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NON-DEGENERATE k-SPHERE MAPPINGS BETWEEN SPHERES

D. Coram* and P. Duvall*

0. Introduction

Suppose that $f: S^{2k+1} \longrightarrow S^{k+1}$ is a mapping between spheres. We say that f is a *non-degenerate* k-*sphere mapping* provided that each point inverse has the shape of a k-sphere and each point $y \in S^{k+1}$ has a neighborhood U such that the inclusion induced homomorphism $\pi_k(f^{-1}(z)) \longrightarrow \pi_k(f^{-1}(U))$ is non-zero for every $z \in U$ and every base point in $f^{-1}(z)$. For such a map (with $k > 1$) we prove that almost all of the point inverses fit together "regularly," so that f is an approximate fibration on the complement of a finite set E (Thm. 8). The number of points in E is limited by the Hopf invariant (Cor. 15), but can be arbitrarily large (Thm. 16).

The above definition is related to Lacher's k-sphere mappings [L 2], [B-L]. There a mapping $f: S^{2k+1} \longrightarrow N^n$ is a k-sphere mapping provided N^n is a closed topological n-manifold and $f^{-1}(y)$ is homeomorphic to either a point or a k-sphere for each $y \in N$. He proves that there are only two possibilities: either $n = 2k + 1$ and f is a homeomorphism, or $n = k + 1$, N is a homotopy sphere and $f^{-1}(y)$ is a k-sphere for every $y \in N$. In the more general shape setting Lacher's proof shows that again there are only two possibilities: either $n = 2k + 1$ and f is a cell-like mapping, or $n = k + 1$, N is a homotopy sphere and $f^{-1}(y)$ has the

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shape of a sphere for every $y \in N$. Cell-like mappings are fairly well understood [L 3], but the second case is not. This paper takes a closer look there.

One attractive possibility for f in the second case is that f is some kind of fibration. The examples Lacher gives are locally trivial except over a single point. Of course, if such a mapping is a locally trivial fibration, then all point inverses are homeomorphic. However, without extra conditions (such as complete regularity [D-H]) the converse is false even under the nicest conditions. Let us analyze one example of the failure of the converse in order to point out that a kind of converse is possible in the more relaxed setting of shape theory and approximate fibrations.

Example. Let I be the unit interval [0,1] and let A be the arc in $I \times I$ consisting of 3 straight line segments from (1/2,0) to (3/8,3/4), then to (5/8,1/4) and then to (1/2,1). There is a homeomorphism $h: I \times I \longrightarrow I \times I$ which is the identity on the boundary of $I \times I$ and takes A to $\{1/2\} \times I$. Let $\pi: I \times I \longrightarrow I$ be projection onto the second factor, and let $p: I \times I \longrightarrow I$ be defined by $p = \pi h$. Then p is a locally trivial fibration. Now let ϕ_n: [1/n+1,1/n] $\longrightarrow$ [0,1] be the linear homeomorphism $\phi_n(t) = n(n+1)t - n$. Next define $q_k: I \times I \longrightarrow I$ by

$$q_k(x,y) = \begin{cases} \phi_n^{-1}(p(\phi_n(x),y)) & \text{if } 1/n+1 \le x \le 1/n \text{ and } n \le k \\ x & \text{if } x \le 1/k+1 \end{cases}$$

Let $q = \lim_{k\to\infty} q_k$. Then q is not locally trivial at $q^{-1}(0)$.

Now define $r_\ell: I \times I \longrightarrow I$ by

$$r_\ell(x,y) = \begin{cases} \phi_n^{-1}(q_{\ell-n+1}(\phi_n(x),y)) & \text{if } 1/n+1 \le x \le 1/n \text{ and } n < \ell \\ x & \text{if } x \le 1/\ell+1. \end{cases}$$

Let $r = \lim_{\ell\to\infty} r_\ell$. Then r fails to be locally trivial over the infinite set $\{0,1/n|n = 1,2,\cdots\}$, even though every point inverse is an arc. However, r is an approximate fibration since it is the limit of locally trivial fibrations [C-D 2, Prop. 1.1].

In order to have an example of a k-sphere mapping we proceed as follows. If $S^1 \times S^1$ is obtained by the standard identification of opposite faces of $I \times I$, then $r: I \times I \longrightarrow I$ induces $\tilde{r}: S^1 \times S^1 \longrightarrow S^1$. Now define $f: S^3 \longrightarrow S^2$ to be the composition

$$S^3 \cong S^1 * S^1 \xrightarrow{\gamma} \Sigma(S^1 \times S^1) \xrightarrow{\Sigma\tilde{r}} \Sigma S^1 \cong S^2$$

where * denotes join, Σ suspension, and γ the map whose only non-degenerate point inverses are the joined circles which are mapped to the suspension points. Although each point inverse is a simple closed curve, f fails to be locally trivial over a 1-dimensional set. However f is an approximate fibration except at one point.

This is not a coincidental example. In [C-D 4], the authors showed that a non-degenerate 1-sphere mapping $f: S^3 \longrightarrow S^2$ is an approximate fibration over the complement of a set with at most 2 points. Such an f can be approximated by Seifert fiber maps.

The present paper is concerned with non-degenerate k-sphere mappings $f: S^{2k+1} \longrightarrow S^{k+1}$ with $k > 1$. Of course there are, as above, examples of such maps which fail to be completely regular on infinite sets. We prove that in this case too, such maps are approximate fibrations over the complement of a finite set. However, in contrast to the case with k = 1, it is interesting that f may fail to be an approximate fibration at more than two points.

Most of our notation is standard: I denotes the unit interval [0,1]; S^k, the unit sphere; B^k, the unit ball; Z, the integers; and R, the real numbers. The symbol F_y denotes $f^{-1}(y)$. Also we use $H_p(\check{H}^p)$ to denote the p^{th} singular homology (Čech cohomology) with Z coefficients. The p^{th} homotopy group (shape group) is denoted $\pi_p(\underline{\pi}_p)$. If $\phi\colon A \longrightarrow B$ is a homomorphism between groups isomorphic to Z, α and β are generators of A and B, and $\phi(\alpha) = p\beta$, we say that ϕ is a *multiplication by* $|p|$. For definitions and results on approximate fibrations, the reader is referred to [C-D 2] and [C-D 3]. For information on shape theory see [Sg] or [B 2].

1. The Finiteness Theorem

Let $f\colon S^{2k+1} \longrightarrow S^{k+1}$ be a non-degenerate k-sphere mapping and suppose that k > 1. Then f is a 1 - UV mapping so that each F_y satisfies the cellularity criterion [C, Lem. 15], [L 1]. It follows that each F_y has an arbitrarily small neighborhood T, PL-homeomorphic to $S^k \times B^{k+1}$, such that the inclusion $F_y \subset T$ is a shape equivalence [C-D 1], [V]. We shall call such a T a *small tube about* F_y.

Fix a reference point $b \in S^{k+1}$ and let T be a small tube about F_b. Let U be a fixed neighborhood of b such that $f^{-1}(U) \subset T$. If $y \in U$ and T' is a small tube about F_y in T, the inclusion induced map $\pi_k(T') \longrightarrow \pi_k(T)$ is multiplication by p for some integer p. It is easy to see that p is independent of T' and base points. By our non-degeneracy hypothesis, we may assume that U is chosen so that $p > 0$. Define a function $\alpha\colon U \longrightarrow R$ by $\alpha(y) = p$.

Remark. The function α is intended to measure the twisting of F_y about F_b. If we were to assume that each F_y is homeomorphic to a k-sphere, we could define α by choosing a neighborhood U and a retraction $r\colon f^{-1}(U) \longrightarrow F_b$ and setting $\alpha(y) = |\deg(r|F_y)|$. The reader may find it helpful to keep this more concrete situation in mind throughout what follows.

Propositions 1 through 6 exactly parallel lemmas of the same number in [C-D 4]. We will not repeat the proofs here unless significant changes are necessary.

Proposition 1. *If* $y \in U$, *there is a neighborhood* V *of* y *in* U *such that for every* $z \in V$, *there is a positive integer* k *such that* $\alpha(z) = k\alpha(y)$.

Proposition 2. α *is lower semicontinuous.*

Proposition 3. *The set* $C = \{y \in U | \alpha$ *is continuous at* $y\}$ *is open and dense in* U. *The set* $D = U - C$ *can be written as* $D = D_1 \cup D_2$ *where* D_1 *is dense in itself and* D_2 *is countable.*

Proposition 4. $f|f^{-1}(C)\colon f^{-1}(C) \longrightarrow C$ *is an approximate fibration.*

Proof. Let $y \in C$ be given. We will show that $f|f^{-1}(C)$ is completely movable at y [C-D 3]. Given a neighborhood U' of F_y, choose a small tube T_y about F_y in U'. Since y is a point of continuity of α, we may assume that α is constant on $f(T_y)$. Given z such that $F_z \subset \text{int } T_y$, and a neighborhood V' of F_z in T_y, choose a small tube T_z about F_z in V'. Let $W' = \text{int } T_z$. Since α is constant on $f(T_y)$ the inclusion $T_z \subset T_y$ is a homotopy equivalence. Hence there is a deformation of V' into W' within U' keeping a neighborhood of F_z fixed. Hence, by [C-D 3, Th. 3.9] $f\, f^{-1}(C)$ is an approximate fibration.

Proposition 5. *If* A *is an arc in* U *with endpoint* $d \in D$ *such that* $A - \{d\} \subset C$, *then* $Sh(f^{-1}(A)) = Sh(S^k)$. *Furthermore, if* c *is the other endpoint of* A, *then the inclusion induced homomorphism* $\underline{\pi}_k f^{-1}(c) \longrightarrow \underline{\pi}_k f^{-1}(A)$, *is a multiplication by* $\alpha(c)/\alpha(d)$.

Proposition 6. D *is countable.*

Proof. By Proposition 3, $D = D_1 \cup D_2$ where D_2 is countable, so we wish to prove $D_1 = \emptyset$. Suppose $D_1 \neq \emptyset$. As in Lemma 6 of [C-D 4] we can find an arc A in U such that $D \cap A = \text{Bd } A = \{d,e\}$, $\alpha(d) = \alpha(e) = \alpha(c)/p$ for some $p > 1$ and every $c \in \text{Int } A$, and $\check{H}^{k+1}(f^{-1}(A)) = Z_p$. Hence, $H_{k-1}(S^{2k+1} - f^{-1}(A)) = Z_p$ by Alexander duality. On the other hand $H_{k-1}(S^{k+1} - A) = 0$. This is impossible since $f_*\colon H_{k-1}(S^{2k+1} - f^{-1}(A)) \longrightarrow H_{k-1}(S^{k+1} - A)$ is an isomorphism by the Vietoris mapping theorem [L 1].

Proposition 7. D *is finite.*

Proof. Suppose D is infinite. Then D contains infinitely many isolated points $\{d_1, d_2, d_3, \cdots\}$. Arguing as in

Prop. 6, we can show that $\alpha(d_m) \neq \alpha(d_n)$ whenever $m \neq n$. Hence $\alpha(D)$ is an unbounded subset of R. On the other hand by Proposition 1 there is a neighborhood V_n of d_n such that $\alpha(d_n) \leq \alpha(y)$ for each $y \in V_n$. Since d_n is an isolated point, $\alpha(d_n) < \alpha(c)$ for some $c \in C$. Since α is constant on C, this implies that $\{\alpha(d_n)\}$ is bounded, which is a contradiction.

Now we let b vary over S^{k+1}. It is clear that the property of being a point of continuity does not depend on the reference point used to define α, so by covering S^{k+1} with a finite number of U's, we get the following:

Theorem 8. *If* $f: S^{2k+1} \longrightarrow S^{k+1}$ *is a non-degenerate* k-*sphere mapping*, $k > 1$, *then* f *is an approximate fibration over the complement of a finite set of points in* S^{k+1}.

2. Connections with the Hopf Invariant

Suppose that $f: S^{2k+1} \longrightarrow S^{k+1}$ is a non-degenerate k-sphere mapping, $k > 1$, and let $E(f) \subset S^{k+1}$ be the finite set of points which fail to have a neighborhood over which f is an approximate fibration. If $e \in E(f)$, we can use e as a reference point for a map α as in section 1. Then $\alpha(y) = p > 1$ for all y sufficiently close to e, $y \neq e$, and we say that e *is an exceptional point of degree* p.

Proposition 9. *If* d *and* e *are exceptional points of degree* p *and* q, *then the least common divisor* $(p,q) = 1$.

Proof. If $(p,q) = r > 1$, we can find an arc A such that $A \cap E(f) = \text{Bd}A = \{d,e\}$. As in Prop. 6, $\check{H}^{k+1}(f^{-1}(A)) \cong Z_r$ which is a contradiction.

Now suppose that X and Y are compacta in S^{2k+1} such that

X and Y satisfy the cellularity criterion and Sh(X) = Sh(Y) = Sh(S^k). Define the linking number L(X,Y) to be p, where the inclusion induced map $\check{H}_k(X) \longrightarrow H_k(S^{2k+1} - Y)$ is multiplication by p.

Proposition 10. *Let* f: $S^{2k+1} \longrightarrow S^{k+1}$ *be a non-degenerate* k-*sphere mapping which is an approximate fibration over an open set* U *in* S^{k+1}. *Let* y,z ∈ U. *Then* $L(F_y,F_z) = |H(f)|$ *where* H(f) *is the Hopf invariant of* f.

Proof. Choose a tube $T_y \subset S^{2k+1} - F_z$ containing F_y such that the inclusion-induced homomorphism $\check{H}_k F_y \longrightarrow H_k T_y$ is an isomorphism. Next choose ε > 0 such that $f^{-1}(N(y,\varepsilon)) \subset \text{Int } T_y$, and choose δ > 0 such that δ-close maps into S^{k+1} are ε/2-homotopic [B 1, Th. 3.1]. Let f_1: $S^{2k+1} \longrightarrow S^{k+1}$ be a map such that $d(f,f_1) < \delta$ and f_1 is simplicial relative to triangulations of S^{2k+1} and S^{k+1} for which y is in the interior of a (k+1)-simplex. Finally choose an open disk D such that $y \in D \subset N(y,\varepsilon/2)$. Now suppose that K: $S^{2k+1} \times I \longrightarrow S^{k+1}$ is an ε/2-homotopy with $K_0 = f$ and $K_1 = f_1$. Note that $K((S^{2k+1} - \text{Int } T_y) \times I) \subset S^{k+1} - D$ since if $x \in S^{2k+1} - \text{Int } T_y$, then $d(K(x,t),y) \geq d(f(x),y) - d(f(x),K(x,t)) > \varepsilon - \varepsilon/2 = \varepsilon/2$. Therefore, $f \simeq f_1$ as maps of pairs $(S^{2k+1}, S^{2k+1} - \text{Int } T_y) \longrightarrow (S^{k+1}, S^{k+1} - y)$. Now consider the diagram

$$\begin{array}{ccc}
H_0 y & & H_k(F_y) \\
\downarrow \cong & & \downarrow \cong \\
H_0 D & & H_k(\text{Int } T_y) \\
\downarrow \cong & & \downarrow \cong \\
H^{k+1}(S^{k+1}, S^{k+1} - D) & \xrightarrow{f^* = f_1^*} & H^{k+1}(S^{2k+1}, S^{2k+1} - \text{Int } T_y)
\end{array}$$

where the uppermost arrows are inclusion induced isomorphisms, the next lower arrows are duality isomorphisms [Sp, Th.

6.2.17], and the horizontal arrow is induced by either f or f_1. Using f* the composition applied to the generator of $H_0 y$ yields the homology class of F_y. Using f_1^* it yields the homology class of $f_1^{-1}(y)$. But $f^* = f_1^*$ so the homology class of F_y in $S^{2k-1} - F_z$ can be represented by $f_1^{-1}(y)$. Similarly the homology class of F_z in $S^{2k+1} - F_y$ can be represented by $f_1^{-1}(z)$. (The epsilonics can be done simultaneously for y and z to get a single approximation f_1.) Hence $L(F_y,F_z) = \ell(f_1^{-1}(y),f_1^{-1}(z)) = |H(f_1)| = |H(f)|$, [Stn, p. 113], where $\ell(\ ,\ )$ denotes the usual homological linking number.

Proposition 11. *Suppose that* $f\colon S^{2k+1} \longrightarrow S^{k+1}$ *is a nondegenerate* k*-sphere mapping, and* d *and* e *are exceptional points of degrees* p *and* q. *Then* $|H(f)| = pqL(F_d,F_e)$.

Proof. Let T_d and T_e be disjoint small tubes about F_d and F_e, let x and y be non-exceptional points such that $F_x \subset \operatorname{int} T_d$ and $F_y \subset \operatorname{int} T_e$, and let W_x, W_y be small tubes about F_x and F_y inside T_d and T_e. Then if Σ_d, Σ_e, Σ_x, Σ_y denote the k-cycles carried by the cores of T_d, T_e, etc. with suitable orientations, we have $\Sigma_x \sim p\Sigma_d$ in $S^{2k+1} - T_e$, $\Sigma_y \sim q\Sigma_d$ in $S^{2k+1} - T_d$ and $L(F_x,F_y) = |\ell(\Sigma_x,\Sigma_y)| = |\ell(p\Sigma_d,q\Sigma_e)| = pq|\ell(\Sigma_d,\Sigma_e)| = pqL(F_d,F_e)$.

Proposition 12. *If* $f\colon S^{2k+1} \longrightarrow S^{k+1}$ *is a nondegenerate* k*-sphere mapping, and* U *is an open* (k+1)*-cell in* S^{k+1} *with* $y \in U$ *the only possible exceptional point in* U, *then* $f^{-1}(U)$ *is homeomorphic to* $S^k \times R^{k+1}$. *If* W *is a small tube about* F_y, $W \subset f^{-1}(U)$ *is a homotopy equivalence.*

Proof. By the exact homotopy sequence of an approximate fibration, $f^{-1}(U)$ is simply connected at infinity. Thus the first conclusion follows from the second by the *open collar theorem* [Sb]. Using the stationary lifting property of approximate fibrations [C-D 2], we may deform $f^{-1}(U)$ into W keeping a neighborhood of F_y fixed, so that the second conclusion follows.

Theorem 13. *Suppose* $f: S^{2k+1} \longrightarrow S^{k+1}$ *is a nondegenerate* k-*sphere mapping with exceptional set* $\{e_1, \cdots, e_r\}$, *where* e_i *has degree* p_i. *Then* $|H(f)| = \Pi_{i=1}^r p_i$.

Proof. Let y and z be points in $S^{k+1} - E(f)$. Note that $S^{2k+1} - F_y$ is homeomorphic to $S^k \times R^{k+1}$ [C-D 1], [V]. By working with open regular neighborhoods of arcs joining e_i to z, we can find open (k+1) - cells U_z, U_i in $S^{k+1} - \{y\}$ such that $z \in U_z$, $U_i \cap E(f) = \{e_i\}$, and $U_i \cap U_j = U_z$, $i \neq j$. By stationary lifting of a deformation of $S^{k+1} - \{y\}$ into $\cup_{i=1}^r U_i$, one can show as in Proposition 12 that the inclusion of $f^{-1}(\cup_{i=1}^r U_i)$ into $S^{2k+1} - F_y$ induces an isomorphism on k^{th}-homology. Let G_z, G_i and K_j denote $H_k(f^{-1}(U_z))$, $H_k(f^{-1}(U_i))$ and $H_k(f^{-1}(\cup_{i=1}^j U_i))$ respectively. Choose generators η_z, η_i of G_z, G_i such that $\eta_z = p_i \eta_i$ in G_i. Also let $s_j = \Pi_{i=1}^j p_i$ and $s_{j,i} = s_j/p_i$. We wish to prove the following statement inductively.

Statement S_j. The group K_j is cyclic with generator $\xi_j = \sum_{i=1}^j q_{j,i}\eta_i$ for some integers $q_{j,i}$ such that $\sum_{i=1}^j q_{j,i}s_{j,i} = 1$, and the image of η_z in K_j is $s_j \xi_j$.

Note that S_1 is immediate. Now suppose that S_j is true for some $j < r$. The Mayer-Vietoris Theorem gives the exact sequence

$$0 \longrightarrow G_z \xrightarrow{g \to (g,-g)} K_j \oplus G_{j+1} \xrightarrow{(g,h) \to g+h} K_{j+1} \longrightarrow 0$$

(Throughout this proof we use the same notation for a homology class and its image under an inclusion induced map.) Since the p_i's are relatively prime, there exist integers a and b such that $ap_{j+1} + bs_j = 1$. It is easy to see that $\mu_1 = (s_j\xi_j, -p_{j+1}\eta_{j+1})$ and $\mu_2 = (a\xi_j, b\eta_{j+1})$ form a basis for $K_j \oplus G_{j+1}$. Therefore $a\xi_j + b\eta_{j+1}$ is a generator for K_{j+1}. We claim that S_{j+1} is satisfied with $q_{j+1,i} = aq_{j,i}$ for $i \le j$ and $q_{j+1,j+1} = b$.

First

$$\xi_{j+1} = \textstyle\sum_{i=1}^{j+1} q_{j+1,i}\eta_i = a\sum_{i=1}^{j} q_{j,i}\eta_i + b\eta_{j+1} = a\xi_j + b\eta_{j+1}$$

so ξ_{j+1} generates K_{j+1}. Secondly

$$\begin{aligned}\textstyle\sum_{i=1}^{j+1} q_{j+1,i}s_{j+1,i} &= ap_{j+1}(\textstyle\sum_{i=1}^{j} q_{j,i}s_{j,i}) + bs_j \\ &= ap_{j+1} + bs_j = 1\end{aligned}$$

Finally $\eta_z = (s_j\xi_j, 0) = s_jb\mu_1 + s_jp_{j+1}\mu_2$ in $K_j \oplus G_{j+1}$, so that $\eta_z = s_{j+1}\xi_{j+1}$ in K_{j+1}. Consequently we conclude that S_j is true for each $j \le r$.

Now let $\xi = \xi_r$, $q_i = q_{r,i}$, and $s = s_r$. Since $S^{2k+1} - F_y \cong S^k \times R^{k+1}$, there is a k-sphere Σ embedded in $S^{2k+1} - F_y$ with $L(\Sigma, F_y) = 1$. We may assume that ξ, η are generators of the k^{th} homology of small tubes about Σ, F_{e_i} respectively. If η is a suitably oriented generator of the k^{th} homology of a small tube about F_y, we have $1 = \ell(\xi,\eta) = \sum_{i=1}^{r} q_i\ell(\eta_i\gamma)$. By Proposition 11 (thinking of y as an exceptional point of degree 1), $\ell(\eta_i,\eta) = \pm L(F_{e_i}, F_y) = \pm|H(f)|/p_i$. Thus, $1 = \sum_{i=1}^{r} q_i(\pm|H(f)|/p_i)$ and

$$s = |H(f)|\textstyle\sum_{i=1}^{r} \pm q_i s_{r,i}.$$

However, by Propositions 9 and 11, s divides $|H(f)|$, so

$s = |H(f)|$.

Corollary 14. *If* $f: S^{2k+1} \longrightarrow S^{k+1}$ *is as above*, k *is odd.*

Proof. Since we may consider nonexceptional points as exceptional points of degree 1, we have $H(f) > 0$, but $H(f) = 0$ if k is even [H].

Corollary 15. *If* f *is as above*, $|E(f)| \leq r$, *where* r *is the number of prime divisors of* $H(f)$.

3. Some Examples

First we extend the technique used for constructing the example in the introduction of a 1-sphere mapping to obtain k-sphere mappings. The technique for obtaining examples with 0, 1, or 2 exceptional points is not new: see [H], [E], and [L 2].

Let $h: S^k \times S^k \longrightarrow S^k \times S^k$ be any homeomorphism, and let $\pi: S^k \times S^k \longrightarrow S^k$ be projection onto the second factor. Define $f_h: S^{2k+1} \longrightarrow S^{k+1}$ to be the composition $S^{2k+1} \cong S^k * S^k \xrightarrow{\gamma} \Sigma(S^k \times S^k) \xrightarrow{\Sigma(\pi h)} \Sigma S^k \cong S^{k+1}$. Then f_h is a k-sphere mapping. Furthermore, if α and β are the generators of $H_k(S^k \times S^k)$ corresponding to $S^k \times \{x\}$ and $\{x\} \times S^k$ for some $x \in S^k$ and $h_*(\alpha) = p\alpha + q\beta$, then f_h has Hopf invariant $|pq|$ and two exceptional fibers of degrees $|p|$ and $|q|$.

To construct more complicated examples, we are led to the question of which automorphisms of $H_k(S^k \times S^k)$ are induced by homeomorphisms. For simplicity, we will assume from now on that k is odd and $k \neq 1, 3, 7$. (If $k = 1, 3, 7$, one can use the multiplication on S^k to construct homeomorphisms.)

If ϕ is an automorphism of $H_k(S^k \times S^k)$ such that $\phi(\alpha) = p\alpha + q\beta$, $\phi(\beta) = r\alpha + s\beta$, we identify ϕ with the matrix $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$. Clearly, the automorphisms $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$ can always be realized as maps induced by homeomorphisms. Let C_k: $S^k \longrightarrow SO(k+1)$ be the characteristic map of the tangent bundle of S^{k+1} [Hs, p. 87] and define h: $S^k \times S^k \longrightarrow S^k \times S^k$ by $h(x,y) = (x, C_k(x)y)$. By [Hs, p. 89], h induces $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Similarly, we can realize $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. (It is not hard to show that the realizations of the elementary matrices we have listed can be composed to realize any automorphism of the form $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ where $p \equiv s \bmod 2$, $q \equiv r \bmod 2$, and $p \not\equiv q \bmod 2$, but we shall not use this fact.) We are now ready to construct examples with many exceptional points.

Theorem 16. *For any* $n \geq 1$, *there are non-degenerate* k-*sphere mappings of* S^{2k+1} *to* S^{k+1} *with* n *exceptional fibers for* $k > 1$, k *odd*.

Proof. For n = 1, 2, we can use the maps f_h constructed above for suitably chosen h. Suppose that $n \geq 2$ and f: $S^{2k+1} \longrightarrow S^{k+1}$ is a nondegenerate k-sphere mapping which has n exceptional points and is locally trivial over the complement of E(f). Write $S^{2k+1} = (S^k \times B^{k+1}) \cup (B^{k+1} \times S^k)$ where the union is along $S^k \times S^k$; and write $S^{k+1} \cong \Sigma S^k = S^k \times [-1,1]/\sim$ where $\sim$ identifies $S^k \times \{-1\}$ to a point called $-\infty$ and $S^k \times \{1\}$ to a point called $+\infty$. We may assume without loss of generality that the exceptional fibers are contained in the interior of $S^k \times B^{k+1}$, $f^{-1}(S^k \times \{0\}) = S^k \times S^k$ and $f^{-1}(S^k \times [-1,0]) = S^k \times B^{k+1}$. Let $b \in S^k \times \{0\}$ be a point of S^{k+1} and let η be a generator of $H_k(F_b)$. It follows from a linking argument that $\eta = \pm p\alpha \pm \beta$, where $p = H(f)$. We assume that $\eta = p\alpha + \beta$; the other cases are similar. Let h be a homeomorphism of $S^k \times S^k$ which realizes $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, and let $X = (S^k \times B^{k+1}) \cup_h (B^{k+1} \times S^k)$. It follows that X is a homotopy sphere and thus X is homeomorphic to S^{2k+1} [St1]. Define $g: X \longrightarrow S^{k+1}$ by

$$g(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in S^k \times B^{k+1} \\ (fh^{-1}(\frac{x}{|x|},y), 1-|x|), & \text{if } (x,y) \in (B^{k+1}-\{0\}) \times S^k; \\ +\infty, & \text{if } (x,y) \in \{0\} \times S^k. \end{cases}$$

Then g is a non-degenerate k-sphere mapping with (n+1)-exceptional fibers and is locally trivial over the complement of the exceptional set. The new exceptional point has degree 2p + 1.

The authors can now remove the non-degeneracy condition from the hypotheses in the paper. The main results, Theorems 8 and 13, remain true with the only condition on the point inverses being that each has the shape of a k-sphere. These results will appear elsewhere.

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