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by

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1. Background

Is there an American general topologist who has not heard of the normal Moore space problem? The unique position this problem has long occupied in our minds may be attributed mostly to the unique influence R. L. Moore and his Texas school have had on the course of general topology here. Even for people who have gone on to other areas of topology, the problem seems to have an aura which is not explainable by its intrinsic qualities. Had history unfolded otherwise, we might be hearing far more about the more general problem of whether every first countable normal space is collectionwise normal; or about the more special problem of whether every normal space with a uniform base is metrizable. However, one way or the other, general topologists could not, I believe, have helped but become intrigued by this little circle of related problems. Why should first countability have any bearing on whether a normal space is collectionwise normal? Why is it that the tricks we use to destroy metrizability in a space with a uniform base, always seem to destroy normality as well?

The main milestones in the history of the problem give us many other things to puzzle over. Right at the very beginning, set theory intruded: F. B. Jones, who first posed the problem in print [8] in 1937, showed that if one assumes $2^{\aleph_0} < 2^{\aleph_1}$, then one can prove that every separable normal

Moore space is metrizable. His paper contains the statement, so quaint-sounding to our modern ears, that "the author has tried for some time without success to prove that $2^{\aleph_1} > 2^{\aleph_0}$." We have known since 1963 that this natural-seeming axiom is actually independent of the usual axioms of set theory. As for Jones' result about separable normal Moore spaces, that has been known to be independent since 1967. On the topological side, Bing showed [1] that the existence of a Q-set on the real line is sufficient, and Heath [7] that it is necessary, for the existence of a separable nonmetrizable normal Moore space. On the set-theoretic side, the credit is not so easy to assign, and the account in [15, Chapter III] makes one wonder just what it means to be the discoverer of a theorem. Silver and Solovay are the names most often mentioned in connection with the consistency of there being a Q-set; but a great deal of the work was already done by Rothberger in 1948 [12], [15]. And the contributions of Franklin Tall to this phase of the problem have yet to be adequately chronicled.

Once one gets over the shock of set theory intruding into the existence of separable nonmetrizable normal Moore spaces, the results do arrange themselves rather neatly. The nonseparable case is much more chaotic. One would think that since the continuum hypothesis (CH) has to be negated so strongly to get a separable example, perhaps the assumption of CH would destroy all examples; but Shelah has recently shown that the existence of a nonmetrizable normal Moore space is at least consistent with CH. Still more recently, Devlin and Shelah have shown that it is consistent with $\diamondsuit$ + GCH. Fleissner seemed to be making progress with the stronger axiom

V = L, showing [3], [4] that it implies every normal space of character $\leq c$ is collectionwise Hausdorff, and we have long known:

Theorem 1. (Bing [1]) *A Moore space is metrizable if, and only if, it is collectionwise normal.*

It may still turn out that V = L implies every normal Moore space is metrizable, but the big breakthrough of which I will speak in the next section came at the opposite end of the spectrum: an axiom which implies that the cardinality of the continuum is so great that the aleph which expresses it has a subscript equal to itself (And this is only the beginning! See [14] for further details on how big c must be.), an axiom which implies the consistency of there being a measurable cardinal, and so its own consistency is not provable in ZFC--such is the axiom which I found to imply that every normal Moore space is metrizable, and indeed that every first countable normal space is collectionwise normal.

I have already proven an even more general result elsewhere [11], but I will give a slightly rearranged proof in Section 2, because there is a detail in that proof which will lead us to a number of set-theoretic statements in Section 4, statements implying every first countable normal space is collectionwise normal. If one of these could be shown to hold in some "nice" model of set theory (that is, a model whose consistency follows from that of ZFC) the normal Moore space problem would be finally laid to rest. On the other hand, if it could be shown that they imply the consistency of there being an inaccessible cardinal, then it will become reasonable to try to show "every normal Moore space is metrizable" implies that also.

2. What Large Cardinals Can Do

This section includes a proof that the Product Measure Extension Axiom (PMEA) implies every first countable normal space is collectionwise normal. This axiom, whose consistency was shown by Kunen over five years ago [unpublished result!] to follow from the consistency of there being a strongly compact [2] cardinal, has to do with the usual measure on the product of two-point sets. In what follows, let λ be a cardinal number, defined as the set of all ordinals whose cardinality $<\lambda$. I will use the natural correspondence between the space of functions, ${}^{\lambda}2$, and the power set $\mathcal{P}(\lambda)$, to make the argument simpler: with each function one associates its support, and with each subset of λ, one associates its characteristic function. Thus a basic clopen subset of $\mathcal{P}(\lambda)$ will consist of all subsets of λ which contain a fixed finite subset of λ and miss another fixed finite subset, and its product measure will be $1/2^n$, where n is the cardinality of the union of these two finite sets. We are primarily concerned with clopen sets of the form $\mathcal{B}_\alpha = \{A \subset \lambda | \alpha \in A\}$, whose measure is 1/2, of the form $\mathcal{B}_\alpha - \mathcal{B}_\beta = \{A \subset \lambda | \alpha \in A, \beta \notin A\}$, whose measure is 1/4, and $\mathcal{B}_\alpha \Delta \mathcal{B}_\beta = (\mathcal{B}_\alpha - \mathcal{B}_\beta) \cup (\mathcal{B}_\beta - \mathcal{B}_\alpha)$, whose measure is 1/2; and of course $\mu(\mathcal{P}(\lambda)) = 1$. This measure can be extended in the usual way to all Baire subsets of the product space. [6, §58]

The *Product Measure Extnesion Axiom* (PMEA) is that, no matter what the cardinality λ, this measure can be extended to a c-additive measure μ defined on all subsets of ${}^{\lambda}2 \simeq \mathcal{P}(\lambda)$. A measure is called c-additive if the union of fewer than c sets of measure 0 is likewise of measure 0. Actually, for first countable spaces, countable additivity is enough; but we cannot escape "large cardinals" by weakening the axiom in this way. Solovay has shown [14] that the following two axioms are equiconsistent, i.e. if there is a model of one, there is a model of the other:

A. ZFC + "there exists a measurable cardinal"

B. ZFC + "there exists a countably additive measure μ on the real line $\mathbf{R}$ extending Lebesgue measure and defined on all subsets of $\mathbf{R}$."

Now, when $\lambda = \aleph_0$, ${}^{\lambda}2$ is homeomorphic to the Cantor set, and the product measure can be tied together with Lebesgue measure by using the Cantor function. So another equivalent condition is:

C. ZFC + "PMEA holds for $\lambda = \aleph_0$, with c-additivity weakened to countable additivity."

The following theorem, coupled with Theorem 1, shows that every normal Moore space is metrizable under PMEA.

Theorem 2. [PMEA] *Every first countable normal space is collectionwise normal.*

Proof. Let μ be a countably additive measure, extending the measure on $\mathcal{P}(\lambda)$ just described, such that $\mu(\mathcal{A})$ exists for all $\mathcal{A} \subset \mathcal{P}(\lambda)$.

Let X be a first countable normal space and let $\{C_\alpha \mid \alpha < \lambda\}$ be a discrete collection of closed subsets of X. For each $A \subset \lambda$, there exist disjoint open subsets U_A and V_A of X containing $\cup\{C_\alpha | \alpha \in A\}$ and $\cup\{C_\alpha | \alpha \notin A\}$ respectively, because X is normal and these are disjoint closed sets.

For each α and each $p \in C_\alpha$, let $\{U_n(p)\}_{n=1}^\infty$ be a base of open neighborhoods at p. Let

$$\mathcal{A}[p,n] = \{A \subset \lambda | U_n(p) \subset U_A \text{ or } U_n(p) \subset V_A\}.$$

Because the $U_n(p)$ form a local base, there exists for each $p \in C_\alpha$ and each $A \subset \lambda$ an integer n such that $U_n(p) \subset U_A$ if $\alpha \in A$ or $U_n(p) \subset V_A$ if $\alpha \notin A$. We will now show, using PMEA:

Claim. For each α *and each* $p \in C_\alpha$, *it is possible to choose an integer* n_p *such that, if* $q \in C_\beta$, $\beta \neq \alpha$, *the set* $\mathcal{A}[p,n_p] \cap \mathcal{A}[q,n_q]$ *contains a set splitting* α *from* β, *i.e. a set* A *such that either* $\alpha \in A$, $\beta \notin A$, *or* $\beta \in A$, $\alpha \notin A$.

Once the claim in proven, we simply surround each point p by the open set $U_{n_p}(p)$. Then, given any $p \in C_\alpha$, $q \in C_\beta$, $\beta \notin \alpha$, we have either $U_{n_p}(p) \subset U_A$, $U_{n_q}(q) \subset V_A$ (if $\alpha \in A$, $\beta \notin A$) or $U_{n_p}(p) \subset V_A$, $U_{n_q}(q) \subset U_A$ (if $\alpha \notin A$, $\beta \in A$), where A is as in the claim. Then, since U_A and V_A are disjoint, so are $U_{n_p}(p)$ and $U_{n_q}(q)$. Now let $U_\alpha = \cup\{U_{n_p}(p) | p \in C_\alpha\}$. If $\alpha \neq \beta$, we have $U_\alpha \cap U_\beta = \emptyset$, and so we have put the C_α into disjoint open sets U_α, as desired.

Proof of Claim. [This is the only place where PMEA is used.] By the statement just before the claim, $\mathcal{A}[p,n] \nearrow \mathcal{P}(\lambda)$ for all p. [The notation is from measure theory; it says that $\mathcal{A}[p,n] \subset \mathcal{A}[p,n+1]$ for all n and $\cup_{n=1}^\infty \mathcal{A}[p,n] = \mathcal{P}(\lambda)$.] Hence $\mu(\mathcal{A}[p,n]) \to 1$ by countable additivity. Thus there exists n_p such that $\mu(\mathcal{A}[p,n_p]) > 7/8$. If we pick n_p for all p, then $\mu(\mathcal{A}[p,n_p] \cap \mathcal{A}[q,n_q]) > 3/4$. Suppose $p \in C_\alpha$, $q \in C_\beta$, $\alpha \neq \beta$. Because $\mu(\mathcal{B}_\alpha - \mathcal{B}_\beta) = 1/4$, there exists

$$A \in \mathcal{A}[p,n_p] \cap \mathcal{A}[q,n_q] \cap (\mathcal{B}_\alpha - \mathcal{B}_\beta).$$

But A splits α from β, as desired.

Actually, we have done more than prove the claim. We have gotten the sets $\mathcal{A}[p,n_p] \cap \mathcal{A}[q,n_q]$ to contain sets splitting any given α from any given β. It would be nice if we could show the two tasks are equivalent, because the set-theoretic reformulations of this second task (see section 4) are much simpler than any I have come up with for the first.

The proof of Theorem 2 made use of the huge variety of pairs of disjoint open sets which were available, one pair for each element of $\mathcal{P}(\lambda)$. The underlying idea was to pick a neighborhood of each point which is contained in some member of "enough" pairs, and hence disjoint from the other member. The measure gave us a highly quantitative way of interpreting "enough." [How often do you see that number 7/8 in a topology paper?!] Is there any way of interpreting "enough" which will prove the claim, yet bypass the theory of large cardinals? That problem is what the rest of this paper centers on.

3. Baire Category Arguments

Except for Lemma 1, the material in this section actually goes back to the summer of 1976, when I had some fascinating discussions about the normal Moore space problem with Mike Starbird. Through them, I already realized back then that if the claim in Theorem 2 can be made to hold in general, the theorem itself would follow. But instead of attacking the claim with measures, I looked at a Baire category interpretation of a "large enough" subset of ${}^\lambda 2$. This may explain in part why I was so long in thinking about measures: we usually think of "first category" sets and "measure zero" sets about on a par with each other, even though neither property implies the other on the real line; and since the Baire category results below are very far from a solution, why should one expect more from measures? What I failed to take into account was that measures give us many possible interpretations of "large"; being "not of measure zero" is actually one of the crudest, and it would not have been enough for Theorem 2.

Another contributing factor in my blindness was the fact that the Baire category arguments below are done in ZFC, and in those days I preferred to avoid even simple axioms like CH, let alone measurable cardinals.

Let us begin with a sequence of subsets of ${}^{\lambda}2$, $\{\mathcal{A}(n)\}_{n=1}^{\infty}$, filling up ${}^{\lambda}2$ monotonically. By the Baire category theorem on the compact space ${}^{\lambda}2$, there is an integer n such that $\mathcal{A}(n)$ contains a dense subset $\mathcal{D}$ of some basic clopen set $\mathcal{B}$. The elements of $\mathcal{B}$ are required to take on the value 0 on a finite subset Z of λ and the value 1 on another finite subset S; and every function that does this will be in $\mathcal{B}$. Because $\mathcal{D}$ is dense in $\mathcal{B}$, it is possible to find a function in $\mathcal{D}$ which will behave any preassigned way on a finite set of indices in $\lambda - (Z \cup S)$. In particular, if $\alpha, \beta \notin Z \cup S$, there is a function $f \in \mathcal{D}$ such that $f(\alpha) = 1$, $f(\beta) = 0$; the corresponding member of $\mathcal{P}(\lambda)$ thus splits α from β. Also, if $\alpha \in Z \cup S$, $\beta \notin Z \cup S$, there is a function $f \in \mathcal{D}$ which behaves on β in the opposite way from its behavior on α. The only pairs of ordinals that give us trouble are those for which both members are in Z, or both in S. These we can split by "climbing a little higher," finding an integer $m \geq n$ such that for any pair $\alpha, \beta \in Z$, there is an $f \in \mathcal{A}(m)$ which takes on the value 1 on α and 0 on β; and similarly for S.

Of course, this argument does much better than just split pairs of ordinals. The following lemma, which will be used in Section 4, explores another possibility. Its statement and proof are strongly reminiscent of what we did using PMEA, except that it has to do with individual $\mathcal{A}(n)$'s instead of their pairwise intersections.

Lemma 1. *Let* N *be a fixed positive integer and let* $\{\mathcal{A}(n)\}_{n=1}^{\infty}$ *be a sequence of subsets of* ${}^{\lambda}2$ *such that* $\mathcal{A}(n) \nearrow {}^{\lambda}2$. *There exists an integer* m *such that* $\mathcal{A}(m)$ *contains functions* f *satisfying any set of* N *or fewer conditions of the form* $f(\alpha) = 1$, $f(\beta) = 0$ *which are not mutually contradictory.*

In other words, $\mathcal{A}(m)$ *meets every basic clopen set of product measure* $\geq 1/2^N$.

Proof. Let the product (Haar) measure be defined on all Borel sets [6, §58]. Since $\mathcal{A}(n) \nearrow {}^{\lambda}2$, we have $\overline{\mathcal{A}(n)} \nearrow {}^{\lambda}2$ also, and $\mu(\overline{\mathcal{A}(m)}) \to 1$. Pick m such that $\mu(\overline{\mathcal{A}(m)}) > 1-1/2^N$. Now $\mathcal{A}(m)$ meets every open set which $\overline{\mathcal{A}(m)}$ does, and $\overline{\mathcal{A}(m)}$ meets every Borel open set of measure $\geq 1/2^N$.

[There are also purely combinatorial and topological proofs of this lemma. The shortest I know was suggested by van Douwen: suppose that for each n, there exists a basic clopen set U_n such that $\mathcal{A}(n) \cap U_n = \emptyset$, $\mu(U_n) \geq 1/2^N$. Without loss of generality, we may assume each U_n is restricted in a set F_n of exactly k coordinates for some $k \leq N$. Since these sets of coordinates are of the same (finite) cardinality, the usual Δ-system argument, which would ordinarily require uncountably many finite sets, goes through: there is a quasi-disjoint subfamily $\{F_{n_i}\}_{i=1}^{\infty}$ [9, Appendix A]. Take any function f which agrees with U_{n_i} on F_{n_i}; these conditions do not conflict with one another, and so $f \in \bigcap_{n=1}^{\infty} U_{n_i}$. But this contradicts $\mathcal{A}(n) \nearrow {}^{\lambda}2$.]

As we bring other sequences into the picture, we have to sharpen our tools. Two dense subsets of the same basic clopen set can easily be disjoint, and what is needed is the "iterated" Baire category theorem:

Lemma 2. *Let* X *be a compact* T_2 *space, and let* $X = \bigcup_{n=1}^{\infty} A_n$. *Then there exists an* n *such that* A_n *is not the union of countably many nowhere dense subsets of* X.

The proof is trivial, and the lemma enables us to take care of any countable collection of sequences filling up ${}^{\lambda}2$. For our first sequence, which I will suggestively label $\{\mathcal{A}[1,n]\}_{n=1}^{\infty}$, we pick n_1 so that $\mathcal{A}[1,n_1]$ is not the union of countably many nowhere dense sets, and in addition contains sets splitting any two elements of λ. For the next sequence, pick n_2 so that $\mathcal{A}[2,n_2]$ is not the union of countably many nowhere dense sets; and so that, in addition, its intersection with $\mathcal{A}[1,n_1]$ is dense in some basic open subset of ${}^{\lambda}2$ *and* contains functions splitting any two elements of λ. [As before, all but finitely many pairs of elements are split merely because of the density of the intersection in some basic open set.]

In general, for the kth sequence $\{\mathcal{A}[k,n]\}_{n=1}^{\infty}$, pick n_k such that $\mathcal{A}[k,n_k]$ is not the union of countably many nowhere dense subsets of ${}^{\lambda}2$; such that, in addition, $\mathcal{A}[k,n_k] \cap \mathcal{A}[i,n_i]$ is dense in some basic open subset of ${}^{\lambda}2$ for any $i < k$; and such that each of these $k - 1$ intersections also contains functions splitting any α from any β.

However, even though we can choose n_k for all k so that $\mathcal{A}[i,n_i] \cap \mathcal{A}[j,n_j]$ contains functions splitting any α from any β, there is no way we can be sure of adding an ω-th sequence. It may even be that for any fixed n, $\mathcal{A}[\omega,n] \cap \mathcal{A}[k,n_k]$ "works" for only finitely many k. True, we can always include this new sequence by re-numbering all the sequences using only the finite ordinals; but this means giving up all hope of extending the induction out to ω_1. *And in fact, we cannot get out to* ω_1 *in* ZFC: if we use one of the "consistent" nonmetrizable normal Moore spaces of cardinal ω_1 to guide us in our definition of $\{\mathcal{A}[\alpha,n]\}_{n=1}^{\infty}$ for each countable α, all hope of defining n_α cleverly enough evaporates.

Even if we use models of set theory where nonmetrizable normal Moore spaces are unknown, Lemma 2 just does not seem to be strong enough. This is suggested, though not rigorously, by the fact that the very axiom which seems to give us the largest subsets of ${}^{\lambda}2$ via a Lemma 2 route, also gives us nonmetrizable normal Moore spaces of cardinal ω_1! I am referring to MA + $\neg$CH, which tells us that a compact T_2 space satisfying ccc (as does ${}^{\lambda}2$, for all λ) is not the union of fewer than $\mathfrak{c}$ nowhere dense sets.

So in the next section we will turn to another way of defining "large enough" subsets, but we will also make some use of the work in this section.

4. Linked Systems

The material in this section also predates Theorem 2, but by only a few days. It involves turning our intuition upside down: instead of thinking of climbing high enough on sequences that fill up ${}^{\lambda}2$, we think of setting up a system of subsets of ${}^{\lambda}2$ which does not let any sequence filling up ${}^{\lambda}2$ slip through. This inversion was inspired by a paper of S. Mrówka [10]. In this paper, Mrówka attempted to construct first countable compact T_1 spaces of arbitrarily large non-measurable cardinal. Unfortunately, as I. Juhász observed, the convergence structure defined in that paper need not be topological. In fact, it remains a fascinating unsolved problem whether there is a first countable compact T_1 space of cardinal $> c$. (The corresponding problem for T_2 spaces remained unsolved for fifty years before Arhangel'skii showed the answer is no!)

At any rate, the following result of Mrowka's is valid:

Lemma 3. *Let* λ *be a nonmeasurable cardinal. For each sequence* $\mathcal{A} = \{\mathcal{A}(n)\}_{n=1}^{\infty}$ *of subsets of* ${}^{\lambda}2$ *such that* $\mathcal{A}(n) \nearrow {}^{\lambda}2$, *let* $n_{\mathcal{A}}$ *be a positive integer. No matter what the choice of* $n_{\mathcal{A}}$, *there will be a finite collection* $\mathcal{A}^1, \cdots, \mathcal{A}^n$ *of sequences such that* $\mathcal{A}^1(n_{\mathcal{A}1}) \cap \cdots \cap \mathcal{A}^n(n_{\mathcal{A}n})$ *is finite.*

The proof is extremely simple: suppose it were possible to choose $n_{\mathcal{A}}$ for each $\mathcal{A}$ so that every finite intersection of sets $\mathcal{A}(n_{\mathcal{A}})$ is infinite. Let Γ be the collection of all sets of the form $\mathcal{A}(n_{\mathcal{A}})$, and let $\mathcal{U}$ be any free ultrafilter containing Γ. Then every ascending sequence of subsets of ${}^{\lambda}2$ whose union is ${}^{\lambda}2$ contains a member of Γ, hence of $\mathcal{U}$, so that $\mathcal{U}$ has the countable intersection property.

This result seems tantalizingly close to negating the strengthened version of the claim in the proof of Theorem 2--the one that was actually proven, as pointed out at the end of Section 2. If we could just find a *pair* of sequences whose chosen terms have finite intersections, then that intersection could not possibly contain functions splitting any pair α, β. An infinite collection is needed for that, and the necessary size increases as λ does, although log (λ) is an adequate size [${}^{\kappa 2}2$ contains a dense subspace of cardinal κ].

Now, just as the negation of Lemma 3 leads to a free ultrafilter on ${}^{\lambda}2$ with the c.i.p., so the negation of the statement that there is a *pair* $\mathcal{A}_1, \mathcal{A}_2$ such that $\mathcal{A}^1(n_{\mathcal{A}^1}) \cap \mathcal{A}^2(n_{\mathcal{A}^2})$ is finite will lead to a free maximal linked system $\mathcal{L}$ which "catches every ascending sequence that fills up ${}^{\lambda}2$." (A collection of sets is a *linked system* if any *pair* has nonempty intersection.) Actually, maximality is immaterial: once we have a linked system which "catches" all such sequences, any maximal system to which it is extended will do the same.

We have a beautiful theory of how large a set must be to admit a free ultrafilter with the c.i.p., much of it going back to Ulam's 1930 paper [16]. Can we work out a similar theory for maximal linked systems which "catch every ascending sequence which fills up the set," and such that every binary intersection is infinite? As far as I know, no one has done so; some Dutch topologists have worked very successfully with linked systems in constructing supercompactifications, but they do not seem to have gotten around to such a study.

It is not yet clear how much bearing such a study would have upon the normal Moore space problem. All that is clear right now is that in a model where no such linked system exists, the strengthened version of the claim in Theorem 2 fails; but the claim itself allows finite binary intersections, so such a study might not have any bearing on the claim. Even if it could destroy the claim in "all reasonable models" of set theory, it would "only" give us a metacompact Moore space with a normalized collection of closed subsets that cannot be put into disjoint open sets, but the space we have available now would not be normal in such a case. This is the space T_λ of Rudin and Starbird, as a close look at the construction will show [13]. In an upcoming paper I will give a detailed proof of this, as well as a normal Moore space whose condition for being metrizable bears at least a superficial resemblance to the claim in Theorem 2 (but is, of course, even less demanding). To negate that condition in "all reasonable models" of set theory would be to settle the normal Moore space problem--but that is three steps removed from where we are now.

The other possibility is that such a linked system can be constructed in a "reasonable" model of set theory. But this in itself is even less satisfactory, because the claim in the proof of Theorem 2 is very specific about what is to be found in each binary intersection, and it gets more demanding with increasing values of λ. What is needed is a more highly structured linked system, and that is what will be described below.

Let us suppose we could find, for a given λ, an assignment such as in the strengthened version of the claim. To put it one way:

Property 1. *For each sequence* $\mathcal{A} = \{\mathcal{A}(n)\}_{n=1}^{\infty}$ *of subsets of* ${}^{\lambda}2$ *such that* $\mathcal{A}(n) \nearrow {}^{\lambda}2$, *there is an assignment* $n_{\mathcal{A}}$ *of a positive integer such that for any* $\alpha, \beta \in \lambda$, $\alpha \neq \beta$, *and any pair of sequences* $\mathcal{A}^1$, $\mathcal{A}^2$, *the set* $\mathcal{A}^1(n_{\mathcal{A}1}) \cap \mathcal{A}^2(n_{\mathcal{A}2}) \cap (\mathcal{B}_\alpha \Delta \mathcal{B}_\beta)$ *is nonempty.*

Then it would follow that the sets of the form $\mathcal{A}(n_{\mathcal{A}})$ and $(\mathcal{B}_\alpha \Delta \mathcal{B}_\beta)$ form a linked system which "catches every ascending sequence that fills up ${}^{\lambda}2$." We can tidy up this system by pushing $n_{\mathcal{A}}$ up enough to include the function χ_λ which sends every point of λ to 1. In this way the sets $\mathcal{B}_\alpha$ can be added to the system, leaving it linked. [Automatically, such intersections as $\mathcal{B}_\alpha \cap (\mathcal{B}_\lambda \Delta \mathcal{B}_\delta)$ and $(\mathcal{B}_\alpha \Delta \mathcal{B}_\beta) \cap (\mathcal{B}_\lambda \Delta \mathcal{B}_\delta)$ are nonempty as long as $\alpha \neq \beta$, $\gamma \neq \delta$.] Also, all triple intersections involving only sets of the form $\mathcal{A}(n_{\mathcal{A}})$ or $\mathcal{B}_\beta$ are nonempty. Thus Property 1 gives rise to:

Property 2. *There exists a linked system* Λ *of subsets of a set* X *and a subcollection* Γ *of* Λ *such that:*

(i) $|\Gamma| = \lambda$

(ii) For all distinct $B_1, B_2 \in \Gamma$, $B_1 \Delta B_2 \in \Lambda$.

(iii) For each sequence $\{A(n)\}_{n=1}^{\infty}$ *of subsets of* X *satisfying* $A(n) \nearrow X$, *there exists* n *such that* $A(n) \in \Lambda$.

(iv) $L_1 \cap L_2 \cap L_3 \neq \emptyset$ *for all* $L_i \in \Lambda$, *with the possible exception of when two or more are of the form* B *or* $B_1 \Delta B_2$, *where* B, B_1, $B_2 \in \Gamma$.

The first two conditions can be thought of as an index of how badly Λ fails to be a filter subbase--clearly, if $B_1 \Delta B_2 \in \Lambda$, then $B_1 \cap B_2$ cannot be added to Λ without destroying linkedness. The possible exceptions in the fourth condition encompass this obvious case $B_1 \cap B_2 \cap (B_1 \Delta B_2)$ as well as the less obvious $(B_1 \Delta B_2) \cap (B_2 \Delta B_3) \cap (B_3 \Delta B_1)$, both of which are empty by pure Boolean algebra. They also encompass cases where one of the terms is not of the special Γ-form. However, the $\mathcal{A}(n)$ mentioned in Property 1 can always be pushed high enough, through the use of Lemma 1, to make all such ternary intersections nonempty. So we can replace (iv) by a more elegant condition which makes Λ as close to being triply linked as (ii) allows:

(iv⁺) $L_1 \cap L_2 \cap L_3 \neq \emptyset$ *for all* $L_i \in \Lambda$, *except in cases involving the abstract algebraic identities* $B_1 \cap B_2 \cap (B_1 \Delta B_2) = \emptyset$ *and* $(B_1 \Delta B_2) \cap (B_2 \Delta B_3) \cap (B_3 \Delta B_1) = \emptyset$, *with* $B_i \in \Gamma$ *for all* i.

It is these fourth conditions that make Property 2 special. One can easily get a maximal linked system satisfying the first three by letting $X = {}^{\lambda}2$, $\Gamma = \{\mathcal{B}_\alpha | \alpha < \lambda\}$, and including a set in Λ if it either contains a set of the form $\mathcal{B}_\alpha$ or $\mathcal{B}_\alpha \Delta \mathcal{B}_\beta$, or meets all such sets and in addition contains the function χ_λ. Of course, this last detail makes this a degenerate example, resembling fixed ultrafilters more closely than free ones: there are plenty of pairs in Λ that have only χ_λ in common.

To get from Property 2 back to Property 1, we use a very familiar kind of transformation. With X, Γ, and Λ as in

Property 2, let Γ be arranged in a λ-sequence, $\Gamma = \{B_\alpha | \alpha < \lambda\}$. Let ψ be the function from X to ${}^\lambda 2$ whose α-th coordinate is 1 or 0 depending on whether $x \in B_\alpha$ or not. [Of course, some points of ${}^\lambda 2$ might have more than one preimage in X while others have none.] Then $\psi^{-1}(\mathcal{B}_\alpha) = B_\alpha$ for all α, and similarly for $\mathcal{B}_\alpha \Delta \mathcal{B}_\beta$. For each sequence $\{\mathcal{A}(n)\}_{n=1}^\infty$ such that $\mathcal{A}(n) \nearrow {}^\lambda 2$, let $n_\mathcal{A}$ be the least integer such that $\psi^{-1}(\mathcal{A}(n_\mathcal{A}))$ is in Λ; such an $n_\mathcal{A}$ always exists because $\psi^{-1}(\mathcal{A}(n_\mathcal{A})) \nearrow X$. Because of (iv) and the preservation of intersection under preimages, Property 1 is satisfied.

We did not use the full force of (iv) in the above, but could have gotten by with, e.g.

(iv$^-$) If neither L_1 *nor* L_2 *is in* Γ*, nor of the form* $B \Delta B'$ *with* $B, B' \in \Gamma$*, then any set of the form* $L_1 \cap L_2 \cap (B_1 \Delta B_2)$ *(where* $L_i \in \Lambda$*,* $B_i \in \Gamma$ *for* $i = 1,2$*) is nonempty.*

Of course, numerous other variations on (iv) are possible, but all seem to have the esthetic defect that Λ need not be a *maximal* linked system. But this may actually be an advantage as far as constructing such Λ's in some model of set theory is concerned, and the significance of such a construction lies in:

Theorem 3. *Let* **M** *be a model of set theory in which Property* 2 *holds. Then every first countable normal space in* **M** *is* λ*-collectionwise normal. If Property* 2 *holds in* **M** *no matter what the choice of* λ*, then every first countable normal space in* **M** *is collectionwise normal.*

Proof. Property 2 implies Property 1, which implies the truth of the claim in Theorem 2.

5. Personal-historical Note

One might think that the word "measurable" in Mrówka's theorems [10] was enough to tip me off to the use of product measures, especially since I had reviewed much of the material in Section 3 a few days earlier, and more yet in the days following. However, I was so mesmerized by the idea of strengthening Lemma 3 to pairwise intersections that the literal meaning of the word "measurable" did not register; whenever I did think along those lines for the next nine days, I always thought in terms of $\{0,1\}$-valued measures. Moreover, there were all kinds of other promising avenues to investigate.

What finally got me going on the right track was a closer look at the usual proof that if λ does not support a countably additive $\{0,1\}$-valued measure, neither does 2^λ [16, [5, p. 165]. On the same day I'd read Mrówka's papers, I had looked at this proof and seen that it did not go through for maximal linked systems. Now, ten days later, I focused on the way that proof used those subsets β_α of the product set ${}^\lambda 2$. I got to wondering which clopen subsets of, say, the Cantor set ${}^\omega 2$ would have to belong to a maximal linked system. Once I realized that it need not contain any sets of measure $<1/2$, there was no holding me back from the conjecture that PMEA was true for $\lambda = \omega_1$ and hence that it implies every first countable normal space is ω_1-collectionwise normal.

There I got stuck for one more day, being misled by the following folklore "theorem":

"Every cardinal $\kappa > c$ on which there is a real-valued measure is ($\{0,1\}$-valued) measurable."

Now there is a valid theorem which set theorists are fond of phrasing this way, assuming everyone knows that the qualifier "κ-additive" is to be understood. (Of course, all I needed for first countable spaces was countable additivity.) After searching in vain for this exact "theorem" but finding its correctly stated counterpart, I suddenly realized that one could simply take a subset A of cardinal c for any set B of cardinal $\geq c$, define (if possible) a real-valued measure on all subsets of A, and then let the measure of each subset of B be the measure of its trace on A.

A drawback of this trick is that such a measure could not possibly extend the product measure on ${}^{\lambda}2$ when $\lambda > 2^c$. This is because ${}^{\lambda}2$ has the collection $\{\mathcal{B}_\alpha | \alpha < \lambda\}$ of subsets such that $\mathcal{B}_\alpha \Delta \mathcal{B}_\beta$ has positive measure for $\alpha \neq \beta$. If a measure is concentrated on a set of cardinal c, no such collection can be of cardinal $>2^c$.

Incidentally, it seems more logical to me to define a "real-valued measurable cardinal" to be one on which there is a countably additive measure defined for every subset, such that every subset of smaller cardinal is of measure 0. It would make the long stretch between c and the first Ulam-measurable cardinal so much more interesting.

As it was, none of the printed sources I saw gave any indication that real-valued measurable cardinals in this "reformed" sense existed, even assuming any current large cardinal axiom. So, when I wrote to Mary Ellen Rudin on

Nov. 20, 1977, I put most of my emphasis on cardinals $\leq 2^c$, just mentioning the possibility of PMEA holding in some model as a conjecture and then backtracking, "Maybe I'd better not even call this a conjecture--it's too wild a thought--but if it is true, then it is true that in that model, every 1st countable normal space is collectionwise normal."

But Mary Ellen showed the letter to Ken Kunen, and everything fell into place.

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