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Section 1

There are four main results in this paper: (1) a necessary condition for the product of a space with any metric space to be normal, (2) a characterization of compact T_2 spaces, (3) a complete analogue of the Morita-Hoshina Homotopy Extension Theorem (3.7 [13]) for ANR spaces, and (4) a characterization of spaces for which every metric space is an AE. Each of these results involves the notion of M-embedding, which was introduced in [17]. (See also [8], [15])

In what follows, γ will denote an infinite cardinal number, R will denote the reals, p the irrationals, and I the unit interval; all functions and pseudometrics will be assumed continuous. No separation axioms will be assumed unless stated.

We say a subspace S of a topological space X is M^γ-embedded (P^γ-embedded) in X if every function from S to a γ-separable (complete) metrizable AE extends to X. By an AE or ANR we mean an AE or ANR for metric spaces. By dropping the separability condition, we obtain definitions of P- and M-embedding. P-embedding has been extensively studied, for example, see [1, 2, 13, 14]. For definitions of C*- and C-embedding see [6].

There are certain results we will frequently use, and we list them here.

(α) S is P^γ-embedded (M^γ-embedded) in X iff every function from S to a γ-separable Banach space (normed linear space) extends to X (p. 227 [1], Th. 1 [17]).

(β) X is γ-collectionwise normal iff every closed subset is P^{γ}-embedded in X (p. 189 [1]).

(δ) S is $P^{\aleph_0}$-embedded in X iff S is C-embedded in X (p. 200 [1]).

(η) S is M^{γ}-embedded in X iff S is P^{γ}-embedded in X and given a γ-separable pseudometric d on X, there exists a zero set Z of X such that $S \subset Z \subset \{x \in X: d(x,x_0) = 0$ for some $x_0 \in S\}$ (Th. 1 [17]).

(θ) S is M^{γ}-embedded in X iff S is P^{γ}-embedded in X and given a function f from X to a γ-separable metric space, there exists a zero set Z of X such that $S \subset Z \subset f^{-1}f(S)$ (Th. 1 [17]).

(κ) S is P^{γ}-embedded in X iff $S \times Y$ is P^{γ}-embedded in $X \times Y$ for every compact T_2 space Y with $w(Y) \leq \gamma$ (p. 234 [1] (X need not be $T_{3\frac{1}{2}}$)).

(λ) S is P^{γ}-embedded in X iff $S \times Y$ is C*-embedded in $X \times Y$ for every compact T_2 space Y with $w(Y) \leq \gamma$ (p. 234 [1]--for a sharpened version see [14]).

Removing the cardinality restrictions on each of these (except (δ)) produces characterizations of P- and M-embedding and of collectionwise normality.

Section 2

Since $M^{\aleph_0}$-embedding ($P^{\aleph_0}$-embedding) is equivalent to the extendability of every function into a separable (complete) metrizable AE and since $P^{\aleph_0}$-embedding is equivalent to C-embedding (fact (δ) of Section 1), one might wonder whether S is $M^{\aleph_0}$-embedded in X iff (*): every function from S into an AE embedded in R extends to X. Note that a subset of R is an AE iff it is an interval. 2.1 will show that the above conjecture is false as (*) is equivalent to C-embedding. Example 2.4 of [8] (identical with the example on p. 224 of [17]) shows that C-embedding is strictly weaker than $M^{\aleph_0}$-embedding. (2.1 was first shown by R. Arens for closed subsets of normal spaces [2].)

2.1 *Proposition. If* S *is* C-*embedded in* X, *every function from* S *to an interval* K *of* R *extends to* X *with values in* K.

Proof. There is an extension g of f with $g(X) \subset \overline{K}$. Assuming that K is not closed, $\overline{K} - K$ consists of 1 or 2 points and hence is a zero set of R. Hence $g^{-1}(\overline{K} - K)$ is a zero set of X disjoint from S. Hence there exists $h: X \to [0,1]$ such that $h(S) \equiv 1$ and $h(g^{-1}(\overline{K} - K)) \equiv 0$ (p. 19 [6]). Fix $r \in K$ and define $f^* = hg + (1 - h)r$.

This same idea will work if S is P^{γ}-embedded in X and f is a function from S to a convex subset K of a γ-separable Banach space B such that $\overline{K} - K$ is a zero set in B. (See 4.1 [2])

Fact (θ) of Section 1 with $\gamma = \aleph_0$ tells us that S is $M^{\aleph_0}$-embedded in X iff it is $P^{\aleph_0}$-embedded and given a function f from X to a separable metric space, there exists a zero set Z of X such that $S \subset Z \subset f^{-1}f(S)$. One might ask whether $M^{\aleph_0}$-embedding is equivalent to C-embedding plus (**): Given $f: X \to R$, there exists a zero set Z of X such that $S \subset Z \subset f^{-1}f(S)$. The answer is again no.

To see this, let X be the unit disc in the plane (as a set) and $S = \{(x,y): x^2 + y^2 < 1$, or $x^2 + y^2 = 1$ and x is rational$\}$. Let X have the topology that makes the points of X - S discrete. Hence open sets of X are of the form $U \cup V$, where U is an open neighborhood in the ordinary metric topology and V is a subset of X - S. Any space formed in this way is hereditarily paracompact (see [10]). Hence S is a closed C-embedded subset of X. Since S is an AR that is not an absolute G_δ (see p. 382 [7]), we can show that S is not a zero set of X. Since X is submetrizable (i.e. its topology contains a metric topology), it is clear from (η) in Section 1 that S is not $M^{\aleph_0}$-embedded in X. (To see this, let d be the metric topology on X.) However, let $f: X \to R$ and observe that since S is connected, f(S) is an interval and hence is a G_δ. Therefore $f^{-1}f(S)$ is a G_δ set of X containing S; since X is normal, there exists a zero set Z such that $S \subset Z \subset f^{-1}f(S)$.

Section 3

There is considerable interest in spaces whose product with every metric space is normal. A characterization of this class was given by Morita [11, 12]. A theorem due to Morita, Rudin, and Starbird states that if Y is metric and X normal and countably paracompact, then $X \times Y$ is normal iff $X \times Y$ is countably paracompact [16].

This section will produce a necessary condition for the product of a normal space X with every γ-separable metric space to be normal. If S is a subspace of X, we say that (X,S) has the γ-Zero-Set Interpolation Property (γ-ZIP) if whenever d is a γ-separable pseudometric on X, there exists a zero set Z of X such that:

$$S \subset Z \subset \{x \in X: d(x,x_o) = 0 \text{ for some } x_o \in S\}.$$

By (η) in Section 1, we see that S is M^γ-embedded in X iff S is P^γ-embedded and (X,S) has the γ-ZIP. Hence the γ-ZIP is what needs to be added to P^γ-embedding to produce M^γ-embedding. By dropping the separability condition on d, we obtain a definition of the Zero-Set Interpolation Property (ZIP), and observe that S is M-embedded in X iff S is P-embedded in X and (X,S) has the ZIP. The following proposition is a slight generalization of an example communicated to the author by E. Michael (the example is written up in Section 3 of [18]).

3.1 *Proposition. Let* S *be a closed subset of a normal space* X *such that* $S \times Y$ *is* C*-embedded in* $X \times Y$ *for every* γ*-separable metric space* Y. *Then* (X,S) *has the* γ-ZIP.

Proof. Let d be a γ-separable pseudometric on X and let $A = \{x \in X: d(x,x_o) = 0 \text{ for some } x_o \in S\}$. Let (Y,d) be the γ-separable metric space associated with the pseudometric space (X - A,d). For notational ease we will identify points of Y with those of X - A. Define $f: S \times Y \to R$ by $f(x,y) = 1/d(x,y)$. The map f is well-defined and continuous hence extends to $g: X \times Y \to R$.

Let $H_n = \{x \in X - A: d(x,y) < 1/n \Rightarrow g(x,y) < n\}$. We claim $X - A = \bigcup_n H_n$. Let $x_o \in X - A$ and choose m such that $g(x_o,x_o) < m$. Since g is continuous there exists an open set U of X containing x_o and an $\varepsilon > 0$ such that if $x \in U$ and $d(x_o,y) < \varepsilon$, then $g(x,y) < m$. Choose n such that $n \geq m$ and $1/n \leq \varepsilon$. Then $x_o \in H_n$.

Hence we have $H = \bigcap_n (X - \overline{H}_n) \subset A$. We claim that $S \subset H$.

This will finish the proof, for since S is closed, X is normal, and H is a G_δ, we will be able to find a zero set Z such that $S \subset Z \subset A$. To show that $S \subset H$, argue by contradiction. Assume there exists $x_o \in S \cap \overline{H}_n$ for some n. Choose $y_o \in X - A$ such that $d(x_o,y_o) < 1/2n$. (We can do this since the topology generated by d is contained in the topology on X and $x_o \in \overline{H}_n$.) Then $g(x_o,y_o) > 2n$. Since g is continuous, there exists an open U containing x_o and $\varepsilon > 0$ such that $x \in U$ and $y \in Y$ with $d(y,y_o) < \varepsilon$ implies $g(x,y) > n$.

Choose $x \in U \cap H_n$ such that $d(x,x_o) < 1/2n$. Then $d(x,y_o) \leq d(x,x_o) + d(x_o,y_o) < 1/n$, hence $g(x,y_o) < n$ (since $x \in H_n$). However, $x \in U$ and hence $g(x,y_o) > n$, which is the desired contradiction.

There are a number of corollaries of this result. For example:

3.2 *Corollary. If* X × Y *is normal for every metric* Y, *then every closed subset of* X *has the* ZIP *with respect to* X.

3.3 *Corollary. If* X × Y *is normal for every separable metric* Y, *then every closed subset of* X *is* $M^{\aleph_0}$*-embedded in* X.

Proof. Use (δ) and (η) of Section 1.

3.4 *Corollary. Let* X *be a collectionwise normal space whose product with every metric space is normal. Then every closed subset of* X *is* M*-embedded in* X.

Proof. Use (β) and (η) of Section 1.

3.5 *Corollary* (Michael). *The following are equivalent for a submetrizable space* X:

(a) X *is perfectly normal.*

(b) $X \times Y$ *is perfectly normal for every metric* Y.

(c) $X \times Y$ *is normal for every metric* Y.

Proof. (b) ⇒ (c) is clear and (a) ⇒ (b) is known [9]. Hence we need only show (c) ⇒ (a). Assume (c) but suppose (a) fails. This implies that X is normal and submetrizable, but not perfectly normal. From the definition of the ZIP, it is clear that a subset of a submetrizable space X has the ZIP with respect to X iff it is a zero set. (One may see this by letting d be the metric whose topology is contained in that of X.) Hence X contains a closed subset S such that (X,S) fails to have the ZIP, so by 3.1 there exists a metric space Y such that $X \times Y$ is not normal, giving a contradiction.

In fact, it is clear from the above that if X contains a γ-separable metric topology and fails to be perfectly normal, then there exists a γ-separable metric Y such that $X \times Y$ is not normal. More specifically, if m is a continuous metric on X and S is a closed non-G_δ subset of X, then $S \times Y$ fails to be C-embedded in $X \times Y$, where Y is the metric space $(X - S, m)$. This shows immediately that $X \times P$ fails to be normal, where X is the Michael line and P the irrationals with their usual topology. A different proof was originally given in [10].

It is an open question whether the converse of 3.1 is true. 4.5 of Section 4 will shed some light on this question.

Section 4

Morita and Hoshina (Theorem 3.7 [13]) proved the following generalization of the Homotopy Extension Theorem:

4.1 *Theorem. For a subspace* S *of a topological space* X *the following are equivalent:*

(1) S *is* P^{γ}*-embedded in* X.

(2) $(S \times Y) \cup (X \times B)$ *is* P^{γ}*-embedded in* $X \times Y$ *for every compact* T_2 *space* Y *with* $w(Y) \leq \gamma$ *and its closed subset* B.

(3) $(S \times I) \cup (X \times \{0\})$ *is* P^{γ}*-embedded in* $X \times I$.

(4) (X,S) *has the* HEP *with respect to every complete* ANR *space of weight* $\leq \gamma$.

The analogue of 4.1 for M^{γ}-embedding is the following:

4.2 *Theorem. The following are equivalent:*

(1) S *is* M^{γ}*-embedded in* X.

(2) $(S \times Y) \cup (X \times B)$ *is* M^{γ}*-embedded in* $X \times Y$ *for every compact* T_2 *space* Y *with* $w(Y) \leq \gamma$ *and its closed subset* B.

(3) $(S \times I) \cup (X \times \{0\})$ *is* M^{γ}*-embedded in* $X \times I$.

(4) (X,S) *has the* HEP *with respect to every* ANR *space of weight* $\leq \gamma$.

Proof. The equivalence of (1), (3), and (4) is Theorem 2 of [17]. To complete the proof it remains to show (1) $\Rightarrow$ (2). We state and prove the next theorem, then use it to show (1) $\Rightarrow$ (2).

4.3 *Theorem* (L. Sennott, R. Levy, M. D. Rice). *The following are equivalent for a* T_2 *space* Y*:*

(1) The space Y *is compact.*

(2) If Y *is embedded in a* $T_{3\frac{1}{2}}$ *space* Z *and* X *is any space, then* $X \times Y$ *is* M-*embedded in* $X \times Z$.

(3) If Y *is embedded in a* $T_{3\frac{1}{2}}$ *space* Z *and* X *is any space, then* $X \times Y$ *is* C*-*embedded in* $X \times Z$.

Proof. To show (1) ⇒ (2) let Y be a compact space embedded in a $T_{3\frac{1}{2}}$ space Z, let X be any space, and let $f: X \times Y \to L$ be a continuous function into a normed linear space L. By (α) of Section 1, it is sufficient to extend f to $X \times Z$. Define $g: X \to C^*(Y,L)$ by $g(x)(y) = f(x,y)$. A standard argument shows that g is continuous when $C^*(Y,L)$ has the sup norm topology. We then define $h: g(X) \times Y \to L$ by $h(g(x),y) = f(x,y)$ and observe that g(X) is a metric space and h is continuous. Now $g(X) \times \beta Z$ is the product of a metric space and a compact space and hence is a paracompact M-space. This implies that the closed subset $g(X) \times Y$ is M-embedded (Proposition 2 of [17]). Hence we can lift h to $h^*: g(X) \times \beta Z \to L$. Defining $f^*: X \times Z \to L$ by $f^*(x,z) = h^*(g(x),z)$, one checks that this defines a continuous extension of f. Note: This proof uses an idea contained in the proof of Theorem 2 of [19] and in fact M. Starbird's Theorem 3 [19] is our (1) ⇒ (3) with C*-embedding replaced by C-embedding.

Clearly (2) ⇒ (3). Now assume (3) holds but Y is not compact. By Problem 6J of [6], the space Y is absolutely C*-embedded and hence is almost compact. Let $\beta Y - Y = \{\infty\}$, and let $\{U_\alpha: \alpha \in D\}$ be a base of open neighborhoods of ∞ in βY. We will define a space X such that $X \times Y$ is not C*-embedded in $X \times \beta Y$. Define an ordering on D: $\alpha \leq \beta$ iff

$U_\beta \subset U_\alpha$. Then D becomes a directed set. Let $X = D \cup \{q\}$, where $q \notin D$, points of D are isolated and basic open neighborhoods of q are of the form $\{q\} \cup \{\alpha: \alpha \geq \alpha_o\}$. Denote this set by $[\alpha_o, q]$.

For each α, choose a function f_α on βY such that $f_\alpha(\beta Y - U_\alpha)$ is identically 1 and $f_\alpha(\infty) = 0$. Define $f: X \times Y \to [0,1]$ by $f(\alpha,y) = f_\alpha(y)$ and $f(q,y) = 1$. Clearly f is continuous at points of the form (α,y). Fix (q,y_o) and choose α_o such that $y_o \notin \overline{U}_{\alpha_o}$. If $(x,y) \in [\alpha_o,q] \times (Y - \overline{U}_{\alpha_o})$, then $f(x,y) = 1$.

If there were an extension of f to all of $X \times \beta Y$, the extension would be 1 at all points of the form (q,y) with $y \in Y$ and 0 at all points (α,∞), which implies that the extension is not continuous at (q,∞).

Note: This proof is a generalization of an example given by Comfort and Negrepontis (Example 4.6 of [4]).

To complete the proof of 4.2, let S be M^γ-embedded in X, and Y and B be as in (2). By 4.3 (2) it is clear that $X \times B$ is M^γ-embedded in $X \times Y$. By proposition 5 of [17], we have that $S \times Y$ is M^γ-embedded in $X \times Y$. By Proposition 6 of [17], to show (2) it is sufficient to show that $(S \times Y) \cup (X \times B)$ is P^γ-embedded in $X \times Y$. But this is true from (1) $\Rightarrow$ (2) of 4.1.

We now use 4.3 to obtain a generalization of (κ) in Section 1, which will throw further light on the results in Section 3.

4.4 *Proposition. The following are equivalent:*

(1) S *is* P^{γ}*-embedded in* X.

(2) $S \times Y$ *is* P^{γ}*-embedded in* $X \times Y$ *for every locally compact, paracompact* T_2 *space* Y *with* $w(Y) \leq \gamma$.

(3) $S \times Y$ *is* C^**-embedded in* $X \times Y$ *for every locally compact, paracompact* T_2 *space* Y *with* $w(Y) \leq \gamma$.

Proof. (2) $\Rightarrow$ (3) is clear and (3) $\Rightarrow$ (1) is clear from (λ) of Section 1. It remains to show (1) $\Rightarrow$ (2). Let S be P^{γ}-embedded in X and Y as in (2). For each $y \in Y$, let U_y denote an open neighborhood of y whose closure is compact. Let $\{f_\alpha: \alpha \in A\}$ be a locally finite partition of unity subordinate to the cover $\{U_y: y \in Y\}$, and let K_α denote the compact set $cl(Y - Z(f_\alpha))$.

Let $g: S \times Y \to B$ be a function into a γ-separable Banach space B. By (α) of Section 1, it is sufficient to extend g to $X \times Y$. For each α, the function $g_\alpha = g|S \times K_\alpha$ has an extension to $h_\alpha: X \times K_\alpha \to B$ by (κ) of Section 1. By 4.3 (2), h_α extends to $k_\alpha: X \times Y \to B$. Then $g^*(x,y) = \sum_\alpha f_\alpha(y)k_\alpha(x,y)$ is the desired extension of g.

4.5 *Corollary. If* S *is* C*-embedded in* X*, then* $S \times Y$ *is* C*-embedded in* $X \times Y$ *for any locally compact metric space* Y.

Proof. Let the compact sets K_α be constructed as in the proof of (1) $\Rightarrow$ (2) of 4.4. If Y is metric, then K_α is compact metric. Let $g: S \times Y \to R$. By (δ) and (κ) of Section 1, $g_\alpha = g|S \times K_\alpha$ has an extension to $h_\alpha: X \times K_\alpha \to R$. The proof proceeds as in 4.4.

Comparing 3.1 and 4.5, we see that if S is a closed subset of a normal space X such that (X,S) fails to have ZIP, then there exists a non-locally compact metric space Y such that $S \times Y$ is not C-embedded in $X \times Y$.

4.6 *Corollary. If* S *is* M^{γ}*-embedded in* X*, then* $S \times Y$ *is* M^{γ}*-embedded in* $X \times Y$ *for any locally compact paracompact* T_2 *space* Y *with* $w(Y) \leq \gamma$.

Proof. The proof of (1) ⇒ (2) of 4.4 goes through with B replaced by a γ-separable normed linear space. (To lift g_α use 4.2 (1) ⇒ (2).)

Section 5

As a final application of M-embedding, we generalize two results of E. Chang [3]. (Also see results of Ellis [5].) Although the results deal with ultranormal spaces, they are equivalent to the following:

5.1 *Proposition* (Chang, p. 38, 40 [3]). *Let* X *be nonempty. The following are equivalent.*

(1) X *is a* 0*-dim collectionwise normal (normal) space.*

(2) Every complete (separable) metric space is an AE *for* X.

5.2 *Proposition* (Chang, p. 43 [3]). *Let* S *be a closed* G_δ *subset of a* 0*-dim collectionwise normal (normal) space* X*,* Y *a (separable) metric space and* $f: S \to Y$. *Then* f *extends to* X.

5.3 *Proposition. Let* X *be nonempty. The following are equivalent.*

(1) Every (separable) metric space is an AE *for* X.

(2) X *is a* 0*-dim space in which every closed subset is* M-*(*$M^{\aleph_0}$*-)embedded.*

Proof. We prove the unbracketed equivalence. (1) ⇒ (2) is clear from 5.1 and the definition of M-embedding. To show (2) ⇒ (1), let Y be a metric space, S a closed subset of X, and $f: S \to Y$. Let $\tilde{Y}$ denote the completion of Y with injection map i. Since X is a 0-dim collectionwise normal space, the map $i \circ f: S \to \tilde{Y}$ has an extension $\tilde{f}$ to X, by 5.1. By (θ) of Section 1, there exists a zero set Z of X such that $S \subset Z \subset \tilde{f}^{-1}f(S)$. Hence $\tilde{f}|Z$ maps Z into Y, so by 5.2 it can be lifted to $f^*: X \to Y$, completing the proof.

In [15], Morita remarks that the following generalizations of known results can be proved: If dim $X/S \leq n+1$, then S is M^{γ}-embedded (P^{γ}-embedded) in X iff any map from S into a metric (complete metric) space of weight $\leq \gamma$ which is LC^n and C^n can be extended to X. If dim $X/S \leq n$, then S is M^{γ}-embedded (P^{γ}-embedded) in X iff (X,S) has the homotopy extension property with respect to every metric (complete metric) space of weight $\leq \gamma$ which is LC^n.

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