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QUASI-METRIZABILITY AND THE γ-SPACE PROPERTY IN CERTAIN GENERALIZED ORDERED SPACES

H. R. Bennett

1. Introduction

In [W] W. A. Wilson introduced the concept of a quasi-metric space and such spaces have been studied extensively. (See [A], [B], [R], [St], [SZ] and [G]). The notion of a γ-space first appeared in [H] and it is now known [FL] that the γ-space notion is equivalent to topological properties arising in a multitude of unrelated studies, e.g. Co-Nagata spaces [Ma], spaces having a co-convergent open neighborhood assignment [Sa], locally quasi-uniform spaces with countable bases [FL], spaces admitting an o-metric that satisfies property π[NC] and Nagata first countable spaces [H].

The problem of whether or not each γ-space is quasi-metrizable is a classic problem which goes "back to Ribeiro's paper of 1943 where a theorem which says in effect that every γ-space is quasi-metrizable is given, but the proof is at best incomplete." [TP]

It will be obvious from the definitions (given in section 2) that each quasi-metric space is a γ-space but the converse is not known even if additional hypotheses are imposed (see [G], [LN]). Experience has shown that the process of modifying the real line, or one of its subspaces, to create generalized ordered spaces (see section 2) can produce useful counterexamples and it might be conjectured that such a process could be used to create a γ-space which is not a quasi-metric space. The main result of this note shows that such a strategy cannot work and that the concepts of a γ-space and a quasi-metric space are equivalent in this context, and even in a more general setting. This represents a partial solution to the related problems in [TP] Classic Problem VIII (page 687).

2. Preliminaries and Definitions

A linearly ordered space (= LOTS) is a linearly ordered set $(Y,\leq)$ whose topology λ coincides with the usual open interval topology associated with $\leq$. A generalized ordered space (= GO-space) is a subspace of a LOTS. The standard reference for such spaces is [L] and much of the notation here will follow the notation in [L].

(2.1) *Definition*. If $(Y,\leq)$ is a LOTS with topology λ, select three disjoint, possibly empty, subsets A, B, and C of Y and let τ be the topology on Y having the collection $\lambda \cup \{[x,y[\,|\,x \in A,\ x < y\} \cup \{]x,y]\,|\,y \in B,\ x < y\} \cup \{\{x\}\,|\,x \in C\}$ as a base. Then (Y,τ) is a GO-space on Y and is denoted by $X = GO_Y(R,E,I,L)$ where $I = \{x \in Y\,|\,\{x\}$ is open$\}$, $R = \{x \in Y - I\,|\,[x,\rightarrow[$ is open$\}$, $L = \{x \in Y - I\,|\,]\leftarrow,x]$ is open$\}$, and $E = Y - (R \cup L \cup I)$. Notice if $\leq$ is order dense then $A = R$, $B = L$ and $C = I$.

Common examples of GO-spaces are the Michael Line $M = GO_{E^1}(\phi,Q,E^1-Q,\phi)$ [Mi] (where E^1 denotes the real line and Q the set of rational numbers) and the Sorgenfrey Line $S = GO_{E^1}(E^1,\phi,\phi,\phi)$ [So].

It is convenient throughout this paper to think of Y being embedded in its order completion Y^+. The closure of a set S in a space Y is denoted by cl(S,Y).

(2.2) *Definition*. If X is a set and d is a function from $X \times X$ into the non-negative real numbers, then d is a quasi-metric in X if

(i) $d(x,y) = 0$ if and only if $x = y$, and

(ii) $d(x,z) \leq d(x,y) + d(y,z)$ for each $x,y,z \in X$.

A topological space (X,τ) is a quasi-metric space if there is a quasi-metric in X that is compatible with τ (i.e. the ε-neighborhoods form a base for τ).

In [H] Hodel gave an equivalent definition of a quasi-metrizable space (X,τ) in terms of a function $g\colon N \times X \to \tau$ where N denotes the natural numbers, as follows.

(2.3) *Definition*. If g satisfies

(i) $\{g(n,x) | n \in N\}$ is a local base at x for each $x \in X$, and

(ii) if $y \in g(n+1,x)$ then $g(n+1,y) \subseteq g(n,x)$

then (X,τ) is a quasi-metric space and g is called a quasi-metric function in X.

If (ii) is replaced by

(iii) for each $x \in X$ and $n \in N$ there is a natural number $\alpha(n,x) = m \geq n$ such that if $y \in g(m,x)$, then $g(m,y) \subseteq g(n,x)$, then (X,τ) is a γ-space and g is called a γ-function on X.

By letting $\alpha(n,x) = n + 1$ for each $x \in X$ and $n \in N$ it is clear that each quasi-metric space is a γ-space.

If g is a γ-function on a GO-space X, then, by replacing

$g(n,x)$ by a smaller set if necessary, it may be assumed that each $g(n,x)$ is order-convex (i.e. if $y,z \in g(n,x)$ then $[y,z] \subseteq g(n,x)$).

The following technical definition will simplify the statement of the main theorem.

(2.4) *Defintion.* Let $(Y,\leq)$ be a LOTS and let R, I, E, and L be defined as in (2.1). If $X = GO_Y(R,E,I,L)$, then X satisfies condition C if

(i) $R = \cup\{R_n | n \in N\}$ such that if $x \in L$ and $n \in N$, there is a point x' (possibly in Y^+) such that $x' < x$ and $]x',x[\cap cl(R_n,Y) = \phi$ (where $cl(R_n,Y)$ is the closure of R_n in the LOTS Y),

(ii) $L = \cup\{L_n | n \in N\}$ such that if $x \in R$ and $n \in N$ there is a point x' (possibly in Y^+) such that $x' > x$ and $]x,x'[\cap cl(L_n,Y) = \phi$,

(iii) if $p \in E \cap cl(R_n,Y)$ for some $n \in N$, then there is a point p' (possibly in Y^+) such that $p' < p$ and $]p',p[\cap cl(R_n,Y) = \phi$, and

(iv) if $p \in E \cap cl(L_n,Y)$ for some $n \in N$, then there is a point p' (possibly in Y^+) such that $p < p'$ and $]p,p'[\cap cl(L_n,Y) = \phi$.

3. The Theorem

(3.1) *Theorem. If* (X,τ) *is a* GO-*space constructed from a separable* LOTS$(Y,\leq)$*, then the following are equivalent*

(i) X *is a* γ*-space,*

(ii) X *satisfies condition* C*, and*

(iii) X *is a quasi-metric space.*

Proof. Let R, I, E and L be defined as in (2.1). Let $Q = \{q_1, q_2, \cdots\}$ be a countable dense subset of Y and, for each $n \in N$, let $Q_k = \{q_1, \cdots, q_k\}$.

(i) $\rightarrow$ (ii). Let $g: N \times X \times \tau$ be a γ-function on X and decompose R into countably many subsets by letting $R(n,m,k) = \{x \in R | [x,\rightarrow[\supseteq g(n,x), \alpha(n,x) = m, g(m,x) \cap (Q_k - \{x\}) \neq \phi\}$. Let $\langle x_1, x_2, \cdots \rangle$ be a sequence of points of $R(n,m,k)$ that converges, in Y, to a point p.

If, for $i \in N$, $x_i < p$, then

(a) if $p \in R \cup I$, then the sequence does not converge to p in X, and

(b) $p \notin L \cup E$ since if $p \in L \cup E$, then either there exists $q \in Q_k$ such that $p \in]\leftarrow, q]$ and $]\leftarrow, q] \cap (Q_k - \{p\}) = \emptyset$ or there exists $q', q \in Q_k, q' < q$ such that $p \in]q', q]$ and $]q', q] \cap (Q_k - \{p\}) = \emptyset$. There is a natural number N_1 such that if $i \geq N_1$, then $q \in g(m, x_i)$. Since $p \leq q$ and $g(m, x_i)$ is order-convex, it follows that $p \in g(m, x_i)$. There is a natural number N_2 such that if $i \geq N_2$, then $x_i \in g(m,p)$. Let i and j be larger than $N_1 + N_2$ such that $x_1 < x_j$. Since $p \in g(m, x_j)$, it follows that $g(m,p) \subseteq g(n, x_j) \subseteq [x_j, \rightarrow[$. But $x_i \in g(m,p) \subseteq [x_j, \rightarrow[$. From this contradiction it follows that $p \notin L \cup R$.

If, for $i \in N$, $p < x_i$, then

(c) if $p \in L \cup I$, then the sequence does not converge to p in X, and

(d) if $p \in R \cup E$, then the sequence does converge to p in X.

It is clear from (b) and (c) that no point of L is a limit point of $R(n,m,k)$. Thus if $x \in L$, there is a point x' (possibly in Y^+) such that $x' < x$ and $]x',x[\cap cl(R(n,m,k),Y) = \phi$. Also from (b) and (d) if $p \in E \cap cl(R(n,m,k),Y)$ then there is a point p' (possibly in Y^+) such that $p' < p$ and $]p',p[\cap cl(R(n,m,k),Y) = \phi$.

Decompose L into countably many subsets by letting $L(n,m,k) = \{x \in L \mid]\leftarrow,x] \supseteq g(n,x), \alpha(n,x) = m, g(m,x) \cap (Q_k-\{x\}) \neq \phi\}$. Let $<x_1,x_2,\cdots>$ be a sequence of points of $L(n,m,k)$ that converges, in Y, to a point p.

If, for $i \in N$, $x_i < p$, then

(e) if $p \in L \cup E$, then the sequence converges to p in X, and

(f) if $p \in R \cup I$, then the sequence does not converge to p in X.

If, for $i \in N$, $p < x_i$, then

(g) if $p \in L \cup I$, then the sequence does not converge to p in X, and

(h) $p \in R \cup E$(argue as in (b)).

It is clear from (f) and (h) that no point of R is a limit point of $L(n,m,k)$. Thus, if $x \in R$, there is a point x' (possibly in Y^+) such that $x < x'$ and $]x,x'[\cap cl(L(n,m,k),Y) = \phi$. Also from (e) and (h) if $p \in E \cap cl(L(n,m,k),Y)$ then there is a point p' (possibly in Y+) such that $p < p'$ and $]p,p'[\cap cl(L(n,m,k),Y) = \phi$.

Clearly $R = \cup\{R(n,m,k) \mid (n,m,k) \in N^3\}$ and $L = \cup\{L(n,m,k) \mid (n,m,k) \in N^3\}$. It follows that X satisfies condition C.

(ii) $\rightarrow$ (iii). For each $k \in N$ and $x \in X$ let

$$\ell(k,x) = \max\{q \in Q_k \mid q < x\} \text{ if } \min Q_k < x$$

and let $\ell(k,x)$ be the symbol $\leftarrow$ otherwise.

Let

$$u(k,x) = \begin{cases} \min \{q \in Q_k | x < q\} & \text{if } x < \max Q_k \\ \rightarrow & \text{otherwise.} \end{cases}$$

Let $R = \cup\{R_n | n \in N\}$ and $L = \cup\{L_n | n \in N\}$ be the decomposition of R and L respectively guaranteed by condition C. In order to show that X is a quasi-metric space, a quasi-metric function is constructed as follows:

(i) if $x \in I$, let $h(k,x) = \{x\}$ for each $k \in N$

(ii) if $x \in cl(R_1,Y) \cap R$ and $k \in N$ then there is a point x_k (possibly in Y^+) such that $]x,x_k[\cap (\cup\{cl(L_i,Y) | 1 \leq i \leq k\}) = \phi$. For each $k \in N$, let

$$h(k,x) = [x,x_k[\cap [x,u(k,x)[$$

(iii) if $x \in cl(L_1,Y) \cap L$ and $k \in N$, then there is a point x_k such that $]x_k,x[\cap (\cup\{cl(R_i,Y) | 1 \leq i \leq k\}) = \phi$. For each $k \in N$, let

$$h(k,x) =]x_k,x] \cap]\ell(k,x),x]$$

(iv) if $x \in E \cap cl(L_1,Y) \cap cl(R_1,Y)$ then there are maximal intervals $]x^-,x[$ and $]x,x^+[$ such that

$$]x^-,x[\cap R_1 = \phi =]x,x^+[\cap L_1$$

For each $k \in N$, let

$$h(k,x) =]x^-,x^+[\cap]\ell(k,x),u(k,x)[$$

Let $\{I(1,n) | n \in N\}$ be a pairwise disjoint collection of maximal intervals that are open in Y such that

$$\cup\{I(1,n) | n \in N\} = Y - cl(R_1 \cup L_1,Y).$$

(v) if $x \in (cl(R_1,Y) \cap E) - cl(L_1,Y)$ then there is an $n \in N$ such that $I(1,n) =]x',x[$ for some x' (possibly in Y^+). To see this notice that there is an $x_1 < x$ such that $]x_1,x[\cap cl(R_1,Y) = \phi$ and, since $x \notin cl(L_1,Y)$, there is an $x_2 < x$ such that $]x_2,x[\cap cl(L_1,Y) = \phi$. Thus if x' is the least element of Y^+ such that $]x',x[\cap (R_1 \cup L_1) = \phi$, then $]x',x[= I(1,n)$. If $k \in N$, then there is a maximal interval $]x,x_k[$ such that $]x,x_k[\cap (\cup\{cl(L_i,Y)\ 1 \le i \le k\}) = \phi$. Then, if $k \in N$, let

$$h(k,x) =]x',x_k[\cap]\ell(k,x),u(k,x)[.$$

(vi) In a fashion similar to (v), if $x \in (cl(L_1,Y) \cap E) - cl(R_1,Y)$ then there is an $n \in N$ such that $I(1,n) =]x,x'[$ and a maximal interval $]x_k,x[$ such that $]x_k,x[\cap (\cup\{cl(R_i,Y) | 1 \le i \le k\}) = \phi$. Then, if $k \in N$, let

$$h(k,x) =]x_k,x'[\cap]\ell(k,x),u(k,x)[.$$

(vii) if, for some $n \in N$, $I(1,n) =]x',x[$ and $x \in L \cap cl(R_1,Y)$, then, if $k \in N$, let

$$h(k,x) =]x',x] \cap]\ell(k,x),x].$$

(viii) if, for some $n \in N$, $I(1,n) =]x,x'[$ and $x \in R \cap cl(L_1,Y)$, then, if $k \in N$, let

$$h(k,x) = [x,x'[\cap [x,u(k,x)[.$$

At this stage $h(k,x)$ is defined for each $k \in N$ and each $x \notin \cup\{I(1,n)|n \in N\}$

(ix) if $x \in I(1,n)$ for some $n \in N$, let

$$h(1,x) = I(1,n) \cap]\ell(1,x),u(1,x)[.$$

Now, for each $x \in X$, and whatever values of K such that $h(k,x)$ is defined let

$$g(k,x) = X \cap h(k,x).$$

Inductively, suppose $g(k,x)$ is defined for all $k \in N$ and $x \notin \cup\{I(i,n)|n \in N\}$ and $g(1,x),\cdots,g(i,x)$ is defined of $x \in \cup\{I(i,n)|n \in N\}$. Let $x \in I(i,n)$ for some $n \in N$. If $x \in cl(R_{i+1},Y) \cap R$, $x \in cl(L_{i+1},Y) \cap L$, or $x \in E \cap cl(L_{i+1},Y) \cap cl(R_{i+1},Y)$ define $h(k,x)$ for $k \ge i + 1$ exactly as $h(k,x)$ was defined in (ii), (iii) and (iv) respectively with R_1

replaced by R_{i+1} and L_1 replaced by L_{i+1}. Then, for $k \geq i + 1$, let

$$g(k,x) = X \cap I(i,n) \cap h(k,x).$$

Let $\{I(i + 1,n) | n \in N\}$ be a pairwise disjoint collection of maximal intervals that are open in Y such that $\cup\{I(i+1,n) | n \in N\} = Y - cl(R_1 \cup \cdots \cup R_{i+1} \cup L_1 \cup \cdots L_{i+1},Y)$. If $x \in (cl(R_{i+1},Y) \cap E) - cl(L_{i+1},Y)$, $x \in (cl(L_{i+1},Y) \cap E) - cl(R_{i+1},Y)$, $x \in L \cap cl(R_{i+1},Y)$ and x a right endpoint of some $I(i+1,n)$, or $x \in R \cap cl(L_{i+1},Y)$ and x a left endpoint of some $I(i+1,n)$, define $h(k,x)$ for $k \geq i + 1$ exactly as $h(k,x)$ in (v), (vi), (vii) or (viii) respectively with R_1 replaced by R_{i+1} and L_1 replaced by L_{i+1}. If $k \geq i + 1$, let

$$g(k,x) = X \cap I(i,n) \cap h(k,x).$$

If $x \in I(i+1,n)$ for some $n \in N$, let

$$g(i+1,x) = X \cap I(i+1,x) \cap]\ell(i+1,x),u(i+1,x)[.$$

Thus, by induction $g(k,x)$ is defined for each $(k,x) \in N \times X$.

To show that g is indeed a quasi-metric function involves many cases. The following illustrates how the proof of these cases goes.

If $x \in R$ and n is the first natural number such that $x \in cl(R_n,Y) \cap R$, then, if $y \in g(k+1,x)$, $y \notin I$ and $k + 1 < n$ it follows that $g(k+1,y) = g(k+1,x) \subseteq g(k,x)$. If $y \in I$, then for all $k \in N$ it follows that $g(k+1,y) = \{y\} \subseteq g(k+1,x) \subseteq g(k,x)$. If $n \leq k + 1$, then several cases must be determined. Recall there exists $m \in N$ such that $x \in I(n-1,m)$. Thus if $y \in g(k+1,x)$ then $x < y$, $y \notin \cup\{cl(R_i \cup L_i,Y) | 1 \leq i \leq n - 1\}$ and $y \in I(n-1,m)$.

(i) if $y \in R$ and $y \notin cl(R_n,Y)$, then there is a $j \in N$

such that $y \in I(n,j) \subseteq [x,\rightarrow[$. It is clear that $u(k+1,y) = u(k+1,x) \leq u(k,x)$ and $y_{k+1} = x_{k+1} \leq x_k$. Thus $g(k+1,y) \subseteq g(k+1,x) \subseteq g(k,x)$

(ii) if $y \in cl(R_n,Y) \cap R$, then $u(k+1,y) = u(k+1,x)$ and $y_{k+1} = x_{k+1}$. Thus

$$g(k+1,y) = X \cap I(n-1,m) \cap [y,u(k+1,x)[\cap [y,x_{k-1}[$$
$$\subseteq g(k+1,x) \subseteq g(k,x).$$

(iii) if $y \in L$ then $y \notin \cup\{cl(L_i,Y) | 1 \leq i \leq k+1\}$. Thus, there exists $m \in N$ such that $y \in I(k+1,m) \subseteq [x,\rightarrow[$. Since $u(k+1,y) = u(k+1,x) \leq u(k,x)$ it follows that $g(k+1,y) \subseteq g(k,x)$.

(iv) if $y \in (E \cap cl(L_n,Y)) - cl(R_n,Y)$ then $y \notin g(k+1,x)$. This follows since y is the left endpoint of $I(n,m)$ for some $m \in N$ and there is an interval $]y_n,y[$ such that $]y_n,y[\cap cl(R_n,Y) = \phi$. But there is a $t \in L_n$ such that $x < t < y$. Thus $t \in g(k+1,x)$ but $g(k+1,x) \cap (\cup\{cl(L_i,Y) | 1 \leq i \leq n\}) = \phi$. Thus $y \notin g(k+1,x)$

(v) if $y \in E$ and $y \notin cl(R_n,Y) \cup cl(L_n,Y)$, then $y \in I(n,m)$ for some $m \in N$. Thus, since $[x,\rightarrow[\supseteq I(n,m)$ and $u(k+1,y) = u(k+1,x) \leq u(k,x)$, it is clear that $g(k+1,y) \subseteq g(k,x)$.

(vi) if $y \in (E \cap cl(R_n,Y)) - cl(L_n,Y)$, then y is the right endpoint of $I(n,m)$ for some $m \in N$. Since $y < x_{k+1} \leq x_k$, $u(k+1,y) = u(k+1,x) \leq u(k,x)$ and $[x,\rightarrow[\supseteq I(n,m)$ it follows that $g(k+1,y) \subseteq g(k,x)$.

(vii) if $y \in cl(R_n,Y) \cap cl(L_n,Y)$ then $y \notin g(k+1,x)$ (argue as in (iv)).

The case where $x \in L$ and n is the first natural number such that $x \in cl(L_n,Y)$ follows in a symmetric fashion.

The arguments when $x \in E$ also follow in a similar manner.

Thus g is a quasi-metric function and the theorem is proved.

It is easily seen from (3.1) that the Michael Line and the Sorgenfrey Line are both quasi-metrizable.

(3.2) *Corollary. Let* $(Y,\leq)$ *be a separable* LOTS *and* X *a* GO-*space constructed from* Y. *If* X *is a quasi-metric space, then both* R *and* L *are* F_σ*-subsets in* $R \cup L$ *considered as a subspace of* X.

Proof. Parts (a) through (h) of (i) $\rightarrow$ (ii) of (3.1).

In [EL] an example is given a c-semi-stratifiable GO-space $X = GO_{E^1}(E^1 - Q,\phi,\phi,Q)$. It is an easy consequence of (3.2) that X is not a γ-space.

(3.3) *Remark*. A slight modification of the above proof will show that if X is a generalized ordered space constructed from a LOTS $(Y,\leq)$ where Y has a σ-discrete dense subset, then X is a γ-space if and only if X is quasi-metrizable.

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