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# PROBLEM SECTION

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**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
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# PROBLEM SECTION

Most of the problems in the "Contributed Problems" subsection were raised during the Special Session on General Topology at the 761st meeting of the AMS in Charleston, South Carolina, Nov. 3-4, 1978. We are indebted to Jerry Vaughan for organizing that session, compiling these problems, and giving his permission for them to be printed here. Special thanks are also due to Mary Ellen Rudin for sending Vaughan a long list of problems contributed to her by others.

The subsection "Classic Problems" has been discontinued. Instead, we will publish problem surveys as part of our regularly refereed papers, such as "A survey of two problems" in this issue.

## CONTRIBUTED PROBLEMS

As usual, the person whose name is associated with each of the following problems is someone who is interested in the problem and can supply further information about it. It is not necessarily the person who first raised the problem, nor the one who has done the most research on it.

A bracketed P after a problem indicates that there is a related article in the current volume of these PROCEEDINGS. A bracketed N indicates that there is an abstract in the October, 1978 AMS *Notices* related to the problem, while an [R] indicates the problem was relayed by Mary Ellen Rudin. Numbering of problems in each category picks up from the previous volume.

The words "normal" and "regular" will always be assumed to include "Hausdorff." "Compact" will also be assumed to include "Hausdorff" unless followed by "$T_1$" or "topological."

## A. Cardinal Invariants

5. (*Arhangel'skij*) Let c(X) denote the cellularity of X. Does there exist a space X such that $c(X^2) > c(X)$? [Yes if CH or there exists a Souslin line.] [R]

6. (*Arhangel'skij*) Let d(X) denote the density of X and let t(X) denote the tightness of X, $\omega \cdot \min\{\kappa$: for each $A \subset X$ and each $x \in \mathrm{cl}A$, there exists $B \subset A$ such that $x \in \mathrm{cl}B$, $|B| \leq \kappa\}$. Does there exist a compact space X such that $c(X) = t(X) < d(X)$? [R]

7. (*Przymusiński*) Does there exist for every cardinal $\lambda$ an isometrically universal metric space of weight $\lambda$? [Yes if GCH.] [R]

See also B14, C15, C20, C21, C22, D19, D21, K4, R1, S1, and the article "A survey of two problems," by P. Nyikos.

## B. Generalized Metric Spaces and Metrization

5. (*Wicke*) Is every monotonically semi-stratifiable hereditarily submetacompact [former name: $\theta$-refinable] space semi-stratifiable? [N] [P]

6. (*Wicke*) Is every monotonic $\beta$-space which is hereditarily submetacompact a $\beta$-space? [N] [P]

7. (*Wicke*) Does every primitive q-space with a $\Theta$-diagonal have a primitive base? [N] [P]

8. (*Aull*) For all base axioms such that countably compact regular + base axiom $\Rightarrow$ metrizable, is it true that regular + $\beta$ + collectionwise normal + base axiom $\Rightarrow$ metrizable? In particular, what about quasi-developable spaces, or those

with $\delta\theta$-bases or point-countable bases? [P]

9. (*Aull*) Is every space in the class MOBI quasi-developable? [P]

10. (*Aull*) Is every space with a $\sigma$-locally countable base quasi-developable? [P]

11. (*Aull*) Is every collectionwise normal space with a $\sigma$-locally countable base metrizable (equivalently, paracompact)? [P]

12. (*Aull*) Is every first countable space with a weak uniform base (WUB) quasi-developable? [P]

13. (*Aull*) Does every developable space with a WUB and without isolated points have a uniform base? (Equivalently, is it metacompact?) [P]

14. (*Arhangel'skij*) Let X be regular, Lindelöf, and symmetrizable. Is X separable? Does X have a $G_\delta$-diagonal? [R]

See also C18, C19, D17, D21, D22, H3, H4, M3, P5, P8, and the article "A survey of two problems," by P. Nyikos.

## C. Compactness and Generalizations

15. (*Nyikos*) Does there exist a first countable compact $T_1$ space of cardinality > c? A compact $T_1$ space with points $G_\delta$ and cardinality > c? ["Yes" to the second question is consistent: Shelah]. How large can the cardinality be in either case? [P]

16. (*J. Hagler*) Does there exist a compact space K with a countable dense subset D such that every sequence in D has a convergent subsequence, but K is not sequentially compact? (We may assume without loss of generality that K is a

compactification of $\omega$, i.e. that the points of D are isolated.) [Yes if MA] [R]

17. (*Nyikos*) If a compact space has the property that all countably compact subsets are compact, is the space sequentially compact? [Yes, if P(c) or $2^{\aleph_0} < 2^{\aleph_1}$.] [N]

18. (*Nyikos*) Is there a compact non-metrizable space X such that $X^2$ is hereditarily normal? [Yes if MA + ¬CH: Nyikos.] [N]

19. (*van Douwen*) Is a compact space metrizable if its square is:

(a) hereditarily collectionwise normal? [Yes if MA + ¬CH: Nyikos.]

(b) hereditarily collectionwise Hausdorff? [No if CH: Kunen.]

20. (*van Douwen*) Consider the following statements about an infinite compact space X:

(a) there are $Y \subset X$ and $y \in Y$ such that $\chi(y,Y) \in \{\omega,\omega_1\}$

(b) there is a decreasing family $\mathcal{F}$ of closed sets with $|\mathcal{F}| \in \{\omega,\omega_1\}$ and $|\cap\mathcal{F}| = 1$.

Without loss of generality, X is separable, hence CH ⇒ (a). Clearly (a) ⇒ (b). What happens under CH?

21. (*van Douwen*) Is it true that for all infinite cardinals $\kappa$ we have: $\kappa$ is singular iff initial $\kappa$-compactness is productive iff initial $\kappa$-compactness is finitely productive? [It is consistent.] [N]

22. (*van Douwen*) Is initial $\kappa$-compactness productive if $\kappa$ is singular? [Yes if for all $\mu < \kappa$, $2^\mu < \kappa$ (Saks and Stephenson) hence yes if GCH.] [N]

23. (*van Douwen*) Does there exist a normal space which

is not initially $\kappa$-compact but which has a dense initially $\kappa$-compact subspace, for some (each) $\kappa > \omega$? [This cannot happen if $\kappa = \omega$, of course.] [N]

24. (*M. Pouzet*) A space X is called *impartible* if for every partition {A,B} of X, there is a homeomorphism from X into A or into B. Is there a compact impartible space? [R]

See also A6, H6, K4, K5, O5, O6, O7, P1, P2, P3, P4, and the article "A survey of two problems," by P. Nyikos.

## D. Paracompactness and Generalizations

17. (*Aull*) (a) Is every collectionwise normal space with an ortho-base paracompact? (b) Is there a model of set theory in which every normal space with an ortho-base is paracompact? [P]

18. (*Junnila*) Is a space submetacompact if every directed open cover has a $\sigma$-cushioned refinement? [P]

19.* (*Aull*) (a) For Tychonoff spaces, does pseudocompact plus metacompact equal compact? (b) In a pseudocompact Tychonoff space, does every point-finite collection $\mathcal{U}$ of open sets have a finite subcollection $\mathcal{V}$ such that $\cup\mathcal{V}$ is dense in $\cup\mathcal{U}$?

20. (*Reed*) Does MA + $\neg$CH imply either

(a) perfect (normal), locally compact spaces are subparacompact or

(b) there is no Dowker manifold? [N]

21. (*Reed*) Does there exist in ZFC a normal space of cardinality $\omega_1$ with a point countable base which is not perfect? [N]

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*Recently solved by B. Scott: Yes to (a), no to (b).

22. (*Reed*) Does there exist a strongly collectionwise Hausdorff Moore space which is not normal? [N]

See also B11, B13, C18, C19, K4, P7, and the article "A survey of two problems," by P. Nyikos.

**E. Separation and Disconnectedness**

5. (*Arhangel'skij*) Does every zero-dimensional space have a strongly zero-dimensional subtopology? [R]

6. (*Przymusiński*) If $\mathcal{F}[X]$ is the Pixley-Roy hyperspace over X, then is $\mathcal{F}[X]$ strongly zero-dimensional? [R]

See also R1.

**F. Continua Theory**

6. (*J. T. Rogers*) Suppose M and N are solenoids of pseudo-arcs that decompose to the same solenoid. Are M and N homeomorphic? [P]

See also G9, G10.

**G. Mappings of Continua and Euclidean Spaces**

9. (*Hagopian*) Is every continuous image of every $\lambda$-connected plane continuum $\lambda$-connected? [P]

10. (*Heath and Fletcher*) Is there a Euclidean non-Galois homogeneous continuum? [N]

**H. Homogeneity and Mappings of General Spaces**

3. (*Aull*) Are $\gamma$-spaces, quasi-metrizable spaces, or spaces with $\sigma$-Q bases preserved under compact open maps? What about spaces with ortho-bases? [P]

4. (*Aull*) Are $\theta$-spaces or spaces with a $\delta\theta$-base preserved under perfect mappings? [P]

5. (*van Douwen*) Does there exist a homogeneous

zero-dimensional separable metrizable space which

(a) cannot be given the structure of a topological group or, more strongly,

(b) has the fixed-point property for autohomeomorphisms?

6. (*van Douwen*) Does there exist an infinite homogeneous compact zero-dimensional space which has the fixed-point property for autohomeomorphisms?

See also P1 through P5, P7, S3, and S4.

**K. Connectedness**

4.* (*Zenor*) Does MA + $\neg$CH imply that there is no locally connected, rim compact L-space? [N]

5. (*P. Collins*) Is a locally compact, $\sigma$-compact connected and locally connected space always the union of a countable sequence of compact, connected, locally connected subsets such that $C_i \subset \text{Int}(C_{i+1})$ for all i? [R]

See also D20 and M3.

**M. Manifolds**

3. (*Rudin*) Is there a complex analytic, perfectly normal, non-metrizable manifold? [No if MA + $\neg$CH.]

See also D20.

**O. Theory of Retracts; Extension of Continuous Functions**

5. (*Gruenhage, Kozlowski, and Nyikos*) A compact space is an *absolute retract* (AR) if it is a retract of every compact (equivalently, Tychonoff) space in which it is embedded, and a *Boolean absolute retract* (BAR) if it is zero-dimensional, and a retract of every zero-dimensional (compact) space in

*Recently solved in the affirmative by G. Gruenhage.

which it is embedded.

Is a non-metrizable AR homeomorphic to $I^\kappa$ for some $\kappa$?

6. (*Nyikos*) If X is a BAR, does there exist a BAR Y such that $X \times Y \approx 2^\kappa$ for some $\kappa$? Is $X^\kappa \approx 2^\kappa$ for large enough $\kappa$? (Here $\approx$ denotes homeomorphism.)

7. (*Nyikos*) Does there exist an intrinsic characterization of BAR's either among compact spaces or among dyadic spaces? (This is an old question.)

See also P9.

**P. Products, Hyperspaces, and Similar Constructions**

1. (*Williams*) Are $\beta\mathbf{N} - \mathbf{N}$ and $\beta\mathbf{R} - \mathbf{R}$ co-absolute? [N]

2. (*Williams*) Are $\beta\mathbf{N} - \mathbf{N}$ and $(\beta\mathbf{N} - \mathbf{N}) \times (\beta\mathbf{N} - \mathbf{N})$ co-absolute? [Yes if MA: Balcar, Pelant and Simon.] [N]

3. (*Williams*) Are $\beta\mathbf{R} - \mathbf{R}$ and $(\beta\mathbf{R} - \mathbf{R}) \times (\beta\mathbf{R} - \mathbf{R})$ co-absolute? [N]

4. (*Williams*) Is there a locally compact non-compact metric space X of density at most $2^\omega$ such that $\beta X - X$ fails to be co-absolute with either $\beta\mathbf{N} - \mathbf{N}$ or $\beta\mathbf{R} - \mathbf{R}$? [N]

5. (*Heath*) Is the Pixley-Roy hyperspace of $\mathbb{R}$ homogeneous? [N]

6.* (*Heath*) Does there exist an uncountable, non-discrete space X which is homeomorphic to its Pixley-Roy hyperspace $\mathcal{F}[X]$? [N]

7. (*Rudin*) Can the perfect image of a normal subspace of a $\Sigma$-product of lines be embedded in the $\Sigma$-product?

8. (*Przymusiński*) Can every space with a point-countable base be embedded into a $\Sigma$-product of intervals? [R]

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*Recently answered in the affirmative by P. Nyikos and E. van Douwen.

9. (*Nyikos*) If a product of two spaces is homeomorphic to $2^\kappa$, must one of the factors be homeomorphic to $2^\kappa$? [This is true for $\kappa = \omega$, of course.]

See also E6, O5, O6, S3, and S4.

## Q. Generalizations of Topological Spaces

1. (*Price*) Does there exist a "Čech function" $f: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $f \neq id$, $A \subset f(A)$ for all A, $f(A \cup B) = f(A) \cup f(B)$ for all A, B, and f is onto? (In other words, is there a countable closure space in which every subset is the closure of a subset?) ["Yes" is consistent: Price, Galvin.] [R]

## R. Dimension Theory

1. (*Przymusiński*) If X is a metric space in which every subset is an $F_\sigma$, then is dim X = 0? [Yes if V = L: Reed.] [R]

See also E5 and E6.

## S. Problems Closely Related to Set Theory

1. (*Rudin and Lutzer*) Is every Q-set strong? (In other words, are its finite powers Q-sets?) [P]

2. (*van Douwen and Rudin*) In ZFC, are there two free ultrafilters on $\omega$ with no common finite-to-one image?

3. (*K. Hoffman*) Let $f^k$ be the permutation on the discrete space Z of integers which takes n to n + k. For $k \in Z$ and $p \in \beta(Z)$, let $p^k = \{f^k(M): M \in p\}$, and $O_p = \{p^k: k \in Z\}$ the *orbit* of p. Let $\mathcal{O} = \{\overline{O}p: p \in \beta Z - Z\}$, and let $\mathcal{M}$ be the set of maximal members of $\mathcal{O}$. Is there an infinite strictly increasing sequence of members of $\mathcal{O}$? How long can such be? What can be said in general about $\mathcal{O}$ and $\mathcal{M}$? [R]

4. (*van Douwen*) If $D \subset \beta\omega - \omega$ is nowhere dense, is it true in ZFC that $\{(\beta\pi)^{\rightarrow}D: \pi$ is a permutation of $\omega\}$ is not all of $\beta\omega - \omega$? What if $D = \{\bar{A}: A \in \mathcal{A}\}$ where $\mathcal{A}$ is one of $\{A \subset \omega: \lim|A \cap n|/_n = 1\}$ or $\{A \subset \omega: \Sigma_{n \in A-\{0\}} \frac{1}{n} = \infty\}$?

See also Q1.

**T. Algebraic and Geometric Topology**

In the following problems, let $\Theta_3^H$ denote the abelian group obtained from the set of oriented 3-dimensional PL homology spheres using the operation of connected sum, modulo those which bound acyclic PL 4-manifolds. Let $\alpha: \Theta_3^H \rightarrow Z_2$ denote the Kervaire-Milnor-Rohlin surjection.

1. (*Stern*) Is $\Theta_3^H$ finitely generated? [P]

2. (*Stern*) Does $\Theta_3^H$ contain an element of nontrivial finite order? [P]

3. (*Stern*) Is $\alpha$ an isomorphism? [P]

4. (*Stern*) Suppose a homology 3-sphere $H^3$ admits an orientation reversing PL homeomorphism. Is it true that

(a) $\alpha(H^3) = 0$

(b) $[H^3] = 0$ in $\Theta_3^H$? [P]

**INFORMATION ON EARLIER PROBLEMS**

D6, vol. 1 (*Alster and Zenor*) Is every perfectly normal, locally Euclidean space collectionwise normal? *Consistency result by M. E. Rudin.* Yes if MA + $\neg$CH--in fact, under MA + $\neg$CH, every perfectly normal, locally Euclidean space is metrizable.

C8, vol. 2 (*Comfort*) For p, q $\in \beta N$, define p + q by $A \in p + q$ if $\{n: A - n \in p\} \in q$. Define $p \cdot q$ by $A \in p \cdot q$ if $\{n: A/n \in p\} \in q$. Is there $p \in \beta N - N$ such that p + p = p·p = p? *Solution.* No, Hindman, AMS Notices 25 (1978), A-612.

D10, vol. 2 (*Nyikos*) Is the product of a metacompact space and a metacompact scattered space likewise metacompact? *Solution.* Yes for regular spaces, Hasan Hdeib. In fact (Hdeib) the following result is true. A space is called C-*scattered* if each closed subspace has a point with a neighborhood in the relative topology which is locally compact. A subspace A of a space X is *metacompact relative to* X if for each open (in X) cover of A there is a point-finite (in X) open (in X) refinement which covers A. *Theorem.* Let A be a closed C-scattered subset of a regular metacompact space S. Then $A \times Y$ is metacompact relative to $X \times Y$ for any regular metacompact space Y. *Corollary.* The product of a regular metacompact C-scattered (in particular, scattered) space with a regular metacompact space is metacompact.

Classic Problem II, vol. 1. A related problem was whether every regular hereditarily Lindelöf space with a point countable base is metrizable. *Solution.* No if CH or "there exists a Souslin line." (For the former, see the paper by van Douwen, Tall, and Weiss, AMS Proceedings 64 (1977), 139-145.) Yes if MA + $\neg$CH: Szentmiklóssy. In fact (Szentmiklóssy) under MA + $\neg$CH, every hereditarily Lindelöf first countable regular space is hereditarily separable.

Classic Problem III, vol. 1. Is every normal screenable space paracompact? *Remarks.* M. E. Rudin has found a hole in her "consistency" counterexample, so this problem is back to where it started: we do not even have consistency results pertaining to whether normal + screenable = paracompact.

Classic Problem V, vol. 2. Does every compact hereditarily normal space of countable tightness contain a nontrivial converging sequence? *Consistency Results.* If MA + ¬CH, then (Szentmiklóssy) a compact space can not contain an S-space or an L-space if it is hereditarily normal *or* of countable tightness. On the other hand (Nyikos) if $2^{\aleph_0} < 2^{\aleph_1}$, a compact hereditarily normal space which does not contain an S-space is sequentially compact, and if it has countable tightness it is Fréchet-Urysohn. So the best that can be hoped for in the way of a "real counterexample" is a space which contains an S-space under $2^{\aleph_0} < 2^{\aleph_1}$ but does not contain one under MA + ¬CH. On the other hand, there do exist "consistency counterexamples," constructed by Fedorchuk assuming $\diamondsuit$.

## AMS (MOS) CLASSIFICATION OF PROBLEMS

We present here a listing of the problems that have appeared in the first three volumes of these PROCEEDINGS, according to the AMS classification headings they come under, to the best of our judgement. Many problems are listed under several headings, and no attempt has been made to distinguish between primary and secondary listings.

### 54-00 Difficult to Classify at the Second Level

*vol. 2.* C8, D16. *vol. 3.* S2, S3, S4.

### 54 Axx Generalities

A 05 Topological spaces and generalizations (closure spaces, etc.) *vol. 3.* Q1.

C 55 ANE, AE, ANR, AR spaces (general properties) *vol. 3.* O5, O6.

C 99 None of the above, but in this section *vol. 1.* G1 through G5. *vol. 2.* O2. *vol. 3.* H5, H6.

## 54 Dxx Fairly General Properties

D 05 Connected and locally connected spaces (general aspects) *vol. 1.* B3, D7. *vol. 2.* K1, K2, K3. *vol. 3.* K4, K5.

D 10 Separation axioms, $T_0$-$T_3$ *vol. 2.* K1, K2.

D 15 Higher separation axioms *vol. 1.* D1, D3, D5, D6, D7, E1, Classic Problems II and III. *vol. 2.* D12, D13, D14, K3, Classic Problems V, VII. *vol. 3.* C18, C19, C23, D17, D21, D22.

D 20 Covering properties *vol. 1.* A1, D1, D2, D3, E2, Classic Problems II and III. *vol. 2.* C7, D8 through D12, D14, D15, K3. *vol. 3.* C21, C22, C23, D18, D19, D20.

D 30 Compact spaces and generalizations *vol. 1.* C1, C2, Classic Problem I. *vol. 2.* A4, C3, C7, C10 through C14, Classic Problems V, VI. *vol. 3.* C15 through C20, C24, D19, H6.

D 35 Compactifications, etc. *vol. 2.* C9. *vol. 3.* C16.

D 40 Remainders *vol. 2.* C3, C4, C5, C6, C8, E4. *vol. 3.* P1, P2, P3, P4, S4.

D 45 Local compactness, σ-compactness *vol. 1.* B3, D6, D7. *vol. 3.* D20, K4, K5.

D 50 k-spaces *vol. 2.* E4.

D 55 Sequential spaces *vol. 2.* Classic Problem VI.

G 05 Extremely disconnected spaces, F-spaces, etc. *vol. 1.* C2, D1, E1, E2. *vol. 2.* P1, P2, P3, P4, S3.

G 15 Pathological spaces *vol. 1.* Classic Problem I. *vol. 2.* D12 through D16.

G 20 Counterexamples *vol. 3.* P5, S4.

**54 Hxx Connections with Other Structures, Applications**

H 25 Fixed-point and coincidence theorems *vol. 3.* H5, H6.

H 99 None of the above, but in this section *vol. 1.* G2. *vol. 2.* N1, N2. *vol. 3.* S2, S3.

**55 Dxx Homotopy Theory**

D 99 None of the above, but in this section *vol. 2.* I6.

**57 Axx Topological Manifolds**

A 05 Topology of $E^2$, 2-manifolds *vol. 2.* M1, M2.

A 20 Topology of infinite dimensional manifolds *vol. 1.* I1 through I5. *vol. 2.* J1.

A 99 None of the above, but in this section *vol. 1.* D6. *vol. 3.* M3.

**57 Cxx PL-Topology**

C 15 Triangulating manifolds *vol. 3.* T1, T2, T3, T4.

**Late Remarks**

Problem C. 20(b). Hušek has recently shown this is always true.

Problem S. 1. "No" is consistent: Fleissner.

Problem C8, Vol. 2. (Hindman) there is no pair p,q satisfying $p + q = p \cdot q$.

Problem E. 4. Vol. 2. E. K. van Douwen has shown that the answer to E4, vol. 2 (If X is a $k_w$-space, is $\beta X-X$ necessarily an F-space?) is negative. A proof similar to that for the rationals shows that no countable dense-in-itself $k_w$-space has an F-space for its growth.