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REFINEMENTS OF LOCALLY COUNTABLE COLLECTIONS

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Several questions concerning spaces with a σ-locally countable base and paralindelöf spaces have proved to be surprisingly difficult. It is not known, for example, whether paralindelöf spaces must be paracompact or whether spaces with a σ-locally countable base must be screenable. Recent results appearing in [FR], and examples in [DGN] and [F], have contributed significantly to this area but many fundamental problems remain. Part of the reason for this seems to be that, in contrast with locally finite collections, there are only a small number of suitable techniques available for handling or refining locally countable collections. In this note, we give a result which allows for σ-closure preserving refinements of locally countable collections under certain conditions. By applying this theorem we obtain several new results, including the result that all regular θ-refinable spaces with a σ-locally countable base are developable.

For convenience all regular spaces are assumed to be T_1 but, unless otherwise stated, no separation axioms are assumed. The set of natural numbers is denoted by N. We begin immediately with the statement and proof of the main theorem; applications of this result and relationships to known results will be discussed later.

1. *Theorem. If $\mathcal{P}$ is a collection of closed subsets of* X *and $\mathcal{K}$ is a point-finite open cover of* X *such that each* $K \in \mathcal{K}$ *intersects at most countably many elements of $\mathcal{P}$ then $\mathcal{P}$ has a σ-closure preserving refinement.*

Proof. Assume $\mathcal{K} = \{K(\alpha): \alpha \in \Lambda\}$ where Λ is well-ordered and $K(\alpha) \neq K(\beta)$ if $\alpha \neq \beta$. For each $\alpha \in \Lambda$ the set $\mathcal{H}(\alpha) = \{P \in \mathcal{P}: P \cap K(\alpha) \neq \emptyset\}$ is countable, so express as

$$\mathcal{H}(\alpha) = \{P(1,\alpha), P(2,\alpha), \cdots\}.$$

(Make necessary adjustments in notation if $\mathcal{H}(\alpha)$ is finite or empty.) For each $n \in N$ let $F_n = \{x \in X: \text{ord}\,(x,\mathcal{K}) \leq n\}$. For each finite sequence $(i_1, i_2, \cdots, i_n)$ of natural numbers and each $\beta \in \Lambda$, let

$A(i_1,\cdots,i_n,\beta) = \{(\alpha_1,\cdots,\alpha_n) \in \Lambda^n: \alpha_1 < \alpha_2 < \cdots < \alpha_n = \beta$ and $P(i_1,\alpha_1) = P(i_2,\alpha_2) = \cdots = P(i_n,\alpha_n)\}$.

For each sequence $(i_1,\cdots,i_n) \in N^n$, we will define a closure preserving collection $\mathcal{D}(i_1,\cdots,i_n)$ - this will be done by induction on n.

Let $i \in N$ (a sequence of length 1), and $\beta \in \Lambda$. Define

$$D(i,\beta) = F_1 \cap P(i,\beta) \cap K(\beta), \text{ and}$$

$$\mathcal{D}(i) = \{D(i,\beta): \beta \in \Lambda\}.$$

Then $\mathcal{D}(i)$ is a closure preserving collection (in fact, $\mathcal{D}(i)$ is actually discrete). Now let $n \in N$, $n > 1$ and assume that for any sequence $(j_1,\cdots,j_k) \in N^k$, with $1 \leq k < n$, that $\mathcal{D}(j_1,\cdots,j_k)$ is defined and is a closure preserving collection of subsets of F_k. For any $(i_1,\cdots,i_n) \in N^n$, $\beta \in \Lambda$, define $E(i_1,\cdots,i_n,\beta) = \cup\{F_n \cap P(i_n,\beta) \cap K(\alpha_1) \cap \cdots \cap K(\alpha_n):$ $(\alpha_1,\cdots,\alpha_n) \in A(i_1,\cdots,i_n,\beta)\}$, $H(i_1,\cdots,i_n,\beta) = \cup\{D(i_{j1},\cdots,$ $i_{jk},\alpha_{jk}): (i_{j1},\cdots,i_{jk})$ is a subsequence of $(i_1,\cdots,i_n)$,

$1 \leq k < n$ and $(\alpha_1,\cdots,\alpha_n) \in A(i_1,\cdots,i_n,\beta)\}$, $D(i_1,\cdots,i_n,\beta) = E(i_1,\cdots,i_n,\beta) \cup H(i_1,\cdots,i_n,\beta)$, and $\mathcal{D}(i_1,\cdots,i_n) = \{D(i_1,\cdots,i_n,\beta)\colon \beta \in \Lambda\}$.

To show $\mathcal{D}(i_1,\cdots,i_n)$ is closure preserving let $\Lambda' \subseteq \Lambda$ and suppose $x \in \mathrm{cl}(\cup\{D(i_1,\cdots,i_n,\beta)\colon \beta \in \Lambda'\})$. If $x \in F_n - F_{n-1}$, then there exists $(\gamma_1,\gamma_2,\cdots,\gamma_n) \in \Lambda^n$, with $\gamma_1 < \cdots < \gamma_n$ such that

$$x \in W = K(\gamma_1) \cap K(\gamma_2) \cap \cdots \cap K(\gamma_n).$$

Then $W \cap D(i_1,\cdots,i_n,\beta) \neq \emptyset$, for some $\beta \in \Lambda'$, implies $W \cap E(i_1,\cdots,i_n,\beta) \neq \emptyset$ (since $F_n \cap W \subset F_n - F_{n-1}$) which implies $(\gamma_1,\cdots,\gamma_n) \in A(i_1,\cdots,i_n,\beta)$ (so $\gamma_n = \beta$). This says there is only one $\beta \in \Lambda'$ such that $W \cap D(i_1,\cdots,i_n,\beta) \neq \emptyset$; it follows that $x \in \mathrm{cl}(D(i_1,\cdots,i_n,\beta))$, for $\beta = \gamma_n$ and $\gamma_n \in \Lambda'$. Now suppose $\mathrm{ord}\,(x,\mathcal{K}) = k$, for $1 \leq k < n$; then there exists $(\gamma_1,\cdots,\gamma_k) \in \Lambda^k$, with $\gamma_1 < \cdots < \gamma_k$ such that $x \in V = K(\gamma_1) \cap \ldots \cap K(\gamma_k)$. If $x \in \mathrm{cl}(\cup\{H(i_1,\cdots,i_n,\beta)\colon \beta \in \Lambda'\})$, then $x \in \mathrm{cl}(D(i_{j1},\cdots,i_{jr},\alpha_{jr})) \subset \mathrm{cl}(D(i_1,\cdots,i_n,\beta))$ for some subsequence $(i_{j1},\cdots,i_{jr})$ of $(i_1,\cdots,i_n)$ with $(\alpha_1,\cdots,\alpha_n) \in A(i_1,\cdots,i_n,\beta)$, since $\{D(i_{j1},\cdots,i_{jr},\alpha_{jr})\colon \beta \in \Lambda', (i_{j1},\cdots,i_{jr})$ is a subsequence of $(i_1,\cdots,i_n)$,

$$1 \leq r < n \text{ and } (\alpha_1,\cdots,\alpha_n) \in A(i_1,\cdots,i_n,\beta)\}$$

is closure preserving. Otherwise we have $x \in \mathrm{cl}(\cup\{E(i_1,\cdots,i_n,\beta)\colon \beta \in \Lambda'\})$. Now note that $V \cap E(i_1,\cdots,i_n,\beta) \neq \emptyset$, for some $\beta \in \Lambda'$, implies there is $(\alpha_1,\cdots,\alpha_n) \in A(i_1,\cdots,i_n,\beta)$ and a subsequence $(i_{j1},\cdots,i_{jk})$ of $(i_1,\cdots,i_n)$ such that $\gamma_1 = \alpha_{j1}, \gamma_2 = \alpha_{j2},\cdots,\gamma_k = \alpha_{jk} \leq \beta$. For every subsequence $(i_{j1},\cdots,i_{jk})$ (of length k) of $(i_1,\cdots,i_n)$ let $\Lambda(i_{j1},\cdots,i_{jk}) = \{\beta \in \Lambda'\colon$ there is

$(\alpha_1,\cdots,\alpha_n) \in A(i_1,\cdots,i_n,\beta)$ such that

$$\gamma_1 = \alpha_{j1},\cdots,\gamma_k = \alpha_{jk}\}.$$

Now, since there are only a finite number of such subsequences, there is some subsequence $(i_{j1},\cdots,i_{jk})$ such that $x \in cl(\cup\{E(i_1,\cdots,i_n,\beta): \beta \in \Lambda(i_{j1},\cdots,i_{jk})\})$. For each $\beta \in \Lambda(i_{j1},\cdots,i_{jk})$ we have $E(i_1,\cdots,i_n,\beta) \subset P(i_n,\beta) = P(i_{jk},\gamma_k)$. So $x \in P(i_{jk},\gamma_k)$ (since $P(i_{jk},\gamma_k)$ is closed) and $x \in F_k \cap K(\gamma_1) \cap \cdots \cap K(\gamma_k)$; hence $x \in E(i_{j1},\cdots,i_{jk},\gamma_k) \subset D(i_{j1},\cdots,i_{jk},\gamma_k) \subset D(i_1,\cdots,i_n,\beta)$ for any $\beta \in \Lambda(i_{j1},\cdots,i_{jk})$. This shows $\mathcal{D}(i_1,\cdots,i_n)$ is closure preserving and $\mathcal{D} = \cup\{\mathcal{D}(i_1,\cdots,i_n): n \in N, (i_1,\cdots,i_n) \in N^n\}$ is σ-closure preserving.

If $D(i_1,\cdots,i_n,\beta) \in \mathcal{D}$, it follows by construction of $D(i_1,\cdots,i_n,\beta)$ that

$$D(i_1,\cdots,i_n,\beta) \subset P(i_n,\beta) \in \mathcal{P}.$$

To complete the proof we need to show that $\mathcal{D}$ covers $\cup\mathcal{P}$. Let $x \in \cup\mathcal{P}$ and suppose ord $(x,\mathcal{K}) = n$. There exist elements $K(\alpha_1),\cdots,K(\alpha_n)$ of $\mathcal{K}$ such that $x \in K(\alpha_1) \cap \cdots \cap K(\alpha_n)$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. For each j, $1 \leq j \leq n$, there is $i_j \in N$ so that $x \in P(i_j,\alpha_j) \in \mathcal{H}(\alpha_j)$ and $P(i_1,\alpha_1) = P(i_2,\alpha_2) = \cdots = P(i_n,\alpha_n)$. It follows that $x \in D(i_1,\cdots,i_n,\alpha_n) \in \mathcal{D}$ and the theorem is proved.

A direct application of Theorem 1 shows that in a metacompact space X any locally countable collection of closed sets has a σ-closure preserving refinement. A little more work gives a sharpened version of this in θ-refinable spaces. Recall that a space X is θ-*refinable* [WoW] if for any open cover $\mathcal{U}$ of X there is a sequence $\{\mathcal{G}_n\}_1^\infty$ of open covers of X, each refining $\mathcal{U}$, such that for any $x \in X$ there is $n \in N$ where $0 < \text{ord}\,(x, \mathcal{G}_n) < \omega$. The sequence $\{\mathcal{G}_n\}_1^\infty$ is called a θ-refinement of $\mathcal{U}$. If, in the above definition, the collections $\mathcal{G}_n$ are not required to cover X, then X is said to be *weakly* θ*-refinable* [BL].

2. *Corollary. In a* θ*-refinable space* X *any locally countable collection of closed subsets has a* σ*-closure preserving refinement. Hence every* σ*-locally countable closed collection has a* σ*-closure preserving (closed) refinement.*

Proof. Suppose $\mathcal{P}$ is a locally countable collection of closed subsets of X. There is an open cover $\mathcal{U}$ of X such that each $U \in \mathcal{U}$ intersects at most countably many elements of $\mathcal{P}$. Let $\{\mathcal{G}_n\}_1^\infty$ be a θ-refinement of $\mathcal{U}$. For each $n,k \in N$, let

$$Y_{n,k} = \{x \in X: \text{ord}\,(x,\mathcal{G}_n) \leq k\},$$
$$\mathcal{K}_{n,k} = \{G \cap Y_{n,k}: G \in \mathcal{G}_n\}, \text{ and}$$
$$\mathcal{P}_{n,k} = \{P \cap Y_{n,k}: P \in \mathcal{P}\}.$$

By applying Theorem 1 to the space $Y_{n,k}$ it follows that $\mathcal{P}_{n,k}$ has a σ-closure preserving refinement $\mathcal{D}_{n,k}$ (relative to $Y_{n,k}$), and since $Y_{n,k}$ is closed in X it follows that $\mathcal{D} = \cup\{\mathcal{D}_{n,k}: n,k \in N\}$ is a σ-closure preserving refinement of $\mathcal{P}$. That completes the proof.

It is expected that some sort of covering property (such as θ-refinable) would be necessary in Corollary 2. This is illustrated by Example 3 and Example 4 below. Example 3 is very simple and shows that locally countable covers need not have any "nice refinements". Example 4, due to G. Gruenhage, is described in [DGN] and shows that the θ-refinable condition cannot be weakened to weakly θ-refinable in Corollary 2 (and Corollaries 5, 6, and 7 below).

3. *Example. There is a completely regular space* X *with a locally countable cover* $\mathcal{U}$ *of open and closed sets such that* $\mathcal{U}$ *does not have a* σ*-closure preserving refinement.*

Proof. Let $X = \{(\alpha,\beta) \in \omega_1 \times \omega_1 : \alpha < \beta\}$ with the relative topology inherited from $\omega_1 \times \omega_1$. For each $\alpha \in \omega_1$, let

$$U_\alpha = [0,\alpha] \times (\alpha,\omega_1).$$

Then each U_α is open and closed, and the collection $\mathcal{U} = \{U: \alpha < \omega_1\}$ is a locally countable cover of X with no σ-closure preserving refinement. The details are left to the reader.

4. *Example.* There is an example of a completely regular nondevelopable space Z with a σ-locally countable base. This space is screenable (hence weakly θ-refinable) but not θ-refinable.

See Example 3.3 in [DGN] for the details. The following properties (with some repetition) of Z are easily verified.

(a) Z is weakly θ-refinable and every open cover of Z has a σ-locally countable refinement but Z is not subparacompact.

(b) Z is weakly σ-refinable and has a σ-locally countable network but Z is not a σ-space.

(c) Z is weakly θ-refinable and has a σ-locally countable base but Z is not developable.

The above statements (a), (b), and (c) should be contrasted with Corollaries 5, 6, and 7 below in order to see that these results cannot be significantly improved by weakening the θ-refinable condition. We view Corollary 7 as the main result in this group; this extends results given by Fleissner and Reed in [FR] where it was shown that a regular space X, with a σ-locally countable base, is developable if X is subparacompact or if (under Martin's Axiom) X is metacompact with $|X| < c$.

5. *Corollary. If* X *is a regular* θ*-refinable space in which every open cover has a* σ*-locally countable refinement then* X *is subparacompact.*

Proof. Using regularity we see that every open cover of X has a σ-locally countable closed refinement and by Corollary 2 it follows that every open cover of X has a σ-closure preserving closed refinement. Theorem 1.2 in [Bu] shows that X is subparacompact.

6. *Corollary. A regular* θ*-refinable space* X *with a* σ*-locally countable network is a* σ*-space.*

Proof. It follows from the proofs of Theorem 1 and Corollary 5 that a σ-locally countable closed network for X can be replaced by a σ-closure preserving closed network. According to [NS] this is equivalent to X being a σ-space.

7. *Corollary. A regular* θ*-refinable space* X *with a* σ*-locally countable base is a Moore space.*

Proof. Corollary 5 shows that X is subparacompact; Fleissner and Reed [FR] have shown that a regular subparacompact space with a σ-locally countable base is a Moore space.

Corollary 7 can be viewed as a companion to Fedorčuk's result [Fe] that a paracompact space with a σ-locally countable base is metrizable. C. Aull has also given results [A] where covering properties convert to corresponding base properties in a space with a σ-locally countable base.

A space X is said to be paralindelöf if every open cover of X has a locally countable open refinement. Notice that Corollary 5 says that a regular θ-refinable paralindelöf space X must be subparacompact. It is not known whether paralindelöf spaces must always be subparacompact.

We conclude with a simple result related to the question of whether a space with a σ-locally countable base is screenable (or equivalently has a σ-disjoint base). This proof indicates that if a space X, with a σ-locally countable base $\mathcal{B}$, has a σ-disjoint base then a σ-disjoint base for X can be found by using only unions of elements of $\mathcal{B}$ (as opposed to intersections, differences, etc.).

8. *Proposition. If a space* X *has a* σ*-locally countable base* $\mathcal{B}$ *and a* σ*-disjoint base* $\mathcal{D}$ *then* X *has a base* $\mathcal{G}$ *which is simultaneously* σ*-locally countable and* σ*-disjoint.*

Proof. Suppose $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ where each $\mathcal{B}_n$ is locally countable and $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n$ where each $\mathcal{D}_n$ is a disjoint collection. For any $n,k \in N$ and $D \in \mathcal{D}_n$, let

$$G(D,n,k) = \bigcup\{B: B \in \mathcal{B}_k,\ B \subseteq D\} \text{ and}$$

$$\mathcal{G}(n,k) = \{G(D,n,k): D \in \mathcal{D}_n\}.$$

It is easily shown that each $\mathcal{G}(n,k)$ is simultaneously a locally countable and disjoint collection, and $\mathcal{G} = \bigcup\{\mathcal{G}(n,k): n,k \in N\}$ is a base for X.

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