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THE INVERSE LIMIT OF HOMOTOPY EQUIVALENCES BETWEEN TOWERS OF FIBRATIONS IS A HOMOTOPY EQUIVALENCE - A SIMPLE PROOF

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**THE INVERSE LIMIT OF HOMOTOPY
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OF FIBRATIONS IS A HOMOTOPY
EQUIVALENCE—A SIMPLE PROOF**

Ross Geoghegan¹

The following theorem is due to Edwards and Hastings [1; 3.4.1], but their proof is buried in a considerable amount of machinery, both their own and that of Quillen [3]. For some time I have wanted to see an elementary proof in the literature, both because the theorem is obviously relevant to shape theory and related parts of geometric topology, and because my paper [2] relies upon it. The original version of the present note contained a short proof along the same lines as that of Edwards and Hastings: essentially it separated out the relevant parts of [1] and [3]. On reading that proof, J. Dydak suggested further simplifications, and it is this extremely simple version which will be given here (with Dydak's permission). Let me repeat that the theorem is due to Edwards and Hastings, and that this exposition is intended to be merely a service.

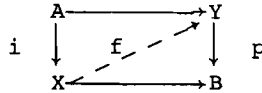
Theorem. Suppose given a strictly commutative diagram of topological spaces and maps

$$\begin{array}{ccccccc}
 \cdots & \leftarrow & X_n & \leftarrow & X_{n+1} & \leftarrow & \cdots \\
 & & \simeq \downarrow f_n & & \simeq \downarrow f_{n+1} & & \\
 \cdots & \leftarrow & Y_n & \leftarrow & Y_{n+1} & \leftarrow & \cdots
 \end{array}$$

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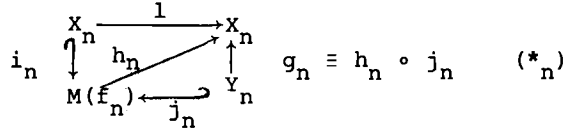
in which each f_n is a homotopy equivalence and each bond is a fibration, then the inverse limit map $f: X \rightarrow Y$ is a homotopy equivalence.

Proof. We will repeatedly use the elementary fact [4; Theorem 3] that if the solid arrow diagram

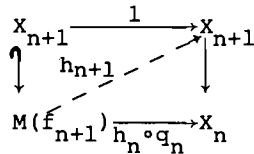


commutes, where i is both a closed cofibration and a homotopy equivalence, and p is a fibration, then f exists making the whole diagram commute.

We claim that maps h_n can be chosen inductively so that the following diagrams $(*_n)$ commute, and commute with the bonds:



Here $M(f_n)$ is the mapping cylinder of f_n , i_n and j_n are the usual inclusions, and the bonding map $q_n: M(f_{n+1}) \rightarrow M(f_n)$ is induced by the given bonds. To prove this claim assume h_1, \dots, h_n exist and obtain h_{n+1} from the following diagram in the obvious way:

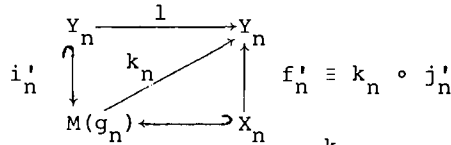


Then let $g_{n+1} = h_{n+1} \circ j_{n+1}$. Claim proved.

The map $H_n: X_n \times I \xrightarrow{\text{identif.}} M(f_n) \xrightarrow{h_n} X_n$ commutes with bonds, and $H_n: 1_{X_n} \simeq g_n \circ f_n$. Let $g = \varprojlim_n g_n$

and $H = \varinjlim_n H_n$. Then $H: l_X \approx g \circ f$.

Now apply the Claim again to get maps k_n so that the following diagrams commute, and commute with the bonds



Define $K_n: Y_n \times I \xrightarrow{\text{identif.}} M(g_n) \xrightarrow{k_n} Y_n$. Let

$f' = \varinjlim_n f'_n$ and $K = \varinjlim_n K_n$. Then $K: l_Y \approx f' \circ g$.

$f' \approx f' \circ g \circ f \approx f$, so g is homotopy inverse for f . The theorem is proved.

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