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Two of the most interesting unsolved problems of general topology are the "S and L problem" and the problem of whether every regular, para-Lindelöf space is paracompact. The purpose of this article is to give an overview of the current status of these problems and many related ones. Since longer surveys on the first problem are to appear in the near future, the treatment here is a straightforward listing of equivalent problems, related problems, and consistency results. Much less is known about the second problem and the problems related to it (and there are no consistency results on it even now!), so I have adopted a looser and more ample format, even passing on a suggestion of Hasan Hdeib concerning a specialized class that may be more amenable to analysis. I wish to thank him for bringing this class to my attention, and to thank Franklin Tall and Michael Wage for their advice on the first and second problems, respectively.

1. The S and L Problem

Is there an S-space? an L-space? ["S-space" will mean "regular hereditarily separable, but not hereditarily Lindelöf space"; "L-space" interchanges "separable" and "Lindelöf".]

[In this problem, "space" will always mean "regular Hausdorff space"].

Equivalent problems. [For S-spaces] Does there exist a hereditarily separable, uncountable, right-separated space?

(A space X is right-separated [resp. left-separated] if it can be arranged in a transfinite sequence, $X = \{x_\alpha : \alpha < \tau\}$ so that every initial segment is open [resp. closed].) Does there exist a zero-dimensional S-space? a Lindelöf S-space?

[For L-spaces] Does there exist a hereditarily Lindelöf, uncountable left-separated space? Does there exist a zero-dimensional L-space? a separable L-space?

[For S-spaces] Does there exist a space of countable spread which is not [hereditarily] Lindelöf? (A space is of *countable spread* if every discrete subspace is countable.)

[For L-spaces] Does there exist a space of countable spread which is not [hereditarily] separable?

There is an S-space [resp. L-space] if, and only if there is a collection of subsets $\{S_\alpha : \alpha < \omega_1\}$ of ω_1, $\alpha \in S_\alpha \subset [\alpha, \omega_1)$ [resp. $\alpha \in S_\alpha \subset (0, \alpha]$] such that for each choice of finite subsets $K_\alpha \subset S_\alpha$, $B_\alpha \subset \omega_1 - S_\alpha$, and each uncountable $S \subset \omega_1$, there exists a pair of distinct ordinals α, β in S such that $K_\alpha \subset S_\beta$ and $B_\alpha \subset \omega_1 - S_\beta$. (Nyikos) This is a set-theoretic translation of the result [15] that there is an S-space [resp. L-space] if, and only if, there is a subspace $\{f_\alpha : \alpha < \omega_1\}$ of 2^{ω_1} with the product topology such that $\alpha \in f_\alpha^{-1}(1) \subset [\alpha, \omega_1)$ (resp. $\alpha \in f_\alpha^{-1}(1) \subset (0, \alpha]$] and such that every discrete set of f_α's is countable.

Remarks. Many other equivalent problems can be concocted, especially by making use of various theorems of the form separable + ________ => Lindelöf (in the blank, one can put e.g. paracompact, meta-Lindelöf...) or countable spread + ________ <=> Lindelöf. One also has theorems of the sort, "If X is separable, then X is of countable spread <=> ________" which can be used to obtain equivalent problems "Is there a separable space satisfying ________ which is not Lindelöf? Few such theorems are available for getting from Lindelöf to separable, spoiling the illusion of duality created above. The illusion is further undermined by results like Tall's "If there is an S-space, there is one which is not completely regular" which have no counterpart for L-spaces.

Related problems. A. Does there exist a countably compact S-space? [Note: this remains unsolved if "regular" is dropped in the definition of S-spaces.] Does there exist an L-space in which every countable subset is closed?

B. Is there an S-space of cardinality > c? an L-space of weight > c? [Note: this is the same question as the one with "hereditarily separable" in place of "S-" and "hereditarily Lindelöf" in place of "L-", since no hereditarily Lindelöf space is of cardinal > c and no separable space is of weight > c.]

C. Does there exist a perfectly normal, or a hereditarily normal S-space?

D. Does there exist a first countable S-space?

E. Does there exist a locally connected S or L space?

F. Does there exist a space of countable spread which is not the union of a hereditarily separable and a hereditarily Lindelöf space?

G. "Are the S and L problems the same?" That is, does the existence of an S-space in a given model of set theory imply the existence of an L-space, and conversely?

H. Does there exist a cardinal α for which there exists a space with no discrete subspace of cardinal α, but which is not α-separable? not α-Lindelöf?

Consistency results. Using a forcing argument, Hajnal and Juhász constructed, for each cardinal α, a model which there exists an α-hereditarily separable space of cardinality 2^{2^α} and an α-hereditarily Lindelöf space of weight 2^{2^α}, such that every subset of cardinality $\leq \alpha$ is closed.

Under $\diamondsuit$, Ostaszewski constructed a locally compact, locally countable, perfectly normal, countably compact S-space, and Fedorchuk constructed a compact, hereditarily normal, hereditarily separable space of cardinality 2^c.

Assuming CH, Hajnal and Juhász constructed an L-space of weight 2^c such that every countable subset is closed; a first countable S-space; and a hereditarily normal, countably compact, non-compact topological group which is an S-space.

Also under CH: Rudin and Zenor constructed perfectly normal S-manifolds of any dimension ≥ 2 (and using $\diamondsuit$, they made them countably compact); Juhász, Kunen, and Rudin constructed a locally compact, locally countable, perfectly normal S-space which can be modified to give a first countable, compact S-space; and Kunen constructed a compact 0-dimensional L-space, and a compact strong S-space.

Assuming ♣, Wage constructed an extremely disconnected S-space. Ginsburg came up with a more general construction, and also one for L-spaces using $\diamondsuit$.

A Dedekind-complete Souslin line with endpoints is a compact L-space, and can be modified to give a compact, locally connected L-space if it is not one already.

Assuming the existence of a Souslin line, Mary Ellen Rudin constructed a normal S-space (this was the first known S-space).

Assuming either CH or the existence of a Souslin line, Roitman constructed a space of countable spread which is not the union of a hereditarily separable and a hereditarily Lindelöf subspace. She has also constructed such spaces in forcing extensions in which neither axiom holds.

Assuming CH, van Douwen, Tall, and Weiss constructed a 0-dimensional L-space with a point-countable base. The construction revolves around the existence of certain Luzin spaces and hence is strictly weaker than CH (for example, it works in any model with a Souslin line) but is destroyed by MA + $\neg$CH (Kunen).

Assuming ($\downarrow$), van Douwen and Kunen have constructed first countable S and L spaces. The axiom ($\downarrow$), which is strictly weaker than CH but is destroyed by MA + $\neg$CH, states that there exists an uncountable Noetherian collection of subsets of ω, such that every incomparable subcollection is countable.

Z. Szentmiklóssy has constructed models by forcing in which MA + $\neg$CH holds, yet there is an S-space.

There are many theorems which say that the existence of certain <u>kinds</u> of S and L spaces is independent of ZFC. The most important are the theorems of Z. Szentmiklóssy which state that under MA + $\neg$CH, no compact space of countable tightness, and no compact hereditarily normal space can contain an S or L subspace, and that under MA + $\neg$CH, there is no first countable L-space. Also of interest are the theorems of Kunen and Zenor, showing (respectively) that MA + ¬CH implies there is no strong S-space, and that the existence of a strong S-space is equivalent to that of strong L-spaces. [A space X is a *strong* S-*space* if X^n is an S-space for each finite n; strong L-spaces are defined analogously.]

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See also K4, Classic Problem II (vol. 1) and Classic Problems VI and VII (vol. 2) [Note: Fedorchuk showed his axiom Φ is equivalent to $\diamondsuit$.] Survey articles by I. Juhász and M. E. Rudin on this problem are to appear in the near future, one in the proceedings of the 1978 Budapest conference (which will include papers by Z. Szentmiklóssy and F. Tall on the problem), the other in a book of surveys in general topology edited by G. M. Reed.

2. Para-Lindelöf Spaces

The main problem in this area is the following: Is every regular para-Lindelöf space paracompact? (A space is *para-Lindelöf* if every open cover has a locally countable open refinement.) Equivalently: Is every regular para-Lindelöf space normal? [This is an observation of J. van Mill: if there exists a para-Lindelöf space X which is not paracompact, then by Tamano's theorem, $X \times \beta X$ is not normal; and clearly, $X \times \beta X$ is still para-Lindelöf.]

The subject of para-Lindelöf spaces is a wide open field, with very little known about which implications hold between covering or separation axioms (regular or beyond: for the duration of this problem, "space" will mean "regular space"), besides those that hold for topological spaces in general. Consider the following properties: regular, completely regular, normal, collectionwise normal; countably metacompact, countably paracompact, realcompact, (weakly) submetacompact, metacompact, paracompact. It is not known whether para-Lindelöf together with any of these properties implies another property if it does not already do so for all spaces.

The list would even have included "subparacompact," were it not for some recent theorems by Dennis Burke:

Theorem. A submetacompact (old name: θ-refinable) *space in which every open cover has a σ-locally countable* (not necessarily open) *refinement is subparacompact.*

Corollary. A submetacompact para-Lindelöf space is subparacompact.

Practically everything else we know about the separation and covering properties of para-Lindelöf spaces is to be found in [1].

Our knowledge of what implications hold between "generalized metric" properties in the presence of the para-Lindelöf property is also very sketchy. We do not even know whether every para-Lindelöf normal Moore space is metrizable, nor whether every para-Lindelöf Moore space is normal (despite being strongly collectionwise Hausdorff [1]) or metacompact.

We are somewhat better off as regards our knowledge of spaces with σ-locally countable bases. We now know that a space with a σ-locally countable base (σ-LCB) need not be para-Lindelöf (even if it is a Moore space [1]) nor need it even be countably metacompact (even if the σ-LCB is also σ-disjoint [4]). We also have a nice conversion of covering properties to base properties, first expounded upon by Aull [2] and extended slightly by Fleissner and Reed [1] and still further by Burke: a consequence of Burke's theorem above and Corollary 2.2 of [1] is:

Corollary. A submetacompact space with a σ-LCB *is a Moore space.*

However, we do not know what happens when normality is brought into the picture - whether, on the one hand, every normal space with a σ-LCB is metrizable or, on the other, whether it is consistent that there be a normal Moore space with a σ-LCB which is not metrizable. Similarly, it is not known whether a collectionwise normal or monotonically normal space with a σ-LCB is metrizable. (However, we do know [6] that every suborderable space with a σ-LCB is metrizable.) For that matter - does anyone know of a "real" example of a normal space with a point-countable base which is not paracompact?

Worst of all, we do not know what para-Lindelöf adds to having a σ-LCB. For all we know, every para-Lindelöf space with a σ-LCB may be metrizable (equivalently, paracompact); on the other hand, there may even be ones that are not countably metacompact, or completely regular.

Those who would like to study these problems but cannot get a good handle on them might want to look at a specialized class for which there is a better structure theory, suggested by Hasan Hdeib. Call a subset T of a space X S-*open* if it can be written in the form $U \setminus S$, where U is open and S is separable. Call a collection $\mathcal{U}$ of sets S-*locally finite* if for each point p of X there exists an S-open set T containing p such that T meets only finitely many members of $\mathcal{U}$. Call a space S-*paracompact* if every open cover has an S-locally finite open refinement.

Every S-paracompact space is metacompact, and para-Lindelöf. [Indeed if one substitutes "countable" for "finite" in the definitions, one obtains a condition *equivalent* to para-Lindelöf.] So, by Burke's theorem, these spaces are subparacompact. Moreover, the points without separable neighborhoods form a closed, paracompact subspace. It also follows from Burke's theorem (but there is also a more direct proof) that every space with a σ-S-locally finite base is a Moore space. And Example 2.5 of [1] is a non-metrizable (and non-para-Lindelöf) space with such a base.

But there is still much that is not known about these spaces; for example, whether every S-paracompact space with a σ-S-locally finite base is metrizable. In fact, all the unsolved problems mentioned above for para-Lindelöf spaces are also open for S-paracompact spaces, unless some implication holds with the addition of "metacompact" or "subparacompact". (For example, it is a theorem, hardly worth mentioning, that every collectionwise normal, S-paracompact space is paracompact.)

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