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SOME REMARKS ON M-EMBEDDING

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Section 1

There are four main results in this paper: (1) a necessary condition for the product of a space with any metric space to be normal, (2) a characterization of compact T_2 spaces, (3) a complete analogue of the Morita-Hoshina Homotopy Extension Theorem (3.7 [13]) for ANR spaces, and (4) a characterization of spaces for which every metric space is an AE. Each of these results involves the notion of M-embedding, which was introduced in [17]. (See also [8], [15])

In what follows, γ will denote an infinite cardinal number, R will denote the reals, p the irrationals, and I the unit interval; all functions and pseudometrics will be assumed continuous. No separation axioms will be assumed unless stated.

We say a subspace S of a topological space X is M^γ -embedded (P^γ -embedded) in X if every function from S to a γ -separable (complete) metrizable AE extends to X . By an AE or ANR we mean an AE or ANR for metric spaces. By dropping the separability condition, we obtain definitions of P - and M -embedding. P -embedding has been extensively studied, for example, see [1, 2, 13, 14]. For definitions of C^* - and C -embedding see [6].

There are certain results we will frequently use, and we list them here.

(α) S is P^γ -embedded (M^γ -embedded) in X iff every function

from S to a γ -separable Banach space (normed linear space) extends to X (p. 227 [1], Th. 1 [17]).

(β) X is γ -collectionwise normal iff every closed subset is P^Y -embedded in X (p. 189 [1]).

(δ) S is P^{\aleph_0} -embedded in X iff S is C -embedded in X (p. 200 [1]).

(η) S is M^Y -embedded in X iff S is P^Y -embedded in X and given a γ -separable pseudometric d on X , there exists a zero set Z of X such that $S \subset Z \subset \{x \in X: d(x, x_0) = 0 \text{ for some } x_0 \in S\}$ (Th. 1 [17]).

(θ) S is M^Y -embedded in X iff S is P^Y -embedded in X and given a function f from X to a γ -separable metric space, there exists a zero set Z of X such that $S \subset Z \subset f^{-1}f(S)$ (Th. 1 [17]).

(κ) S is P^Y -embedded in X iff $S \times Y$ is P^Y -embedded in $X \times Y$ for every compact T_2 space Y with $w(Y) \leq \gamma$ (p. 234 [1] (X need not be $T_{3\frac{1}{2}}$)).

(λ) S is P^Y -embedded in X iff $S \times Y$ is C^* -embedded in $X \times Y$ for every compact T_2 space Y with $w(Y) \leq \gamma$ (p. 234 [1])--for a sharpened version see [14]).

Removing the cardinality restrictions on each of these (except (δ)) produces characterizations of P - and M -embedding and of collectionwise normality.

Section 2

Since M^{\aleph_0} -embedding (P^{\aleph_0} -embedding) is equivalent to the extendability of every function into a separable (complete) metrizable AE and since P^{\aleph_0} -embedding is equivalent to C -embedding (fact (δ) of Section 1), one might wonder whether S

is M^{\aleph_0} -embedded in X iff (*): every function from S into an AE embedded in R extends to X . Note that a subset of R is an AE iff it is an interval. 2.1 will show that the above conjecture is false as (*) is equivalent to C -embedding. Example 2.4 of [8] (identical with the example on p. 224 of [17]) shows that C -embedding is strictly weaker than M^{\aleph_0} -embedding. (2.1 was first shown by R. Arens for closed subsets of normal spaces [2].)

2.1 Proposition. *If S is C -embedded in X , every function from S to an interval K of R extends to X with values in K .*

Proof. There is an extension g of f with $g(X) \subset \bar{K}$.

Assuming that K is not closed, $\bar{K} - K$ consists of 1 or 2 points and hence is a zero set of R . Hence $g^{-1}(\bar{K} - K)$ is a zero set of X disjoint from S . Hence there exists $h: X \rightarrow [0,1]$ such that $h(S) \equiv 1$ and $h(g^{-1}(\bar{K} - K)) \equiv 0$ (p. 19 [6]). Fix $r \in K$ and define $f^* = hg + (1 - h)r$.

This same idea will work if S is P^{γ} -embedded in X and f is a function from S to a convex subset K of a γ -separable Banach space B such that $\bar{K} - K$ is a zero set in B . (See 4.1 [2])

Fact (θ) of Section 1 with $\gamma = \aleph_0$ tells us that S is M^{\aleph_0} -embedded in X iff it is P^{\aleph_0} -embedded and given a function f from X to a separable metric space, there exists a zero set Z of X such that $S \subset Z \subset f^{-1}f(S)$. One might ask whether M^{\aleph_0} -embedding is equivalent to C -embedding plus (**): Given $f: X \rightarrow R$, there exists a zero set Z of X such that $S \subset Z \subset f^{-1}f(S)$. The answer is again no.

To see this, let X be the unit disc in the plane (as a

set) and $S = \{(x,y): x^2 + y^2 < 1, \text{ or } x^2 + y^2 = 1 \text{ and } x \text{ is rational}\}$. Let X have the topology that makes the points of $X - S$ discrete. Hence open sets of X are of the form $U \cup V$, where U is an open neighborhood in the ordinary metric topology and V is a subset of $X - S$. Any space formed in this way is hereditarily paracompact (see [10]). Hence S is a closed C -embedded subset of X . Since S is an AR that is not an absolute G_δ (see p. 382 [7]), we can show that S is not a zero set of X . Since X is submetrizable (i.e. its topology contains a metric topology), it is clear from (η) in Section 1 that S is not M^{\aleph_0} -embedded in X . (To see this, let d be the metric topology on X .) However, let $f: X \rightarrow \mathbb{R}$ and observe that since S is connected, $f(S)$ is an interval and hence is a G_δ . Therefore $f^{-1}f(S)$ is a G_δ set of X containing S ; since X is normal, there exists a zero set Z such that $S \subset Z \subset f^{-1}f(S)$.

Section 3

There is considerable interest in spaces whose product with every metric space is normal. A characterization of this class was given by Morita [11, 12]. A theorem due to Morita, Rudin, and Starbird states that if Y is metric and X normal and countably paracompact, then $X \times Y$ is normal iff $X \times Y$ is countably paracompact [16].

This section will produce a necessary condition for the product of a normal space X with every γ -separable metric space to be normal. If S is a subspace of X , we say that (X,S) has the γ -Zero-Set Interpolation Property (γ -ZIP) if whenever d is a γ -separable pseudometric on X , there exists

a zero set Z of X such that:

$$S \subset Z \subset \{x \in X: d(x, x_0) = 0 \text{ for some } x_0 \in S\}.$$

By (η) in Section 1, we see that S is M^Y -embedded in X iff S is P^Y -embedded and (X, S) has the γ -ZIP. Hence the γ -ZIP is what needs to be added to P^Y -embedding to produce M^Y -embedding. By dropping the separability condition on d , we obtain a definition of the Zero-Set Interpolation Property (ZIP), and observe that S is M -embedded in X iff S is P -embedded in X and (X, S) has the ZIP. The following proposition is a slight generalization of an example communicated to the author by E. Michael (the example is written up in Section 3 of [18]).

3.1 Proposition. Let S be a closed subset of a normal space X such that $S \times Y$ is C -embedded in $X \times Y$ for every γ -separable metric space Y . Then (X, S) has the γ -ZIP.

Proof. Let d be a γ -separable pseudometric on X and let $A = \{x \in X: d(x, x_0) = 0 \text{ for some } x_0 \in S\}$. Let (Y, d) be the γ -separable metric space associated with the pseudometric space $(X - A, d)$. For notational ease we will identify points of Y with those of $X - A$. Define $f: S \times Y \rightarrow \mathbb{R}$ by $f(x, y) = 1/d(x, y)$. The map f is well-defined and continuous hence extends to $g: X \times Y \rightarrow \mathbb{R}$.

Let $H_n = \{x \in X - A: d(x, y) < 1/n \Rightarrow g(x, y) < n\}$. We claim $X - A = \bigcup_n H_n$. Let $x_0 \in X - A$ and choose m such that $g(x_0, x_0) < m$. Since g is continuous there exists an open set U of X containing x_0 and an $\epsilon > 0$ such that if $x \in U$ and $d(x_0, y) < \epsilon$, then $g(x, y) < m$. Choose n such that $n \geq m$ and $1/n \leq \epsilon$. Then $x_0 \in H_n$.

Hence we have $H = \bigcap_n (X - \overline{H}_n) \subset A$. We claim that $S \subset H$.

This will finish the proof, for since S is closed, X is normal, and H is a G_δ , we will be able to find a zero set Z such that $S \subset Z \subset A$. To show that $S \subset H$, argue by contradiction. Assume there exists $x_0 \in S \cap \overline{H}_n$ for some n . Choose $y_0 \in X - A$ such that $d(x_0, y_0) < 1/2n$. (We can do this since the topology generated by d is contained in the topology on X and $x_0 \in \overline{H}_n$.) Then $g(x_0, y_0) > 2n$. Since g is continuous, there exists an open U containing x_0 and $\epsilon > 0$ such that $x \in U$ and $y \in Y$ with $d(y, y_0) < \epsilon$ implies $g(x, y) > n$.

Choose $x \in U \cap H_n$ such that $d(x, x_0) < 1/2n$. Then $d(x, y_0) \leq d(x, x_0) + d(x_0, y_0) < 1/n$, hence $g(x, y_0) < n$ (since $x \in H_n$). However, $x \in U$ and hence $g(x, y_0) > n$, which is the desired contradiction.

There are a number of corollaries of this result. For example:

3.2 Corollary. If $X \times Y$ is normal for every metric Y , then every closed subset of X has the ZIP with respect to X .

3.3 Corollary. If $X \times Y$ is normal for every separable metric Y , then every closed subset of X is M^{\aleph_0} -embedded in X .

Proof. Use (δ) and (η) of Section 1.

3.4 Corollary. Let X be a collectionwise normal space whose product with every metric space is normal. Then every closed subset of X is M -embedded in X .

Proof. Use (β) and (η) of Section 1.

3.5 Corollary (Michael). The following are equivalent for a submetrizable space X :

(a) X is perfectly normal.

(b) $X \times Y$ is perfectly normal for every metric Y .

(c) $X \times Y$ is normal for every metric Y .

Proof. (b) \Rightarrow (c) is clear and (a) \Rightarrow (b) is known [9].

Hence we need only show (c) \Rightarrow (a). Assume (c) but suppose (a) fails. This implies that X is normal and submetrizable, but not perfectly normal. From the definition of the ZIP, it is clear that a subset of a submetrizable space X has the ZIP with respect to X iff it is a zero set. (One may see this by letting d be the metric whose topology is contained in that of X .) Hence X contains a closed subset S such that (X, S) fails to have the ZIP, so by 3.1 there exists a metric space Y such that $X \times Y$ is not normal, giving a contradiction.

In fact, it is clear from the above that if X contains a γ -separable metric topology and fails to be perfectly normal, then there exists a γ -separable metric Y such that $X \times Y$ is not normal. More specifically, if m is a continuous metric on X and S is a closed non- G_δ subset of X , then $S \times Y$ fails to be C -embedded in $X \times Y$, where Y is the metric space $(X - S, m)$. This shows immediately that $X \times P$ fails to be normal, where X is the Michael line and P the irrationals with their usual topology. A different proof was originally given in [10].

It is an open question whether the converse of 3.1 is true. 4.5 of Section 4 will shed some light on this question.

Section 4

Morita and Hoshina (Theorem 3.7 [13]) proved the

following generalization of the Homotopy Extension Theorem:

4.1 Theorem. *For a subspace S of a topological space X the following are equivalent:*

- (1) S is P^Y -embedded in X .
- (2) $(S \times Y) \cup (X \times B)$ is P^Y -embedded in $X \times Y$ for every compact T_2 space Y with $w(Y) \leq \gamma$ and its closed subset B .
- (3) $(S \times I) \cup (X \times \{0\})$ is P^Y -embedded in $X \times I$.
- (4) (X, S) has the HEP with respect to every complete ANR space of weight $\leq \gamma$.

The analogue of 4.1 for M^Y -embedding is the following:

4.2 Theorem. *The following are equivalent:*

- (1) S is M^Y -embedded in X .
- (2) $(S \times Y) \cup (X \times B)$ is M^Y -embedded in $X \times Y$ for every compact T_2 space Y with $w(Y) \leq \gamma$ and its closed subset B .
- (3) $(S \times I) \cup (X \times \{0\})$ is M^Y -embedded in $X \times I$.
- (4) (X, S) has the HEP with respect to every ANR space of weight $\leq \gamma$.

Proof. The equivalence of (1), (3), and (4) is Theorem 2 of [17]. To complete the proof it remains to show (1) \Rightarrow (2). We state and prove the next theorem, then use it to show (1) \Rightarrow (2).

4.3 Theorem (L. Sennott, R. Levy, M. D. Rice). *The following are equivalent for a T_2 space Y :*

- (1) *The space Y is compact.*

(2) If Y is embedded in a $T_{3\frac{1}{2}}$ space Z and X is any space, then $X \times Y$ is M -embedded in $X \times Z$.

(3) If Y is embedded in a $T_{3\frac{1}{2}}$ space Z and X is any space, then $X \times Y$ is C^* -embedded in $X \times Z$.

Proof. To show (1) \Rightarrow (2) let Y be a compact space embedded in a $T_{3\frac{1}{2}}$ space Z , let X be any space, and let $f: X \times Y \rightarrow L$ be a continuous function into a normed linear space L . By (a) of Section 1, it is sufficient to extend f to $X \times Z$. Define $g: X \rightarrow C^*(Y, L)$ by $g(x)(y) = f(x, y)$. A standard argument shows that g is continuous when $C^*(Y, L)$ has the sup norm topology. We then define $h: g(X) \times Y \rightarrow L$ by $h(g(x), y) = f(x, y)$ and observe that $g(X)$ is a metric space and h is continuous. Now $g(X) \times \beta Z$ is the product of a metric space and a compact space and hence is a paracompact M -space. This implies that the closed subset $g(X) \times Y$ is M -embedded (Proposition 2 of [17]). Hence we can lift h to $h^*: g(X) \times \beta Z \rightarrow L$. Defining $f^*: X \times Z \rightarrow L$ by $f^*(x, z) = h^*(g(x), z)$, one checks that this defines a continuous extension of f . Note: This proof uses an idea contained in the proof of Theorem 2 of [19] and in fact M. Starbird's Theorem 3 [19] is our (1) \Rightarrow (3) with C^* -embedding replaced by C -embedding.

Clearly (2) \Rightarrow (3). Now assume (3) holds but Y is not compact. By Problem 6J of [6], the space Y is absolutely C^* -embedded and hence is almost compact. Let $\beta Y - Y = \{\infty\}$, and let $\{U_\alpha: \alpha \in D\}$ be a base of open neighborhoods of ∞ in βY . We will define a space X such that $X \times Y$ is not C^* -embedded in $X \times \beta Y$. Define an ordering on D : $\alpha \leq \beta$ iff

$U_\beta \subset U_\alpha$. Then D becomes a directed set. Let $X = D \cup \{q\}$, where $q \notin D$, points of D are isolated and basic open neighborhoods of q are of the form $\{q\} \cup \{\alpha : \alpha \geq \alpha_0\}$. Denote this set by $[\alpha_0, q]$.

For each α , choose a function f_α on βY such that $f_\alpha(\beta Y - U_\alpha)$ is identically 1 and $f_\alpha(\infty) = 0$. Define $f: X \times Y \rightarrow [0, 1]$ by $f(\alpha, y) = f_\alpha(y)$ and $f(q, y) = 1$. Clearly f is continuous at points of the form (α, y) . Fix (q, y_0) and choose α_0 such that $y_0 \notin \bar{U}_{\alpha_0}$. If $(x, y) \in [\alpha_0, q] \times (Y - \bar{U}_{\alpha_0})$, then $f(x, y) = 1$.

If there were an extension of f to all of $X \times \beta Y$, the extension would be 1 at all points of the form (q, y) with $y \in Y$ and 0 at all points (α, ∞) , which implies that the extension is not continuous at (q, ∞) .

Note: This proof is a generalization of an example given by Comfort and Negreontis (Example 4.6 of [4]).

To complete the proof of 4.2, let S be M^Y -embedded in X , and Y and B be as in (2). By 4.3 (2) it is clear that $X \times B$ is M^Y -embedded in $X \times Y$. By proposition 5 of [17], we have that $S \times Y$ is M^Y -embedded in $X \times Y$. By Proposition 6 of [17], to show (2) it is sufficient to show that $(S \times Y) \cup (X \times B)$ is P^Y -embedded in $X \times Y$. But this is true from (1) \Rightarrow (2) of 4.1.

We now use 4.3 to obtain a generalization of (κ) in Section 1, which will throw further light on the results in Section 3.

4.4 Proposition. *The following are equivalent:*

- (1) S is P^Y -embedded in X .
- (2) $S \times Y$ is P^Y -embedded in $X \times Y$ for every locally compact, paracompact T_2 space Y with $w(Y) \leq \gamma$.
- (3) $S \times Y$ is C^* -embedded in $X \times Y$ for every locally compact, paracompact T_2 space Y with $w(Y) \leq \gamma$.

Proof. (2) \Rightarrow (3) is clear and (3) \Rightarrow (1) is clear from (λ) of Section 1. It remains to show (1) \Rightarrow (2). Let S be P^Y -embedded in X and Y as in (2). For each $y \in Y$, let U_y denote an open neighborhood of y whose closure is compact. Let $\{f_\alpha: \alpha \in A\}$ be a locally finite partition of unity subordinate to the cover $\{U_y: y \in Y\}$, and let K_α denote the compact set $\text{cl}(Y - Z(f_\alpha))$.

Let $g: S \times Y \rightarrow B$ be a function into a γ -separable Banach space B . By (α) of Section 1, it is sufficient to extend g to $X \times Y$. For each α , the function $g_\alpha = g|S \times K_\alpha$ has an extension to $h_\alpha: X \times K_\alpha \rightarrow B$ by (κ) of Section 1. By 4.3 (2), h_α extends to $k_\alpha: X \times Y \rightarrow B$. Then $g^*(x,y) = \sum_\alpha f_\alpha(y)k_\alpha(x,y)$ is the desired extension of g .

4.5 Corollary. *If S is C -embedded in X , then $S \times Y$ is C -embedded in $X \times Y$ for any locally compact metric space Y .*

Proof. Let the compact sets K_α be constructed as in the proof of (1) \Rightarrow (2) of 4.4. If Y is metric, then K_α is compact metric. Let $g: S \times Y \rightarrow R$. By (δ) and (κ) of Section 1, $g_\alpha = g|S \times K_\alpha$ has an extension to $h_\alpha: X \times K_\alpha \rightarrow R$. The proof proceeds as in 4.4.

Comparing 3.1 and 4.5, we see that if S is a closed subset of a normal space X such that (X,S) fails to have ZIP, then there exists a non-locally compact metric space Y such

that $S \times Y$ is not C -embedded in $X \times Y$.

4.6 Corollary. *If S is M^γ -embedded in X , then $S \times Y$ is M^γ -embedded in $X \times Y$ for any locally compact paracompact T_2 space Y with $w(Y) \leq \gamma$.*

Proof. The proof of (1) \Rightarrow (2) of 4.4 goes through with B replaced by a γ -separable normed linear space. (To lift g_α use 4.2 (1) \Rightarrow (2).)

Section 5

As a final application of M -embedding, we generalize two results of E. Chang [3]. (Also see results of Ellis [5].) Although the results deal with ultranormal spaces, they are equivalent to the following:

5.1 Proposition (Chang, p. 38, 40 [3]). *Let X be non-empty. The following are equivalent.*

- (1) *X is a 0-dim collectionwise normal (normal) space.*
- (2) *Every complete (separable) metric space is an AE for X .*

5.2 Proposition (Chang, p. 43 [3]). *Let S be a closed G_δ subset of a 0-dim collectionwise normal (normal) space X , Y a (separable) metric space and $f: S \rightarrow Y$. Then f extends to X .*

5.3 Proposition. *Let X be nonempty. The following are equivalent.*

- (1) *Every (separable) metric space is an AE for X .*
- (2) *X is a 0-dim space in which every closed subset is M - (M^{\aleph_0}) -embedded.*

Proof. We prove the unbracketed equivalence. (1) \Rightarrow (2) is clear from 5.1 and the definition of M -embedding. To show (2) \Rightarrow (1), let Y be a metric space, S a closed subset of X , and $f: S \rightarrow Y$. Let \tilde{Y} denote the completion of Y with injection map i . Since X is a 0-dim collectionwise normal space, the map $i \circ f: S \rightarrow \tilde{Y}$ has an extension \tilde{f} to X , by 5.1. By (θ) of Section 1, there exists a zero set Z of X such that $S \subset Z \subset \tilde{f}^{-1}f(S)$. Hence $\tilde{f}|_Z$ maps Z into Y , so by 5.2 it can be lifted to $f^*: X \rightarrow Y$, completing the proof.

In [15], Morita remarks that the following generalizations of known results can be proved: If $\dim X/S \leq n+1$, then S is M^Y -embedded (P^Y -embedded) in X iff any map from S into a metric (complete metric) space of weight $\leq \gamma$ which is LC^n and C^n can be extended to X . If $\dim X/S \leq n$, then S is M^Y -embedded (P^Y -embedded) in X iff (X, S) has the homotopy extension property with respect to every metric (complete metric) space of weight $\leq \gamma$ which is LC^n .

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