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EXAMPLES OF HEREDITARILY STRONGLY INFINITE-DIMENSIONAL COMPACTA[1]

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Examples are given of strongly infinite dimensional compacta where each non-degenerate subcontinuum is also strongly infinite dimensional. These are by far the easiest of such examples in the literature and in addition a dimension theoretic phenomenon is identified which is used to verify this hereditary property.

1. Introduction

The first example of an infinite dimensional compactum containing no n-dimensional ($n \geq 1$) closed subsets was given by D. W. Henderson [He] in 1967; shortly thereafter, R. H. Bing [Bi] gave a simplified version. In 1971, Zarelua [Z-1], in a relatively unknown article[2] (in Russian), gives probably the simplest construction of this type of example. Later, in 1974, Zarelua [Z-2] constructed more complicated examples which had the property that each non-degenerate subcontinuum was strongly infinite dimensional. In 1977, the authors together with L. Rubin [R-S-W] developed an abstract dimension theoretic approach for constructing these types of examples; a significant feature of the latter approach was that the key concepts of essential families and continuum-wise separators were properly identified. The second author [Wa] used

[1]The first author was partially supported on NSF Grant MCS 76-06522.

[2]The authors only became aware of [Z-1] during the final draft of this paper.

this abstract approach to construct infinite dimensional compacta containing no n-dimensional ($n \geq 1$) subsets (closed or not).

The examples presented in this paper have two important features: first, their construction is particularly simple and clearly illustrates the phenomena underlying all the previous constructions; and second, in spite of the simplicity of their construction, these examples have the property that every non-degenerate subcontinuum is strongly infinite dimensional. A phenomenon is isolated in §7 which shows that these examples are hereditarily strongly infinite dimensional and can be used to show that the "extra care" exercised in [Z-2] and [R-S-W] in order to insure this hereditary property is not necessary. The second example in this paper, see §6, uses the same construction as in [Z-1] where rather technical proofs are used to verify the weaker condition that the example contains no n-dimensional ($n \geq 1$) closed subsets. This property follows rather automatically for us using the theory developed in [R-S-W].

2. Definitions and Basic Concepts

By a *space* we mean a separable metric space, by a *compactum* we mean a compact space, and by a *continuum* we mean a compact connected space. We follow Hurewicz and Wallman [H-W] for basic definitions and results in dimension theory. Specifically, by the *dimension* of a space X, denoted dim X, we mean either the covering dimension or inductive dimension (since these are equivalent for separable metric spaces). A space which is not finite dimensional is said to be *infinite* *dimensional.*

We collect below the definitions and results needed in this paper; the reader is referred to [R-S-W] for a more thorough discussion.

2.1. *Definition.* Let A and B be disjoint closed subsets of a space X. A closed subset S of X is said to *separate* A and B in X if X-S is the union of two disjoint open sets, one containing A and the other containing B. A closed subset S of X is said to *continuum-wise separate* A and B in X provided every continuum in X from A to B meets S.

2.2. *Definition.* Let X be a space and Γ be an indexing set. A family $\{(A_k,B_k): k \in \Gamma\}$ is *essential* in X if, for each $k \in \Gamma$, (A_k,B_k) is a pair of disjoint closed sets in X such that if S_k separates A_k and B_k in X, then $\cap\{S_k: k \in \Gamma\} \neq \emptyset$.

2.3. *Theorem.* [H-W, p. 35 and p. 78]. *For a space* X, *dim* $X \geq n$ *if and only if there exists an essential family* $\{(A_k,B_k): k = 1,\cdots,n\}$ *in* X.

2.4. *Remark.* Using the Hausdorff metric, the set of non-empty closed subsets of a compactum is a compactum. When we refer to a collection of closed subsets being dense, we mean dense with respect to the topology generated by this metric.

2.5. *Proposition.* [R-S-W; Proposition 3.4]. *Let* $\{(A_k,B_k): k = 1,2,\cdots,n\}$ *be a collection of pairs of non-empty, disjoint closed subsets of a compactum* X. *For each*

$k = 1,2,\cdots,n$, *let* $\mathcal{S}_k$ *be a non-empty dense set of separators of* A_k *and* B_k *and let* Y *be a closed subset of* X. *If for each choice of separators* $S_k \in \mathcal{S}_k$, $k = 1,2,\cdots,n$, *we have that* $(\cap\{S_k: k = 1,2,\cdots,n\}) \cap Y \neq \emptyset$, *then* $\{(A_k \cap Y, B_k \cap Y): k = 1,2,\cdots,n\}$ *is an essential family in* Y *and, therefore,* *dim* $Y \geq n$.

2.6. *Definition.* A space X is *strongly infinite dimensional* if there exists a denumerable essential family $\{(A_k,B_k): k = 1,2,\cdots\}$ for X. A space X is *hereditarily strongly infinite dimensional* if each non-degenerate subcontinuum of X is strongly infinite dimensional.

2.7. *Theorem.* [R-S-W; Proposition 5.5]. *Let* X *be a strongly infinite dimensional space with an essential family* $\{(A_k,B_k): k = 1,2,\cdots\}$. *For* $k = 2,3,\cdots$, *let* S_k *be a continuum-wise separator of* A_k *and* B_k *in* X. *If* $Y = \cap\{S_k: k = 2,3,\cdots\}$, *then* Y *contains a continuum meeting* A_1 *and* B_1.

3. Outline of the Example

Let the Hilbert cube be denoted by $Q = \Pi\{I_k: k = 1,2,\cdots\}$ where $I_k = [0,1]$, let $\Pi_k: Q \to I_k$ denote the projection, and let $A_k = \Pi_k^{-1}(1)$ and $B_k = \Pi_k^{-1}(0)$. The family $\{(A_k,B_k): k = 1,2,\cdots\}$ is an essential family in Q [H-W, p. 49].

For each $k = 1,2,\cdots$, a space $Y_k = X_{3k-1} \cap X_{3k}$ will be constructed such that:

3.1. X_j continuum-wise separates A_j and B_j.

3.2. If C is a closed subset of Y_k and $\Pi_k(C) = I_k$, then $\dim C \geq 2$; if fact, $\{(A_{3k-1} \cap C, B_{3k-1} \cap C), (A_{3k} \cap C, B_{3k} \cap C)\}$

is essential in C.

Thus, $Y' = \cap\{Y_k: k = 1,2,\cdots\}$ has the property guaranteed by Theorem 2.7 that Y' contains a continuum meeting A_1 and B_1 (also A_{3k+1} and B_{3k+1}) and if C is a closed subset of Y' such that for some k, $\Pi_k(C) = I_k$, then dim $C \geq 2$.

Also a space X_{3k+1} will be constructed such that 3.1 is satisfied as well as:

3.3. If C is a non-degenerate subcontinuum of $Y'' = \cap\{X_{3k+1}: k = 1,2,\cdots\}$, then there is an integer k such that $\Pi_k(C) = I_k$.

The space $Y = Y' \cap Y'' = \cap\{X_k: k = 2,3,\cdots\}$ will be an example of a hereditarily strongly infinite dimensional space. We will now argue using conditions 3.1-3.3 that it is an infinite dimensional compactum that contains no n-dimensional $(n \geq 1)$ closed subsets. Theorem 2.7 guarantees that Y contains a continuum meeting A_1 and B_1 and hence dim $Y \geq 1$, and 3.2 and 3.3 guarantee that X contains no 1-dimensional subcontinua. Then the compactness insures that X contains no 1-dimensional closed subsets since compact totally disconnected sets are 0-dimensional. This is sufficient since, from the inductive definition of dimension, it is clear that each closed n-dimensional $(n \geq 1)$ set contains k-dimensional closed subsets for each $0 \leq k < n$ and in particular for $k = 1$. Thus, Y is infinite dimensional and contains no n-dimensional $(n \geq 1)$ closed subsets. In section 6 we prove that this example is hereditarily strongly infinite dimensional.

4. Constructing Y_k

Let $\{W_i: i = 1,2,\cdots\}$ be the null sequence of open intervals in I_k indicated in Figure 1. Let $\{S_i^{3k-1}: i = 1,2,\cdots\}$ and $\{S_i^{3k}: i = 1,2,\cdots\}$ be a countable dense sets of separators of A_{3k-1} and B_{3k-1} and A_{3k} and B_{3k}, respectively. Let $\alpha: N \to N \times N$ be a bijection where N denotes the natural numbers and let α_1 and α_2 be α composed with projection onto the first and second factor, respectively.

Let $X_{3k-1} = \Pi_k^{-1}(I_k - \cup\{W_i: i = 1,2,\cdots\}) \cup (\cup\{S_{\alpha_1(i)}^{3k-1} \cap \Pi_k^{-1}(W_i): i = 1,2,\cdots\})$ and let $X_{3k} = \Pi_k^{-1}(I_k - \cup\{W_i: i = 1,2,\cdots\}) \cup (\cup\{S_{\alpha_2(i)}^{3k} \cap \Pi_k^{-1}(W_i): i = 1,2,\cdots\})$; see Figure 2 where k = 1. It is easily seen that X_{3k-1} and X_{3k} continuum-wise separate A_{3k-1} and B_{3k-1} and A_{3k} and B_{3k}, respectively. In addition, if $C \subseteq X_{3k-1} \cap X_{3k}$ with $\Pi_k(C) = I_k$ and $(i,j) \in N \times N$, then $C \cap \Pi_k^{-1}(W_{\alpha^{-1}(i,j)}) \subseteq S_i^{3k-1} \cap S_j^{3k}$; therefore, Proposition 2.5 guarantees that if C is a closed subset of $X_{3k-1} \cap X_{3k}$ with $\Pi_k(C) = I_k$, then $\dim C \geq 2$.

The nature of X_{3k+1} is different than that of X_{3k-1} and X_{3k}; the role of X_{3k+1} is to insure that condition 3.3 will hold. Let $X_{3k+1} = \Pi_{k,3k+1}^{-1}(R_{3k+1})$ where $\Pi_{k,3k+1}$ is the projection onto $I_k \times I_{3k+1}$ and $R_{3k+1} \subseteq I_k \times I_{3k+1}$ is the "rooftop" in Figure 3.

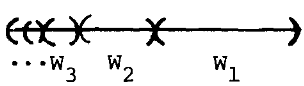

Fig. 1

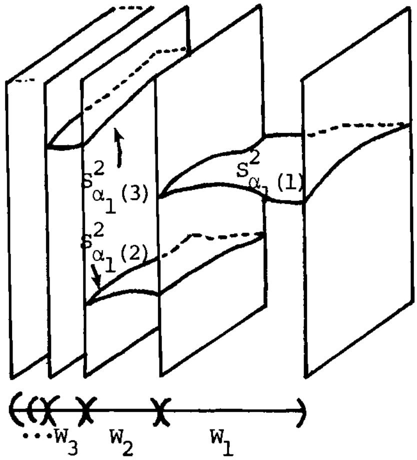

Fig. 2

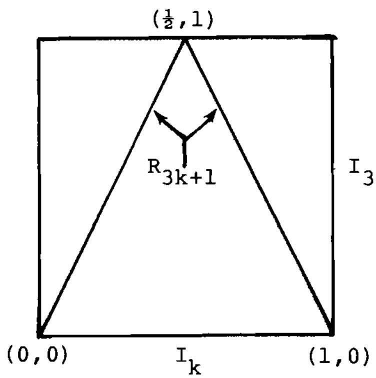

Fig. 3

5. Verifying Condition 3.3

If J is a subinterval of [0,1], let $\ell(J)$ denote the length of J. Let $C \subseteq Y$ be a non-degenerate subcontinuum, let i_1 be such that $\Pi_{i_1}(C)$ is also non-degenerate, and let $\ell(\Pi_{i_1}(C)) = \varepsilon > 0$. Note that since the slopes of the straight line segments of R_{3i+1} are ± 2, and $C \subseteq X_{3i_1+1}$, then $\frac{1}{2} \notin \Pi_{i_1}(C)$

implies that $\ell(\Pi_{3i_1+1}(C)) = 2\varepsilon$. Inductively, let $i_n = 3i_{n-1}+1$, let $J_n = \Pi_{i_n}(C)$ and observe that if $\frac{1}{2} \notin J_{n-1}$, then $\ell(J_n) = n\varepsilon$. Since each I_n has length 1, it follows that there exists an N such that $\frac{1}{2} \in J_N$. Thus, by observing the corresponding properties of R_{3i+1}, it follows that $1 \in J_{N+1}$ and that $0 \in J_{N+2} = [0,b]$ for some $0 < b \leq 1$. Following the above argument we see that if $\frac{1}{2} \leq b < 1$, then $J_{N+3} = [0,1]$ and if $0 < b < \frac{1}{2}$, then $J_{N+3} = [0,2b]$ and hence for some $j > 3$, $J_{N+j} = [0,1]$ which says that for some k, $\Pi_k(C) = I_k$.

6. A Generalization

Let X be a strongly infinite dimensional compactum with essential family $\{(A_k,B_k): k = 1,2,\cdots\}$; let $\{\Pi_k: k = 1,2,\cdots\}$ be a countable dense subset of the space of all mappings from X to $I = [0,1]$; for each k, let $\{S_i^k: i = 1,2,\cdots\}$ be a countable dense set of separators of A_k and B_k, and let $\{W_i: i = 1,2,\cdots\}$ be the null sequence of open intervals in $[\frac{1}{4},\frac{3}{4}]$ indicated in Figure 4.

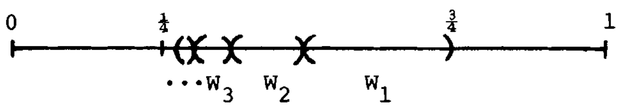

Fig. 4

Let α,α_1,α_2 be as before and, for each k, let $Y_k = X_{2k} \cap X_{2k+1}$ where

$$X_{2k} = \Pi_k^{-1}(I_k - \cup\{W_i: i = 1,2,\cdots\}) \cup (\cup\{S^{2k}_{\alpha_1(i)} \cap \Pi_k^{-1}(W_i): i = 1,2,\cdots\})$$

and

$$X_{2k+1} = \Pi_k^{-1}(I_k - \cup\{W_i: i = 1,2,\cdots\}) \cup (\cup\{S^{2k+1}_{\alpha_2(i)} \cap \Pi_k^{-1}(W_i): i = 1,2,\cdots\}).$$

It is easily seen that condition 3.1 is true and the earlier argument shows that:

6.1. If C is a closed subset of Y_k and $\Pi_k(C) \supseteq [\frac{1}{4},\frac{3}{4}]$, then dim $C \geq 2$, if fact, $\{(A_{2k} \cap C, B_{2k} \cap C), \{A_{2k+1} \cap C, B_{2k+1} \cap C)\}$ is essential in C.

Letting $Y = \cap\{Y_k: k = 1,2,\cdots\} = \cap\{X_k: k = 2,3,\cdots\}$, Theorem 2.7 guarantees that Y contains a continuum meaning A_1 and B_1. Since the Π_k's are a dense set of mappings the following holds:

6.2. If $C \subseteq Y$ is a non-degenerate subcontinuum of Y, then for some $k, \Pi_k(C) \supseteq [\frac{1}{4},\frac{3}{4}]$.

Thus our previous argument shows that we have constructed in an arbitrary strongly infinite dimensional space X a subcompactum Y that is infinite dimensional and contains no n-dimensional ($n \geq 1$) closed subsets. We will show in the next section that in fact Y is hereditarily strongly infinite dimensional.

7. Strong Infinite Dimensionality of Subcontinua

One reason for the additional complexity in the construction in [Z-2] and [R-S-W] was to be able to conclude that the examples had the additional property that each non-degenerate subcontinuum was strongly infinite dimensional. Although we made no effort to construct examples with this hereditary property, the following propositions isolate a phenomenon which forces them to have this property.

Proposition 7.1 gives conditions on a continuum that imply it is strongly infinite dimensional. Observe that conditions 3.2 and 3.3 (resp., 6.1 and 6.2) imply that each nondegenerate subcontinuum of the example constructed in sections 3 and 4 (resp., section 6) satisfies the hypothesis of Proposition 7.1 and thus these examples are hereditarily strongly infinite dimensional. An alternative argument for the example constructed in section 6 can be given using Proposition 7.2.

7.1. *Proposition. Let* $\{(A_k,B_k): k = 1,2,\cdots\}$ *be a family of pairs of disjoint closed subsets of a continuum* X. *Suppose that, for each* k, *there are positive integers* i *and* j *such that, for each continuum* $C \subseteq X$ *meeting* A_k *and* B_k, *the pair* $\{(A_i \cap C, B_i \cap C), (A_j \cap C, B_j \cap C)\}$ *is essential in* C. *If, for some* n, $A_n \neq \phi$ *and* $B_n \neq \phi$, *then* X *is strongly infinite dimensional. Alternately, if for some* i *and* j, $\{(A_i \cap X, B_i \cap X), (A_j \cap X, B_j \cap X)\}$ *is essential in* X, *then* X *is strongly infinite dimensional.*

Proof. Let i_1 and j_1 be such that $\{(A_{i_1},B_{i_1}),(A_{j_1},B_{j_1})\}$ is essential in X. Let i_2 and j_2 be such that for each continuum C meeting A_{j_1} and B_{j_1}, $\{(A_{i_2} \cap C, B_{i_2} \cap C), (A_{j_2} \cap C, B_{j_2} \cap C)\}$ is essential in C. Recursively, for $n \geq 3$, let i_n and j_n be such that for each continuum C meeting $A_{j_{n-1}}$ and $B_{j_{n-1}}$, $\{(A_{i_n} \cap C, B_{i_n} \cap C), (A_{j_n} \cap C, B_{j_n} \cap C)\}$ is essential in C. We now show that the family $\{(A_{i_n},B_{i_n}): n = 1,2,\cdots\}$ is essential in X. For $n = 1,2,\cdots$, let S_n separate A_{i_n} and B_{i_n}.

Since $\{(A_{i_1}, B_{i_1}), (A_{j_1}, B_{j_1})\}$ is essential in X, S_1 contains a continuum from A_{j_1} to B_{j_1}. Since $\{(A_{i_2}, B_{i_2}), (A_{j_2}, B_{j_2})\}$ is essential in this continuum, $S_1 \cap S_2$ contains a continuum from A_{j_2} to B_{j_2}. Since $\{(A_{i_3}, B_{i_3}), (A_{j_3}, B_{j_3})\}$ is essential in this continuum, $S_1 \cap S_2 \cap S_3$ contains a continuum from A_{j_3} to B_{j_3}. Continuing this argument, for each $n \geq 1$, $S_1 \cap \cdots \cap S_n$ contains a continuum from A_{j_n} to B_{j_n} and, therefore, $\cap\{S_n: n = 1,2,\cdots\} \neq \emptyset$.

7.2. *Proposition. Let* X *be a compactum with dim* $X \geq 1$. *Suppose that, for each pair of disjoint closed sets* H *and* K, *there is a family* $\{(A,B),(D,E)\}$ *of pairs of disjoint closed sets such that* $\{(A \cap C, B \cap C), (D \cap C, E \cap C)\}$ *is essential in each continuum* C *from* H *to* K. *Then each non-degenerate subcontinuum of* X *is strongly infinite dimensional.*

Proof. Since the hypotheses are satisfied by non-degenerate subcontinua of X, it suffices to assume that X is a continuum and to show that X is strongly infinite dimensional. Let $\{(A_1,B_1),(D_1,E_1)\}$ be an essential family in X. Recursively, for $n \geq 2$, let $\{(A_n,B_n),(D_n,E_n)\}$ be such that $\{(A_n \cap C, B_n \cap C), (D_n \cap C, E_n \cap C)\}$ is essential in each continuum C from D_{n-1} to E_{n-1}. The argument used in the proof of Proposition 7.1 shows that $\{(A_n, B_n): n = 1,2,\cdots\}$ is essential in X.

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