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Research Announcement:
AN APPLICATION OF TREES TO TOPOLOGY

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AN APPLICATION OF TREES TO TOPOLOGY

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A *tree* is a POSET in which the set of predecessors of any element is well-ordered.

The *Gleason space* [C.N.], $\mathcal{G}(X)$, of a regular space X is the Stone space of the regular-closed set algebra $\mathcal{R}(X)$ of X. Two spaces X and Y are *$\mathcal{G}$-absolute* iff $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ are homeomorphic. Note if X and Y are compact, $\mathcal{G}$-absoluteness is just co-absoluteness [Po; Wo].

Theorem 1. *X is $\mathcal{G}$-absolute with a linearly ordered space if, and only if, $\mathcal{R}(X) - \{X\}$ contains a cofinal tree.*

Application 1. A Moore space X is $\mathcal{G}$-absolute with a linearly ordered space if, and only if, X has a dense metrizable subspace.

Remark 1. A recent result in [Wh] suggests that "Moore" in the preceding might be replaced by "1st countable"; however, if X is a Souslin line, then X provides the counterexample. Further, $X \times [0,1]$ is not $\mathcal{G}$-absolute with any linearly ordered space.

A POSET P is *κ-closed* [B], for a cardinal κ, iff every well-ordered increasing sequence in P, of length κ, is bounded above. P is *separative* [Je] iff given $p \not\leq q \; \exists \; r \geq q \ni \nexists \; s$ $s \geq p$ and $s \geq r$.

Theorem 2. *A κ-closed separative POSET of cardinal $\leq 2^{\kappa}$ has a cofinal tree.*

Application 2. If D is a locally compact non-compact metric space, then $\beta D - D$ is co-absolute with a linearly ordered space having a dense set of P-points.

Remark 2. S. Broverman has verbally communicated a short proof to the author that if D is an infinite discrete space of cardinal $\leq 2^{\omega}$, then $\beta D - D$ and $\beta N - N$ are co-absolute. We note that the same technique shows that if D is a dense-in-itself locally compact non-compact metric space of density $\leq 2^{\omega}$, then $\beta D - D$ and $\beta R - R$ are co-absolute. (N is the space of natural numbers and R is the reals.)

We consider two statements:

(#) $\exists$ precisely one (up to an isomorphism) complete atomless Boolean algebra B with a dense set of cardinal 2^{ω} and ω-closed.

(*) If D is a locally compact non-compact metric space of density $\leq 2^{\omega}$, then $\beta D - D$ is co-absolute with $\beta N - N$.

The following theorem can be proved similarly to [G.J., 13.13]; however, our proof via trees appears new.

Theorem . CH $\Rightarrow$ # *and* $2^{\omega} = 2^{\omega_1} \Rightarrow \neg$#.

Application 3. # $\Rightarrow$ (*) and (*) is consistent with $\neg$#.

Remark 3. If $\mathcal{M}$ is a model of ZFC obtained from a model of CH by adding ω_2 Cohen reals iteratively [B], then (*) is true in $\mathcal{M}$.

Corollary [Wo]. (CH) *If* D *is a dense-in-itself locally compact non-compact metric space of density* $\leq 2^{\omega}$, *then* $\beta D - D$ *is co-absolute with* $\beta N - N$.

Questions. (1) Does there exist a "real" example of a compact 1st countable space not $\mathcal{G}$-absolute with any linearly ordered space?

(2) Does # $\Rightarrow$ CH?

(3) Is $\beta R - R$ co-absolute with $\beta N - N$?

(4) Is $(\beta R - R)^2$ (respectively, $(\beta N - N)^2$) co-absolute with $\beta R - R$ $(\beta N - N)$?

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