteq Q$ be arbitrary embeddings. Consequently, by Remark 9 $kp^{-1}(b) \subseteq Q^+$ is cellular in Q . Let $i: E \rightarrow Q \times Q$ be given by

$$(39) i(x) = (k(x), jp(x)).$$

Clearly, (37) and (38) hold because $ip^{-1}(b) = kp^{-1}(b) \times j(b)$ is cellular in $Q \times j(b)$.

Example 3. By Theorem 14, the Taylor map p , [16], is a shape Q -bundle and therefore a weak shape fibration. However, the Taylor map is not a shape fibration (Example 6 of [11]). Consequently, if $p = \lim \underline{p}$, the maps p_i cannot be bundle maps.

In spite of Example 3, we may pose the following questions for the finite-dimensional case.

Question 1. Is every cell-like map between finite-dimensional metric compacta induced by a level map of ANR-sequences $\underline{p}: \underline{E} \rightarrow \underline{B}$ such that each $p_i: E_i \rightarrow B_i$ is a disk-bundle?

An affirmative answer to this question was answered by T. B. Rushing in "A characterization of inverse limits of n -disk bundle maps," Notices, A.M.S. 193, V. 26, no. 3, abstract 765-627.

Question 2. Is there a notion of a "shape bundle" which satisfactorily generalizes the notion of a shape cell-bundle?

This question has been answered affirmatively by Šime Ungar in "Shape Bundles" (to appear).

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