TOPOLOGY PROCEEDINGS

Volume 4, 1979

Pages 29–49

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN:	0146-4124

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1. Introduction

Let X be a metric space and let d denote the distance-function defined on X. By 2^X we denote the hyperspace of all non-empty compacta lying in X. In [4] Borsuk defined the fundamental metric d_F on 2^X such that two compacta in X which are close with respect to d_F have similar shape properties. In particular, it was demonstrated in [4] and [8] that there is a large number of hereditary shape properties α (i.e., properties preserved by shape dominations) such that the following holds.

(1.1) If $A_0, A_1, \cdots$ is a sequence in 2^X with $\lim d_F(A_k, A_0) = 0$ and $A_k \in \alpha$ for $k = 1,2,\cdots$, then $A_0 \in \alpha$.

On the other hand, Boxer and Sher [5] observed that in general (1.1) is not true for every hereditary shape property α. But, by assuming that A_0 is an FANR, they established (1.1) for every α. Finally, in [8] the same was proved under a weaker assumption that A_0 is a calm compactum. The class of calm compacta was introduced by the author in [6]. It intersects the class of all movable compacta in the class of all FANR's (see Theorem (4.5) in [8]) but calm compacta need not be movable (e.g. solenoids).

In view of the above results, it is interesting to find conditions under which $\lim d_F(A_k, A_0) = 0$ for a sequence $A_0, A_1, \cdots$ in 2^X implies that A_0 is calm.

This paper introduces the notion of a calmly regular (or ca-regular) convergence of compacta lying in X such that if the sequence $A_1, A_2, \cdots$ converges ca-regularly to A_0, then A_0 is calm and $\lim d_F(A_k, A_0) = 0$.

The definition of the ca-regular convergence, besides providing generalizations of theorems about calm compacta from [6] (see §§2 and 3), is justified by the new information that it gives about the collection ca(X) of all calm compacta in a metric space X. The main result (4.6) of this paper shows that one can define a metric d_{ca} on ca(X) such that the convergence with respect to d_{ca} is equivalent to the ca-regular convergence.

The paper is organized as follows. In §2 we recall definitions of $\mathcal{C}$-calmness from [6], where $\mathcal{C}$ is an arbitrary (non-empty) class of topological spaces, and the fundamental metric d_F from [4] and introduce the notion of a $\mathcal{C}$-calmly regular (or $ca_{\mathcal{C}}$-regular) convergence of compacta in a metric space X which lies in an ANR space M. The ca-regular convergence is by definition $ca_{\mathcal{C}}$-regular convergence for $\mathcal{C}$ the class of all finite polyhedra. We first prove (see (2.3)) that in this definition the choice of the embedding of X and of M is immaterial. In (2.4) we prove that if a sequence $\{A_k\}$ in 2^X converges $ca_{\mathcal{C}}$-regularly to a compactum A_0 (in symbols, $A_k \xrightarrow{ca_{\mathcal{C}}} A_0$), then A_0 is $\mathcal{C}$-calm regardless of the nature of A_k's. Then we discuss the role of the class $\mathcal{C}$ ((2.7), (2.8), and (2.9)) and give several examples of situations in which $ca_{\mathcal{C}}$-regular convergence naturally appears ((2.10), (2.12), and (2.14)).

The short §3 shows that taking finite products and suspensions preserves $ca_{\mathcal{C}}$-regular convergence ((3.1) and (3.2), respectively). Also, $A_k \text{—} ca_{\mathcal{C}} \to A_0$ iff for every component C_0 of A_0 there is a component C_k of A_k such that $C_k \text{—} ca_{\mathcal{C}} \to C_0$ ((3.4)).

The final §4 proves, relying heavily on Begle's technique in [1], that the hyperspace $ca_{\mathcal{C}}(X)$ of all $\mathcal{C}$-calm compacta in X with the topology induced by $ca_{\mathcal{C}}$-regular convergence is a metric space. We close by raising two questions about the topological properties of $ca_{\mathcal{C}}(X)$.

This paper is the second in a series in which we study various types of globally regular convergence. In the first [7] we considered $\mathcal{C}$-movably regular convergence.

We assume that the reader is familiar with the theory of shape [2].

Throughout the paper $\mathcal{C}$ and $\mathcal{D}$ will be arbitrary (non-empty) classes of topological spaces. By $\mathcal{P}$ we denote the class of all finite polyhedra.

If not stated otherwise, we reserve X for an arbitrary metric space with a fixed metric d; $A_0, A_1, A_2, \cdots$ are compact subsets of X; M is an absolute neighborhood retract for the class of all metric spaces (in notation, an ANR) which contains X; a neighborhood means an open neighborhood; and $N_\varepsilon(A_0)$ denotes the ε-neighborhood of A_0 in M.

2. $\mathcal{C}$-Calmly Regular Convergence

Let B be a subset of an ANR M, and let V be an open subset of M containing B. We denote by $\mathcal{C}_h(V;B)$ the following statement.

$\boxed{\mathcal{C}_h(V;B)}$ For every neighborhood W of B in M there is a neighborhood W_0, $W_0 \subset V \cap W$, of B in M such that if f,g: $K \to W_0$ are maps of a member K of $\mathcal{C}$ into W_0 which are homotopic in V, then f and g are homotopic in W.

A compactum A is *$\mathcal{C}$-calm* if for some (and hence for every) embedding of A into an ANR M the following holds. There is a neighborhood V of A in M such that $\mathcal{C}_h(V;A)$ is true. It is easy to see that this definition is equivalent to the one given in [6] (see Theorem (4.2) in [6]). $\mathcal{P}$-calm compacta are called *calm*.

Recall [4] (see also [8]) the definition of the fundamental metric d_F on 2^X. Let M be an AR-space containing X metrically.

(2.1) If $A,B \in 2^X$, then $d_F(A,B)$ is the greatest lower bound of those $\varepsilon > 0$ for which there exist ε-fundamental sequences $\underline{f} = \{f_k,A,B\}_{M,M}$ and $\underline{g} = \{g_k,B,A\}_{M,M}$. (By an *$\varepsilon$-fundamental sequence* we mean a fundamental sequence $\underline{f} = \{f_k,A,B\}_{M,M}$ for which there exists a neighborhood U of A in M such that $f_k|U$ is an ε-map (i.e., $d(f_k(x),x) < \varepsilon$ for every $x \in U$) for almost all indices k). It is known [4, Theorem 3.1] that the choice of M is irrelevant when computing $d_F(A,B)$. The hyperspace 2^X with the metric d_F is denoted by 2^X_F.

(2.2) *Definition.* A sequence $\{A_1,A_2,\cdots\}$ of compacta in a metric space X which lies in an ANR M is said to *converge $\mathcal{C}$-calmly regularly* (or ca$_{\mathcal{C}}$*-regularly*) *in* M to a compactum $A \subset X$ provided

(i) $\lim d_F(A_n,A) = 0$, and

(ii) there is a neighborhood V of A in M such that $\mathcal{C}_h(V;A_n)$ holds for almost all indices n.

We first prove that the definition (2.2) is shape theoretic in the sense that $\mathcal{C}$-calmly regular convergence is independent of the choice of M and the embedding of X into M.

(2.3) *Proposition. Let* X *be embedded in* ANR*'s* M *and* M' *and let a sequence* $\{A_n\}$ *of compacta in* X *converge* $\mathcal{C}$*-calmly regularly in* M *to a compactum* $A_0 \subset X$. *Then the convergence* $A_n \to A_0$ *is also* $\mathcal{C}$*-calmly regular in* M'.

Proof. Since $A = \cup_{i=0}^{\infty} A_i$ is compact, there are neighborhoods Z and Z' of A in M and M', respectively, and maps $h: Z \to M'$ and $h': Z' \to M$ such that $h|A = h'|A = id$.

Let V be a neighborhood of A_0 in M and let k_V be an index such that $\mathcal{C}_h(V;A_n)$ holds for each $n \geqq k_V$. Put $V' = (h')^{-1}(V)$ and let $k_{V'} \geqq k_V$ be such that $n \geqq k_{V'}$ implies $A_n \subset V'$.

We claim that $\mathcal{C}_h(V';A_n)$ is true for each $n \geqq k_{V'}$. Indeed, consider an arbitrary neighborhood W' of A_n in M', where $n \geqq k_{V'}$. Let $W = h^{-1}(W')$. Since $n \geqq k_V$ we conclude that there is a neighborhood W_0 of A_n in M, $W_0 \subset V \cap W$, such that if two maps of a member of $\mathcal{C}$ into W_0 are homotopic in V then they are already homotopic in W. Let $W_0^* = (h')^{-1}(W_0)$ and let W_0', $W_0' \subset W_0^* \cap W'$, be a neighborhood of A_n in M' with the property that $h \circ h'|W_0' \simeq i_{W_0',W'}$ in W' ($i_{W_0',W'}$ denotes the inclusion of W_0' into W' and "$\simeq$" stands for "is homotopic to").

Now, if f, g: $K \to W_0'$ are maps of $K \in \mathcal{C}$ into W_0' homotopic in V', then $h' \circ f$ and $h' \circ g$ are maps of K into W_0 homotopic in V. It follows that $h' \circ f \simeq h' \circ g$ in W. But then $h \circ h' \circ f \simeq h \circ h' \circ g$ in W'. The choice of W_0' implies that $h \circ h' \circ f \simeq f$ and

$h \circ h' \circ g \simeq g$ in W'. Hence $f \simeq g$ in W'.

If a sequence $\{A_n\}$ of compacta in X converges ca$\mathcal{C}$-regularly to a compactum A_0 in X in some (and hence, by (2.3), in every) ANR containing X we shall write A_n—ca$\mathcal{C}\to A_0$.

(2.4) *Proposition. Let* A_n—ca$\mathcal{C}\to A_0$. *Then* A_0 *is* $\mathcal{C}$*-calm regardless of the nature of* A_n*'s.*

Proof. We can assume that all compacta under consideration lie in the Hilbert cube Q. Select a neighborhood V of A_0 in Q and an index $k_1 = k_V$ such that $\mathcal{C}_h(V;A_n)$ holds for each $n \geqq k_1$. We shall prove that $\mathcal{C}_h(V;A_0)$ is also true.

Let W be an arbitrary neighborhood of A_0 in Q. Let W', $W' \subset W$, be a compact ANR neighborhood of A_0. Pick $k_2 \geqq k_1$ such that $n \geqq k_2$ implies $A_n \subset \operatorname{int} W'$. Let $\varepsilon > 0$ has the property that ε-close maps into W' are homotopic in W' [10]. Take $k_3 \geqq k_2$ such that $d_F(A_{k_3},A_0) < \varepsilon$. By assumption, there is a neighborhood W_0' of A_{k_3}, $W_0' \subset V \cap W'$, such that any two maps of a member of $\mathcal{C}$ into W_0' homotopic in V are already homotopic in W'.

Now, take an ε-fundamental sequence $\underline{f} = \{f_k, A_0, A_{k_3}\}_{Q,Q}$. We can find an index n_0 and a neighborhood W_0 of A_0 in Q, $W_0 \subset V \cap W'$, such that $d(f_{n_0}(x),x) < \varepsilon$ for each $x \in W_0$ and $f_{n_0}(W_0) \subset W_0'$.

Consider maps $f, g\colon K \to W_0$ of $K \in \mathcal{C}$ into W_0 that are homotopic in V. Then $f_{n_0} \circ f \simeq f$ in W' (because these are ε-close maps into W') and, similarly, $f_{n_0} \circ g \simeq g$ in W'. It follows that $f_{n_0} \circ f$ and $f_{n_0} \circ g$ are maps of K into W_0' homotopic in V. By the choice of W_0' and k_3, we see that they are homotopic in W'. But then f and g are homotopic in W' and therefore also in W.

Let $/C_A/$ denote the number of components of a compactum A.

(2.5) *Proposition. Let* $A_n \text{—ca}_{\mathcal{C}} \rightarrow A_0$. *Then* A_0 *has finitely many components and* $/C_{A_0}/ = /C_{A_n}/$ *for almost all indices* n.

Proof. The first claim follows from (2.4) and Theorem (4.6) in [6]. The second is proved by the method used in the above proof.

(2.6) *Example*. (a) A constant sequence $\{A_n\} = \{A\}$ converges ca$_{\mathcal{C}}$-regularly to A iff A is $\mathcal{C}$-calm.

(b) In the interval $X = [-1,1]$ consider the sequence $A_n = \{-(1/n)\} \cup \{1/n\} (n = 1,2,\cdots)$ of compacta. Then $\lim d_F(A_n,A_0) = 0$, where $A_0 = \{0\}$ is $\mathcal{C}$-calm for every class $\mathcal{C}$, but $\{A_n\}$ does not converge ca$_{\mathcal{C}}$-regularly to A_0. Hence, the converse of (2.4) is not true.

Now we shall discuss the role of a class $\mathcal{C}$ in our definition. With obvious changes the proof of Theorem (4.8) in [6] gives the following.

(2.7) *Theorem. Let* $A_n \text{—ca}_{\mathcal{C}} \rightarrow A_0$ *and let a class* $\mathcal{C}$ *shape dominate a class* $\mathcal{D}$. *Then* $A_n \text{—ca}_{\mathcal{D}} \rightarrow A_0$.

Here a class of topological spaces $\mathcal{C}$ *shape dominates* another such class $\mathcal{D}$ provided for every $X \in \mathcal{D}$ there is $Y \in \mathcal{C}$ such that Y shape dominates X. We use the shape theory of arbitrary topological spaces in the form described by Kozlowski [12] (see also §3 in [13]).

Similarly, when $\mathcal{C}$ and $\mathcal{D}$ are classes of compacta, by replacing in the above definition shape domination with

Borsuk's notion of quasi-domination [3], we say that $\mathcal{C}$ *quasi-dominates* $\mathcal{D}$. With this concept we can improve (2.7) (and by (2.6)(a) also (4.8) in [6]) for classes of compacta.

(2.8) *Theorem. Let* $A_n \text{—ca}_{\mathcal{C}} \to A_0$ *and let a class of compacta* $\mathcal{C}$ *quasi-dominate another such class* $\mathcal{D}$. *Then* $A_n \text{—ca}_{\mathcal{D}} \to A_0$.

Proof. Let V be a neighborhood of A_0 in M and let k_V be an index such that $\mathcal{C}_h(V;A_n)$ holds for each $n \geqq k_V$. We claim that for such indices $\mathcal{D}_h(V;A_n)$ also holds.

Indeed, let $n \geqq k_V$ and let W be an arbitrary neighborhood of A_n in M. Select a neighborhood W_0 of A_n in M, $W_0 \subset V \cap W$, using $\mathcal{C}_h(V;A_n)$. Consider $K \in \mathcal{D}$ and maps f, g: $K \to W_0$ homotopic via a homotopy H_t: $K \to V$ $(0 \leqq t \leqq 1)$. Since the class $\mathcal{C}$ quasi-dominates the class D, there is a compactum $L \in \mathcal{C}$ which quasi-dominates K. We can assume that both K and L lie in the Hilbert cube Q. Since V and W_0 are ANR's, there is a neighborhood Z of K in Q and extensions $\hat{f}$, $\hat{g}$: $Z \to W_0$ of f and g, respectively, and $\hat{H}_t$: $Z \to V$ of H_t such that $\hat{H}_0 = \hat{f}$ and $\hat{H}_1 = \hat{g}$ [10]. Now, we select a neighborhood Z_0 of K in Q, $Z_0 \subset Z$, fundamental sequences $\underline{f} = \{f_k, L, K\}_{Q,Q}$ and $\underline{g} = \{g_k, K, L\}_{Q,Q}$, and an index k_1 such that $k \geqq k_1$ implies $f_k \circ g_k | Z_0 \simeq i_{Z_0,Z}$ in Z. Next, we pick $k_2 \geqq k_1$ and a neighborhood T of L in Q such that $f_k | T \simeq f_{k'} | T$ in Z whenever k, $k' \geqq k_2$.

Note that $\hat{H}_t \circ f_{k_2} | L$ is a homotopy of V connecting maps $\hat{f} \circ f_{k_2} | L$ and $\hat{g} \circ f_{k_2} | L$ of L into W_0. By assumption, these maps are homotopic in W. As before, we conclude that there is a neighborhood T_1 of L in Q, $T_1 \subset T$, and a homotopy $\hat{G}_t$: $T_1 \to W$

$(0 \leqq t \leqq 1)$ such that $\hat{G}_0 = \hat{f} \circ f_{k_2}|T_1$ and $\hat{G}_1 = \hat{g} \circ f_{k_2}|T_1$.

Select $k_3 \geqq k_2$ and a neighborhood T_0 of K in Q, $T_0 \subset Z_0$, such that $g_{k_3}(T_0) \subset T_1$. Then $G_t = \hat{G}_t \circ g_{k_3}|K$ is a homotopy in W connecting $G_0 = \hat{f} \circ f_{k_2} \circ g_{k_3}|K$ and $G_1 = \hat{g} \circ f_{k_2} \circ g_{k_3}|K$. We leave to the reader to check that G_0 is in W homotopic to f and that G_1 is in W homotopic to g. Hence f and g are homotopic in W.

(2.9) *Proposition. Let* $A_n \text{—ca}_{\mathcal{D}} \to A_0$. *If each compactum* A_n $(n = 1,2,\cdots)$ *is* $(\mathcal{C},\mathcal{D})$*-smooth* [6], *then* $A_n \text{—ca}_{\mathcal{C}} \to A_0$.

Proof. Let a neighborhood V of A_0 in M and an index k_V be chosen using the fact that $A_n \text{—ca}_{\mathcal{D}} \to A_0$. We claim that $\mathcal{C}_h(V;A_n)$ holds for every $n \geqq k_V$.

Let W be an arbitrary neighborhood of A_n in M $(n \geqq k_V)$. Since A_n is $(\mathcal{C},\mathcal{D})$-smooth, there is a neighborhood W' of A_n in M, W' $\subset$ W, such that every two $\mathcal{D}$-homotopic [6] maps of a member of $\mathcal{C}$ into W' are homotopic in W. The required neighborhood W_0 is picked with respect to W' using $\mathcal{D}_h(V;A_n)$.

Consider maps f, g: $K \to W_0$ of $K \in \mathcal{C}$ and assume that they are homotopic in V. Then for every $L \in \mathcal{D}$ and every map h: $L \to K$, compositions $f \circ h$ and $g \circ h$ are homotopic in V. The choice of W_0 implies $f \circ h \simeq g \circ h$ in W'. In other words, f and g are $\mathcal{D}$-homotopic in W'. Hence, $f \simeq g$ in W.

We shall give now three examples of situations in which $\text{ca}_{\mathcal{C}}$-regular convergence appears naturally.

(2.10) *Example.* Let $\lim d_F(A_n,A_0) = 0$ and assume that each compactum A_n $(n = 1,2,\cdots)$ is $\mathcal{C}$-trivial [8] and connected. Then $A_n \text{—ca}_{\mathcal{C}} \to A_0$.

We observed in the example (2.6)(b) that the convergence with respect to the fundamental metric to a $\mathcal{C}$-calm compactum is not sufficient for the ca$\mathcal{C}$-regular convergence. We shall now introduce a stronger metric d_{SF} on a subset $X[A]$ of 2^X consisting of all compacta in X with the same shape as a compactum A and prove that if A is $\mathcal{C}$-calm then the convergence with respect d_{SF} implies ca$\mathcal{C}$-regular convergence.

(2.11) *Definition.* If $B, C \in X[A]$, then $d_{SF}(B,C)$ is the greatest lower bound of those $\varepsilon > 0$ for which there exist ε-fundamental sequences $\underline{f} = \{f_k, B, C\}_{M,M}$ and $\underline{g} = \{g_k, C, B\}_{M,M}$ such that $\underline{g} \circ \underline{f} \simeq \underline{id}_B$ and $\underline{f} \circ \underline{g} \simeq \underline{id}_C$.

One easily proves that this is indeed a metric and that the computation of $d_{SF}(B,C)$ is independent of the choice of the AR-space M containing X.

(2.12) *Example.* Let $A_0, A_1, A_2, \cdots$ be elements of $X[A]$ and assume that $\lim d_{SF}(A_n, A_0) = 0$. If A is a $\mathcal{C}$-calm compactum, then $A_n \xrightarrow{ca\mathcal{C}} A_0$.

Proof. Without loss of generality we can assume that X lies in the Hilbert cube Q. Let V be a compact ANR neighborhood of A_0 in Q such that $\mathcal{C}_h(V;A_0)$ holds. Select $\varepsilon > 0$ with the property that ε-close maps into V are homotopic in V. Pick an index k_V such that $n \geq k_V$ implies $A_n \subset \operatorname{int} V$ and $d_{SF}(A_n, A_0) < \varepsilon$. Hence for each $n \geq k_V$ there are ε-fundamental sequences $\underline{f}^n = \{f_k^n, A_0, A_n\}_{Q,Q}$ and $\underline{g}^n = \{g_k^n, A_n, A_0\}_{Q,Q}$, neighborhoods U_0^n and V_0^n of A_0 and A_n, respectively, and an index k_n such that $k \geq k_n$ implies that $f_k^n | U_0^n$ and $g_k^n | V_0^n$ are ε-maps and $\underline{f}^m \circ \underline{g}^n \simeq \underline{id}_{A_m}$.

Consider an arbitrary neighborhood W of A_n $(n \geq k_V)$.

Pick $k_W \geqq k_n$ and a neighborhood W' of A_0 such that $f_k^n(W') \subset W$ for all $k \geqq k_W$. Now, select a neighborhood W_0' of A_0 with respect to W' using $\mathcal{C}_h(V;A_0)$. Then we take a required neighborhood W_0 of A_n inside $W \cap V_0^n \cap V$ and an index $k_0 \geqq k_W$ for which $g_k^n(W_0) \subset W_0'$ and $f_k^n \circ g_k^n | W_0 \simeq i_{W_0,W}$ in W whenever $k \geqq k_0$.

Let f, g: $K \to W_0$ be maps of $K \in \mathcal{C}$ into W_0 and assume that they are homotopic in V. Observe that $g_{k_0}^n \circ f \simeq f$ in V (these are ε-close maps into V) and, similarly, $g_{k_0}^n \circ g \simeq g$ in V. Hence, $g_{k_0}^n \circ f, g_{k_0}^n \circ g: K \to W_0'$ are homotopic in V. The choice of W_0' implies that $g_{k_0}^n \circ f \simeq g_{k_0}^n \circ g$ in W'. But then $f_{k_0}^n \circ g_{k_0}^n \circ f \simeq f_{k_0}^n \circ g_{k_0}^n \circ g$ in W. Finally, since $f_{k_0}^n \circ g_{k_0}^n \circ g \simeq f$ in W and $f_{k_0}^n \circ g_{k_0}^n \circ g \simeq g$ also in W, it follows that $f \simeq g$ in W.

Examples of sequences converging with respect to d_{SF} are provided by sequences of n-dimensional ANR's converging homotopy n-regularly [9]. In fact, one easily checks that the proof of Theorem (4.2) in [9] contains the proof of the following statement.

(2.13) Let $A_0, A_1, A_2, \cdots$ be n-dimensional compact ANR's in a metric space X. If $\{A_n\}$ converges homotopy n-regularly to A_0, then there is a subsequence $\{A_k\}$ of $\{A_n\}$ such that $\lim d_{SF}(A_k, A_0) = 0$.

(2.14) *Example.* Under the assumptions of (2.13) we see from (2.12) that $A_n \xrightarrow{ca\mathcal{C}} A_0$, for every class $\mathcal{C}$.

3. Operations Preserving ca$\mathcal{C}$-Regular Convergence

In this section we shall prove that by taking finite products and suspensions of ca$\mathcal{C}$-regularly converging sequences of compacta we get ca$\mathcal{C}$-regularly converging sequences. We also investigate in what way ca$\mathcal{C}$-regular convergence of components of the members of the sequence $\{A_n\}$ to components of A_0 imply that $A_n \text{—ca}\mathcal{C}\to A_0$.

(3.1) *Theorem. If for each* $i = 1,\cdots,m$, $\{A_n^i\}$ *is a sequence of compacta in a metric space* X_i *converging* ca$\mathcal{C}$*-regularly to a compactum* A_0^i *in* X_i, *then* $A_n = \prod_{i=1}^m A_n^i \text{—ca}\mathcal{C}\to A_0 = \prod_{i=1}^m A_0^i$.

Proof. We can assume that each X_i lies in an ANR space M_i. Then $X = \prod_{i=1}^m X_i$ lies in the ANR space $M = \prod_{i=1}^m M_i$. It suffices to prove that $\{A_n\}$ converges ca$\mathcal{C}$-regularly to A_0 in M.

For each $i = 1,\cdots,m$ pick a neighborhood V_i of A_0^i in M_i and an index k_i such that $\mathcal{C}_h(V_i;A_k^i)$ holds for all $k \geqq k_i$. Let $V = \prod_{i=1}^m V_i$ and let $k_V = \max\{k_1,\cdots,k_m\}$. We claim that $\mathcal{C}_h(V;A_k)$ is true for each $k \geqq k_V$.

Indeed, let $k \geqq k_V$ and consider an arbitrary neighborhood W of A_k in M. We can find neighborhoods $W_1,\cdots,W_m$ of $A_k^1,\cdots,A_k^m$ in $M_1,\cdots,M_m$, respectively, such that $W' = \prod_{i=1}^m W_i \subset W$. For each $i = 1,\cdots,m$ inside $V_i \cap W_i$ select a neighborhood W_{i0} of A_k^i using the property of V_i and k_i. Put $W_0 = \prod_{i=1}^m W_{i0}$.

If f, $g\colon K \to W_0$ are maps of $K \in \mathcal{C}$ into W_0 that are homotopic in V, then the compositions $\pi_i \circ f$, $\pi_i \circ g\colon K \to W_{i0}$ (where π_i is the projection of M onto M_i) are homotopic in V_i ($i = 1,\cdots,m$). It follows that they are homotopic in W_i. Hence, f and g are homotopic in W' and therefore also in W.

(3.2) *Theorem. If* $A_n \text{—ca}\mathcal{C}\to A_0$, *then the sequence* $\{SA_n\}$ *of the (unreduced) suspensions of* A_n ($n = 1,2,\cdots$) *converges* ca$\mathcal{C}$*-regularly to* SA_0.

Proof. We can assume that $A = \cup_{i=0}^{\infty} A_i$ lies in a compact convex infinite-dimensional subset M of a Banach space N_0. Setting $N = N_0 \times R$ (R denotes the real line), let us identify every point $y \in N_0$ with the point $(y,0) \in N$. We select a point $c \in A$ and define the suspension SA_i as the union of all segments $|ay|$ and $|a'y|$, where $a = (c,1), a' = (c,-1)$, and $y \in A_i$ $(i = 0,1,2,\cdots)$.

We shall show that $\{SA_n\} \to SA_0$ ca$\mathcal{C}$-regularly in SM. Observe that both M and SM are homeomorphic to Q [11].

Since $\{A_n\} \to A_0$ ca$\mathcal{C}$-regularly in M, there is a compact ANR neighborhood $\tilde{V}$ of A_0 in M and an index $k_{\tilde{V}}$ such that for every $n \geqq k_{\tilde{V}}$ the statement $\mathcal{C}_h(\tilde{V};A_n)$ holds. Let $V = SM[-1,-(1/2)] \cup S\tilde{V} \cup SM[1/2,1]$, where $SM[\alpha,\beta]$ denotes all points of SM with the second coordinate in the interval $[\alpha,\beta]$, $-1 \leqq \alpha \leqq \beta \leqq 1$, and let $k_V = k_{\tilde{V}}$. We claim that $\mathcal{C}_h(V;SA_n)$ is true for all $n \geqq k_V$.

Indeed, let $n \geqq k_V$ and let W be an arbitrary neighborhood of SA_n in SM. Select a neighborhood $\tilde{W}$ of A_n in M and an ε, $0 < \varepsilon < 1/2$, such that $SM[-1,-1+\varepsilon] \cup S\tilde{W} \cup SM[1-\varepsilon,1] \subset W$. Now take a compact ANR neighborhood $\tilde{W}_0$, $\tilde{W}_0 \subset \tilde{W} \cap \tilde{V}$, of A_n in M using the choice of $\tilde{V}$ and $k_{\tilde{V}}$ and put $W_0 = SM[-1,-1+\varepsilon] \cup S\tilde{W}_0 \cup SM[1-\varepsilon,1]$.

Let $C_-(D_-)$ be the set obtained as the union of all segments $|b'y|$ where $b' = (c,-1+(\varepsilon/2))$ and $y \in S\tilde{W}_0[-1+\varepsilon]$ $(y \in SV[-1/2])$, and let $C_+(D_+)$ be the set obtained as the union of all segments $|by|$ where $b = (c,1-(\varepsilon/2))$ and $y \in S\tilde{W}_0[1-\varepsilon]$ $(y \in SV[1/2])$.

Observe that $(SM[-1,-1+\varepsilon],C_-)$, $(SM[1-\varepsilon,1],C_+)$, $(SM[-1,-1/2],D_-)$ and $(SM[1/2,1],D_+)$ are pairs of AR's.

Hence in each of these pairs there is a strong deformation retraction of the first set onto the second set [10].

Consider maps $f, g: K \to W_0$ of $K \in \mathcal{C}$ into W_0 and assume that they are homotopic in V. Applying strong deformation retractions determined by the first two pairs, we see that f and g are homotopic in W_0 to maps f' and g', respectively, of K into $S\tilde{W}_0[-1+\varepsilon,1-\varepsilon] \cup C_- \cup C_+$. By applying strong deformation retractions determined by the last two pairs, we see that f' and g' are homotopic in $S\tilde{V}[-1/2,1/2] \cup D_- \cup D_+$.

Since both $S\tilde{V}[-1/2,1/2] \cup D_- \cup D_+$ and $S\tilde{W}_0[-1+\varepsilon,1-\varepsilon] \cup C_- \cup C_+$ are contained in $S\tilde{V}[-1+(\varepsilon/2),1-(\varepsilon/2)]$ it follows (after projecting onto $\tilde{V}$) that f' and g' are homotopic in $S\tilde{W}[-1+(\varepsilon/2),1-(\varepsilon/2)]$. Hence, f and g are homotopic in W.

(3.3) *Corollary. The (unreduced) suspension of a $\mathcal{C}$-calm compactum is also $\mathcal{C}$-calm.*

The proof of the following proposition is left to the reader.

(3.4) *Proposition. If $A_n \text{—} ca_{\mathcal{C}} \to A_0$, then for every component C_0 of A_0 there is a component C_n of A_n such that $C_n \text{—} ca_{\mathcal{C}} \to C_0$. Conversely, let $\mathcal{C}$ be a component hereditary class of topological spaces and let every compactum A_n $(n = 0,1,2,\cdots)$ has precisely k $(k < \infty)$ components $C_n^1,\cdots,C_n^k$ such that $C_n^i \text{—} ca_{\mathcal{C}} \to C_0^i$, $1 \leqq i \leqq k$. Then $A_n \text{—} ca_{\mathcal{C}} \to A_0$.*

4. The Metric of $ca_{\mathcal{C}}$-Regular Convergence

The collection of all $\mathcal{C}$-calm compacta in a metric space X can be made into a hyperspace $ca_{\mathcal{C}}(X)$ by defining the notion of convergence by means of $\mathcal{C}$-calmly regular convergence. In this section, using Begle's method in [1], we shall define the metric d_{ca} on the space $ca\mathcal{C}(X)$ in such a way that $\lim d_{ca}(A_n, A_0) = 0$ iff $A_n \xrightarrow{ca\mathcal{C}} A_0$.

In fact, it is clear from the explanation on the page 444 in [1] that such a metric can be introduced provided we can prove the analogues of Lemmas 1, 3, 4, and 5 in [1] for the function $\gamma_{\mathcal{C}}(\varepsilon, A)$ (defined in (4.1)) corresponding to Begle's function $\delta_n(\varepsilon, P)$. The analogue of Lemma 4 was established in (2.4) while Lemmas (4.2), (4.3), and (4.5) below correspond to Lemmas 1, 3, and 5, respectively.

Throughout this section we assume that X lies in an AR space M of diameter 1.

(4.1) *Definition.* For each compact subset A of M and for each $\varepsilon > 0$, let $\gamma_{\mathcal{C}}(\varepsilon, A)$ be the least upper bound of all numbers γ, $0 \leqq \gamma \leqq \varepsilon$, such that $\mathcal{C}_h(N_\gamma(A); A)$ holds.

It is clear that for each compactum A in M, $\gamma_{\mathcal{C}}(\varepsilon, A)$ always exists and is a non-negative monotone non-decreasing, and hence measurable, function on the half-open interval $I^* = (0,1]$. If A is $\mathcal{C}$-calm, then $\gamma_{\mathcal{C}}(\varepsilon, A) > 0$ everywhere in I^* and conversely.

The relation between definitions (2.2) and (4.1) is provided by the following.

(4.2) *Lemma. If* $\lim d_F(A_n, A_0) = 0$, *then the sequence* $\{A_n\}$ *converges* $ca\mathcal{C}$*-regularly to* A_0 *iff* $\liminf \gamma_{\mathcal{C}}(\varepsilon, A) > 0$ *for each* ε *in* I^*.

Proof. The proof is similar to the proof of Lemma (4.2) in [7].

The following two lemmas resemble Lemmas (4.3) and (4.4) in [7]. The proofs are similar in spirit but technically more complicated.

(4.3) *Lemma. Let* $\lim d_F(A_n, A_0) = 0$. *Then* $\limsup \gamma_{\mathcal{C}}(\varepsilon, A_n) \leqq \gamma_{\mathcal{C}}(\varepsilon, A_0)$ *for all but countably many points* ε *in* I^*.

Proof. We shall prove that if $\limsup \gamma_{\mathcal{C}}(\varepsilon_0, A_n) > \gamma_{\mathcal{C}}(\varepsilon_0, A_0)$ at the point $\varepsilon_0 \in I^*$, then the function $\gamma_{\mathcal{C}}(\varepsilon, A_0)$ has a jump at the point ε_0 (in particular, it is not continuous at ε_0). Since $\gamma_{\mathcal{C}}(\varepsilon, A_0)$ is a monotone function, there are at most countably many points of I^* at which this can happen.

Take an $e > 0$ and a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that $\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_i}) \geqq \gamma_{\mathcal{C}}(\varepsilon_0, A_0) + e$ for all $i > 0$. Let b, $0 < b < e/2$, be an arbitrary number and choose an index i_0 so that $i \geqq i_0$ implies $d_F(A_{n_i}, A_0) < b$. Observe that $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_0)+(e/2)}(A_0) \subset N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_i})}(A_{n_i})$. We claim that $\mathcal{C}_h(N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_0)+(e/8)}(A_0); A_0)$ holds. Since $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_i})}(A_{n_i}) \subset N_{\varepsilon_0+b}(A_0)$, this would imply that for every b, $0 < b < e/8$, $\gamma_{\mathcal{C}}(\varepsilon_0+b, A_0) \geqq \gamma_{\mathcal{C}}(\varepsilon_0, A_0)+(e/8)$, i.e., that the function $\gamma_{\mathcal{C}}(\varepsilon, A_0)$ has a jump at least $e/8$ at the point ε_0.

Let W be an arbitrary compact ANR neighborhood of A_0 in M (which we can assume is the Hilbert cube Q). Pick ξ, $0 < \xi < b/4$, such that 2ξ-close maps into W are homotopic in W and ξ-close maps into $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_0)+(e/4)}(A_0)$ are homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_0)+(e/2)}(A_0)$. Select $j \geqq i_0$ such that $d_F(A_{n_j}, A_0) < \xi$. Let $\underline{f} = \{f_k, A_{n_j}, A_0\}_{M,M}$ and $\underline{g} = \{g_k, A_0, A_{n_j}\}_{M,M}$ be ξ-fundamental sequences. Let Z be a neighborhood of A_0 and let $k_{\underline{g}}$ be an index such that $k \geqq k_{\underline{g}}$ and $x \in Z$ implies $d(g_k(x), x) < \xi$.

Similarly, let Z' be a neighborhood of A_{n_j} and let $k_{\underline{f}}$ be an index such that $k \geqq k_{\underline{f}}$ and $x \in Z'$ implies $d(f_k(x),x) < \xi$.

Now we pick a neighborhood W' of A_{n_j} and $k_{W'} \geqq k_{\underline{f}}, k_{\underline{g}}$ such that $f_k(W') \subset W$ for all $k \geqq k_{W'}$. Inside W' let us select a neighborhood W_0' of A_{n_j} with the property that maps of $K \in \mathcal{C}$ into W_0' homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0, A_{n_j})}(A_{n_j})$ are already homotopic in W'. Finally, the required neighborhood W_0 of A_0, $W_0 \subset Z \cap W \cap N_{\gamma_{\mathcal{C}}(\varepsilon_0,A_0)+(e/8)}(A_0)$ and the index $k_0 \geqq k_{W'}$ are picked so that $g_{k_0}(W_0) \subset Z' \cap W_0'$.

Consider maps f, g: $K \to W_0$ defined on a member K of $\mathcal{C}$ and suppose that they are homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0,A_0)+(e/8)}(A_0)$. Since $(g_{k_0}\circ f, f)$ and $(g_{k_0}\circ g, g)$ are two pairs of ξ-close maps into $N_{\gamma_{\mathcal{C}}(\varepsilon_0,A_0)+(e/4)}(A_0)$, it follows that $g_{k_0}\circ f$ and $g_{k_0}\circ g$ are homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0,A_{n_j})}(A_{n_j})$. The choice of W_0' implies that these maps are already homotopic in W'. But, then $f_{k_0}\circ g_{k_0}\circ f$ and $f_{k_0}\circ g_{k_0}\circ g$ are homotopic in W. We conclude that f and g are homotopic in W because $f_{k_0}\circ g_{k_0}\circ f \simeq f$ in W and $f_{k_0}\circ g_{k_0}\circ g \simeq g$ in W.

(4.4) *Lemma. Let* $A_n \overset{ca_{\mathcal{C}}}{\longrightarrow} A_0$. *Then* $\gamma_{\mathcal{C}}(\varepsilon_0,A_0) \leqq \liminf \gamma_{\mathcal{C}}(\varepsilon_0,A_n)$ *at every point* $\varepsilon_0 \in I^*$ *in which the function* $\gamma_{\mathcal{C}}(\varepsilon,A_0)$ *is continuous.*

Proof. Let us consider a point $\varepsilon_0 \in I^*$ at which the function $\gamma_{\mathcal{C}}(\varepsilon,A_0)$ is continuous. Suppose that $\gamma_{\mathcal{C}}(\varepsilon_0,A_0) > \liminf \gamma_{\mathcal{C}}(\varepsilon_0,A_n)$. Then there is an e, $0 < e < \varepsilon_0$, and a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that $\gamma_{\mathcal{C}}(\varepsilon_0,A_{n_i})+2e < \gamma_{\mathcal{C}}(\varepsilon_0,A_0)-2e$, for all $i > 0$. Since the function $\gamma_{\mathcal{C}}(\varepsilon,A_0)$ is continuous at ε_0, there is a number d, $0 < d < e$, such that $\gamma_{\mathcal{C}}(\varepsilon,A_0) \in$

$(\gamma_{\mathcal{C}}(\varepsilon_0,A_0)-2e,\ \gamma_{\mathcal{C}}(\varepsilon_0,A_0)+2e)$ for all $\varepsilon \in (\varepsilon_0-2d,\varepsilon_0+2d)$. In particular, $\gamma_{\mathcal{C}}(\varepsilon_0-d,A_0) > \gamma_{\mathcal{C}}(\varepsilon_0,A_0)-2e > \gamma_{\mathcal{C}}(\varepsilon_0,A_{n_i})+2e$.

We claim that there is an index k such that $N_{\gamma_{\mathcal{C}}(\varepsilon_0,A_{n_k})+e}(A_{n_k}) \subset N_{\varepsilon_0}(A_{n_k})$ and such that $\mathcal{C}_h(N_{\gamma_{\mathcal{C}}(\varepsilon_0,A_{n_k})+e}(A_{n_k});A_{n_k})$ holds. This would imply that $\gamma_{\mathcal{C}}(\varepsilon_0,A_{n_k}) \geqq \gamma_{\mathcal{C}}(\varepsilon_0,A_{n_k})+e$ which is an obvious contradiction.

By using the fact that $A_n \text{---} ca\mathcal{C} \to A_0$, inside $N_{\gamma_{\mathcal{C}}(\varepsilon_0-d,A_0)}(A_0)$ we pick a compact ANR neighborhood V of A_0 and an index k_V so that $\mathcal{C}_h(V;A_i)$ is true for all $i > k_V$. Let ξ, $0 < \xi < d$, has the property that ξ-close maps into V are homotopic in V and that $N_\xi(A_0) \subset V$. Pick an integer k so large that $n_k \geqq k_V$ and $d_F(A_{n_k},A_0) < \xi$. Let $\underline{f} = \{f_k,A_{n_k},A_0\}_{M,M}$ be an ξ-fundamental sequence, let Z' be a neighborhood of A_{n_k}, and let $n_{Z'}$ be such that $d(f_n(x),x) < \xi$ for all $x \in Z'$ and $n \geqq n_{Z'}$.

Now we pick a neighborhood W_0^*, $W_0^* \subset V$, of A_0 in M with the property that maps of $K \in \mathcal{C}$ into W_0^* which are homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0-d,A_0)}(A_0)$ are already homotopic in V.

Consider now an arbitrary neighborhood W' of A_{n_k}. Let a neighborhood $\tilde{W}_0'$ of A_{n_k} be selected with respect to W' and V using $\mathcal{C}_h(V;A_{n_k})$. Finally, let a neighborhood W_0', $W_0' \subset \tilde{W}_0' \cap Z'$, of A_{n_k} and $n_0 \geqq n_{Z'}$ be such that $f_{n_0}(W_0') \subset W_0$.

Suppose f, g: $K \to W_0'$ are maps of a member K of $\mathcal{C}$ homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0,A_{n_k})+e}(A_{n_k})$. Observe that $N_{\gamma_{\mathcal{C}}(\varepsilon_0,A_{n_k})+e}(A_{n_k}) \subset N_{\gamma_{\mathcal{C}}(\varepsilon_0-d,A_0)}(A_0)$ and that $f' = f_{n_0}\circ f$ and f and $g' = f_{n_0}\circ g$ and g are two pairs of ξ-close maps into V. It follows that $f'\colon K \to W_0^*$ and $g'\colon K \to W_0^*$ are homotopic in $N_{\gamma_{\mathcal{C}}(\varepsilon_0-d,A_0)}(A_0)$.

The choice of W_0^* gives that f' and g' are homotopic in V. But then f and g are homotopic in V so that the choice of V and k_V implies that f and g are homotopic in W' and this is what we wanted to prove.

Combining the last two lemmas we have the following theorem.

(4.5) *Theorem. If* $A_n \xrightarrow{ca_{\mathcal{C}}} A_0$, *then* $\lim \gamma_{\mathcal{C}}(\varepsilon, A_n)$ *exists and equals* $\gamma_{\mathcal{C}}(\varepsilon, A_0)$ *almost everywhere in* I*.

We are now ready to introduce the metric d_{ca} on the hyperspace $ca_{\mathcal{C}}(X)$ of all $\mathcal{C}$-calm compacta in a metric space X. Let E be the Banach space of all bounded measurable functions on the interval I*, the norm of an element f in E being defined as:

$$\|f\| = \int_0^1 |f| d\varepsilon.$$

We define a correspondence between $ca_{\mathcal{C}}(X)$ and a subset of $2_F^X \times E$ by assigning to each element A of $ca_{\mathcal{C}}(X)$ the element $(A, \gamma_{\mathcal{C}}(\varepsilon, A))$ of $2_F^X \times E$. This correspondence is one-to-one, so a metric is defined in $ca_{\mathcal{C}}(X)$ by letting the distance between two points in $ca_{\mathcal{C}}(X)$ be the distance between the corresponding points in $2_F^X \times E$. Specifically,

$$d_{ca}(A,B) = [d_F^2(A,B) + (\int_0^1 |\gamma_{\mathcal{C}}(\varepsilon,A) - \gamma_{\mathcal{C}}(\varepsilon,B)| d\varepsilon)^2]^{1/2}.$$

With obvious modifications the argument on the page 444 in [1] shows that this metric induces the same topology on $ca_{\mathcal{C}}(X)$ as that naturally defined in terms of $\mathcal{C}$-calmly regular convergence. Hence, one can prove the following.

(4.6) *Theorem. There is a metric* d_{ca} *on the hyperspace* $ca_{\mathcal{C}}(X)$ *of all* $\mathcal{C}$*-calm compacta in a metric space* X *such that* *for a sequence* $A_0, A_1, A_2, \cdots$ *in* $\mathrm{ca}_{\mathcal{C}}(X)$ $\lim d_{ca}(A_n, A_0) = 0$ *iff* $A_n \xrightarrow{\mathrm{ca}_{\mathcal{C}}} A_0$.

At present we can state only the folloiwng three corollaries that describe topological properties of the metric space $(\mathrm{ca}_{\mathcal{C}}(X), d_{ca})$.

(4.7) *Corollary. If* X *is homeomorphic to* Y, *then* $\mathrm{ca}_{\mathcal{C}}(X)$ *is homeomorphic to* $\mathrm{ca}_{\mathcal{C}}(Y)$.

Proof. The proof is similar to the proof of (4.7) in [7].

(4.8) *Corollary. The identity map* id: $(\mathrm{ca}_{\mathcal{C}}(X), d_{ca}) \to (\mathrm{ca}_{\mathcal{C}}(X), d_F)$ *is continuous.*

Proof. See (2.1).

(4.9) *Corollary. Let* α *be a hereditary shape property. Then the collection of all elements of* $\mathrm{ca}_{\mathcal{P}}(X)$ *which have property* α *constitute a closed subset of* $\mathrm{ca}_{\mathcal{P}}(X)$.

Proof. See (4.4) in [8].

We leave many questions concerning the topological structure of the space $\mathrm{ca}_{\mathcal{C}}(X)$ open. The most natural problem would be to see what properties of X are carried over onto $\mathrm{ca}_{\mathcal{C}}(X)$. In particular, is $\mathrm{ca}_{\mathcal{C}}(X)$ separable (topologically complete) if X is separable (topologically complete)? The last two questions are in view of (4.8) and the method of the proofs for Theorems 2 and 3 in [1] equivalent to the following questions.

(4.10) If X is a separable (topologically complete) metric space, is $(\mathrm{ca}_{\mathcal{C}}(X), d_F)$ also separable (topologically complete) metric space?

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41001 Zagreb, p. p. 187

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