SHAPE THEORY AND GEOMETRIC PROBLEMS

by

Jack Segal

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1. Introduction

In 1968, K. Borsuk [1] introduced a new area of topology called shape theory. This theory gave a classification of compact metric spaces which was coarser than that of homotopy theory but which coincided with it on ANR's and CW complexes. The theory can be viewed as a Čech homotopy theory since its relationship to homotopy theory is analogous to the relationship between Čech homology and singular homology. Shape theory can be applied effectively to very general spaces in contradistinction to homotopy theory which requires spaces with nice local properties such as CW complexes or ANR's in order to capture their global structure.

In 1971, S. Mardešić and J. Segal [21] described an alternate approach to shape theory for compact Hausdorff spaces based on ANR-systems. If $\underline{X} = (X_\alpha, p_\alpha^{\alpha'}, A)$ and $\underline{Y} = (Y_\beta, q_\beta^{\beta'}, B)$ are inverse systems of compact ANR's over cofinite directed sets with inverse limits X and Y, respectively, then a *shape map* from X to Y is given by a homotopy class of maps of systems $(f_\beta, \phi): \underline{X} \to \underline{Y}$, i.e., where $\phi: B \to A$ is an increasing function and $f_\beta: Y_{\phi(\beta)} \to Y_\beta$ satisfies

$$f_\beta p_{\phi(\beta)}^{\phi(\beta')} \simeq q_\beta^{\beta'} f_{\beta'}, \text{ whenever } \beta \le \beta'.$$

Two such maps of systems (f_β, ϕ) and $(g_\beta, \psi): \underline{X} \to \underline{Y}$ are *homotopic* if for each β there is a $\lambda \ge \phi(\beta), \psi(\beta)$ such that

$$f_\beta p_{\phi(\beta)}^\lambda \simeq g_\beta p_{\psi(\beta)}^\lambda.$$

K. Morita [25] generalized this approach to arbitrary topological spaces. The description which follows is the one given by Dydak and Segal in [8]. One associates with each pointed topological space (X,x) a single inverse system, called the Čech system of (X,x) which is denoted by Č(X,x). The construction of this procedes as follows. Let $(U_\alpha)_{\alpha \in A}$ be the set of all locally finite numerable open coverings of X such that each U_α has exactly one member containing x. If $U_{\alpha'}$ is a refinement of U_α and $\alpha' \neq \alpha$, then we put $\alpha < \alpha'$. For each $\alpha \in A$ let (K_α, k_α) be the nerve of U_α (with the weak topology), where k_α is the vertex corresponding to the unique element of U_α containing x. Then one can obtain a map $(X,x) \to (K_\alpha, k_\alpha)$ with homotopy class $p_\alpha: (X,x) \to (K_\alpha, k_\alpha)$ and a simplicial map $(K_\beta, k_\beta) \to (K_\alpha, k_\alpha)$ for $\beta > \alpha$ with unique homotopy class p_α^β such that $p_\alpha^\beta \cdot p_\beta = p_\alpha$ and $p_\alpha^\beta \cdot p_\beta^\gamma = p_\alpha^\gamma$ for $\alpha < \beta < \gamma$. Then the Čech system of (X,x) is

$$\check{C}(X,x) = ((K_\alpha, k_\alpha), p_\alpha^\beta, A).$$

This is the approach by which Čech dealt with homology.

Let $\underline{q} = (q_\beta)_{\beta \in B}: (X,x) \to ((Y_\beta, y_\beta), q_\beta^{\beta'}, B)$ be a morphism of pro-HT (i.e., the pro-category of the homotopy category of pointed topological spaces). Then $\underline{q}$ is said to satisfy the continuity condition, provided: (1) (factoring) for any map f: (X,x) → (K,k) into (K,k) which is homotopy equivalent to a pointed CW complex there is a map $g_\beta: (Y_\beta, y_\beta) \to$ (K,k) with $[g_\beta] \cdot q_\beta = [f]$ and (2) (short tails) if g,h: $(Y_\beta, y_\beta) \to$ (K,k) are two maps such that $[g] \cdot q_\beta = [h] \cdot q_\beta$, then for some $\beta' \geq \beta$, $[g] \cdot q_\beta^{\beta'} = [h] \cdot q_\beta^{\beta'}$. Then the main

property of Čech systems is that $p_X = (p_\alpha)_{\alpha \in A}$: $(X,x) \to$ Č(X,x) satisfies the continuity condition.

Two important shape invariants are Čech homology and cohomology [21]. It is also possible to describe new continuous functors for an arbitrary topological space X such as the shape groups by taking inverse limits of inverse systems of homotopy groups of inverse systems associated with X. Further, one can use the systems themselves without passing to the limit to obtain the homotopy pro-groups. These homotopy pro-groups are an important shape invariant. Borsuk [3] also introduced a shape invariant called movability. This is a generalization of the notion of ANR. Mardešić and Segal [23] described movability in terms of ANR-systems. An inverse system $\underline{X} = (X_\alpha, p_\alpha^{\alpha'}, A)$ is said to be *movable* provided for any $\alpha \in A$ there is an $\alpha' \geq \alpha$ such that for each $\alpha'' \geq \alpha'$ there is a morphism r: $X_{\alpha'} \to X_{\alpha''}$ with $p_\alpha^{\alpha''} r = p_\alpha^{\alpha'}$. This definition applies in any pro-category. A space (X,x) is called *movable* provided Č(X,x) is movable. The importance of movability stems from the fact that in its presence, one may take the inverse limit of a system without losing algebraic information about the system.

To illustrate this consider the following shape version of the classical Whitehead theorem. (This version is due to M. Moszyńska [27] and was improved upon by Mardešić [20] and J. Keesling [15].

Whitehead Theorem. Let $\underline{f}$: $(X,x) \to (Y,y)$ *be a shape map of pointed continua. If* $\underline{f}$ *induces isomorphism* pro-$\pi_k(\underline{f})$: pro-$\pi_k(X,x) \to$ pro-$\pi_k(Y,y)$ *of the homotopy pro-groups for*

$k \le m + 1$ *and* max(ddim X, ddim Y) $\le$ m, *then* $\underline{f}$ *is a shape*

equivalence.

Note that the shape version applies to continua while
the homotopy version only applied to spaces with strong
local properties such as CW complexes and that the homotopy
pro-groups have replaced the homotopy groups.

Recall that the *deformation dimension*, ddim X, of a
space X is the minimum n such that any map of X into a CW
complex K is homotopic to one whose image lies in the n-
skeleton of K.

The following is a movable version of the Whitehead
Theorem.

Corollary. Let $\underline{f}$: (X,x) → (Y,y) *be a shape map of*
pointed continua such that $\check{\pi}_k(\underline{f})$: $\check{\pi}_k(X,x)$ → $\check{\pi}_k(Y,y)$ *is an*
isomorphism for $k \le m + 1$. *If* (X,x) *and* (Y,y) *are movable*
and max(ddim X, ddim Y) $\le$ m, *then* $\underline{f}$ *is a shape equivalence.*

2. Locally Well-Behaved Shape Representatives

In this section we discuss the question of when a con-
tinuum has the shape of (1) a locally connected continuum
or (2) a CW complex. Borsuk [2] introduced an n-dimen-
sional stratification of movability called n-movability.
The pointed 1-movable case is of special interest because
a continuum possessing this property has the shape of some
locally connected continuum and every locally connected
continuum is pointed 1-movable. Moreover, such continua
can be characterized by a purely algebraic property in terms
of their first homotopy pro-group.

An inverse system $\{(X_\alpha, x_\alpha), p_\alpha^{\alpha'}, A\}$ of the homotopy category of pointed topological spaces and maps preserving base points is n-*movable* iff for each α there is an $\alpha' \geq \alpha$ such that for all $\alpha'' > \alpha$, and any homotopy class f: $(K,k) \to (X_{\alpha'}, x_{\alpha'})$ where K is an n-dimensional CW complex, there exists a homotopy class g: $(K,k) \to (X_{\alpha''}, x_{\alpha''})$ with

$$p_\alpha^{\alpha''} \cdot g = p_\alpha^{\alpha'} \cdot f.$$

Then a pointed topological space (X,x) is n-*movable* if its Čech system Č(X,x) is n-movable. Finally, a continuum X is called *pointed 1-movable* provided (X,x) is 1-movable for each x in X.

A pro-group $\underline{G} = \{G_\alpha, p_\alpha^{\alpha'}, A\}$ is said to satisfy the *Mittag-Leffler condition* provided for any α in A, there exists an $\alpha' \geq \alpha$ such that for any $\alpha'' \geq \alpha'$ we have

$$p_\alpha^{\alpha''}(G_{\alpha''}) = p_\alpha^{\alpha'}(G_{\alpha'}).$$

Further, a pro-group $\underline{G}$ is said to be *stable* if it is isomorphic, as a pro-group, to a group.

Lemma. A pointed continuum (X,x) is 1-movable iff pro-π_1(X,x) satisfies the Mittag-Leffler condition.

Theorem 1. [7] Let (X,x) and (Y,y) be pointed continua. If Sh(X) = Sh(Y) and (X,x) is 1-movable, then Sh(X,x) = Sh(Y,y).

Corollary. A continuum X is pointed 1-movable iff (X,x) is 1-movable for some x in X.

Note that in general Sh(X) = Sh(Y) (i.e., X and Y have

the same unpointed shape) does not imply that Sh(X,x) =

Sh(Y,y) (i.e., X and Y have the same pointed shape) for all

x in X and y in Y. However, Dydak [7] has shown that

pointed 1-movability is an invariant of unpointed shape.

The following result due to Krasinkiewicz [18] charac-

terizes those continua which have the shape of a locally

connected continuum.

Theorem 2. For a continuum X the following conditions

are equivalent:

(a)   there exists a decreasing sequence $X_1, X_2, X_3, \ldots$

of locally connected continua such that $X = \bigcap_{j=1}^{\infty} X_j$ and X_{j+1}

is a strong deformation retract of X_j for $j \geq 1$,

(b)   X has the shape of a locally connected continuum,

(c)   X is pointed 1-movable.

By the Lemma and Theorem 1, condition (c) is equivalent

to pro-π_1(X,x) satisfying the Mittag-Leffler condition.

This result provides a direct computational means to deter-

mine which continua have the shape of a locally connected

continuum. Among the known classes of pointed 1-movable

continua are (1) arcwise connected continua [19], (2)

hereditarily decomposable continua [17], (3) subcontinua

of 2-manifolds [17], [24], (4) continuous images of pointed

1-movable continua [17], [24]. On the other hand, the

dyadic solenoid is not pointed 1-movable.

S. Ferry [13] recently generalized Krasinkiewicz's

result to homotopy locally n-connected continua as follows.

Theorem 3. A continuum X has the shape of an LC^n con-

tinuum iff pro-π_k(X) *is stable for* $0 \leq k \leq n$ *and Mittag-Leffler for* $k = n + 1$.

Borsuk [3] also introduced a shape version of ANR's which is more restrictive than movability and which is called FANR's. We refer to these spaces as absolute neighborhood shape retracts (ANSR's). An important property of pointed ANSR's is that they have the shape of CW complexes. This allows the use of the theory of CW complexes to investigate their geometry. D. A. Edwards and R. Geoghegan [10] have shown that a pointed connected space (X,x) of finite shape dimension has the shape of a pointed CW complex iff each of its pro-groups pro-π_n(X,x) is stable. However, it is not known whether ANSR's are pointed ANSR's. This question has received considerable attention. Geoghegan [14] has shown that a connected ANSR X is a pointed ANSR iff the first derived limit $\lim^1$pro-π_1(X,x) = 0. J. Dydak and P. Minc [7] have described some examples which come close to answering the question.

3. Shape and Some Geometric Problems

One way in which shape theory is related to geometric problems is through the consideration of images of ANR's under CE (or cell-like) mappings (a mapping f: X → Y of X onto Y is said to be a CE map if the inverse image of each point has trivial shape). Between ANR's and, in particular, manifolds CE maps worked well and play a central role in the work of L. C. Siebenmann [28], R. D. Edwards, and J. E. West [32]. However, J. L. Taylor's example [29] of a CE map from the Kahn space (an infinite dimensional acyclic

space of nontrivial shape) onto the Hilbert cube which is
not a shape equivalence showed the need to limit CE maps in
some way in the more general setting. To do this G.
Kozlowski [16] introduced the notion of hereditary shape
equivalence. A mapping f: X → Y of X onto Y is a *hereditary
shape equivalence* iff $f|f^{-1}(C): f^{-1}(C) → C$ is a shape equiva-
lence for all closed subsets C of X. By restricting C to
the points of X one sees that hereditary shape equivalences
are CE maps. Hereditary shape equivalences behave well with
respect to quotients and agree with CE maps on spaces which
have a strong local structure (e.g., CE maps are hereditary
shape equivalences when they map between ANR's or when the
range has finite dimension). Kozlowski used this shape
theoretic notion to give the following elegant characteri-
zation of the CE images of ANR's.

   *Theorem 5. If f: X → Y is a CE map and X is an ANR,
then Y is an ANR iff f is a hereditary shape equivalence.*

   The behavior of CE maps with respect to dimension has
been of considerable interest. Formally the question can
be asked as follows.

   Question 1. Does there exist a GE map which raises
dimension? In [31] J. J. Walsh has pointed out how Ques-
tion 1 is related to the following classical problem.

   Question 2. Does there exist an infinite dimensional
compactum with finite cohomological dimension?

If such a compactum existed it could not contain finite

dimensional subsets with dimension greater than the cohomo-
logical dimension of the compactum. Walsh [30] has recently
constructed infinite dimensional compacta which do not con-
tain n-dimensional (n $\geq$ 1) subsets and has given [31] a
method to show many of the known examples have infinite co-
homological dimension. The *cohomological dimension* of a
space X, cdim X, is defined to be the minimum integer n
such that^{t}, whenever m $\geq$ n and A closed in X, the homomor-
phism i*: $\check{H}^m(X;G) \to \check{H}^m(A;G)$ induced by the inclusion i: A $\subset$ X
is onto. Moreover, it follows from the fact that cohomo-
logical dimension and dimension agree on finite dimensional
spaces and the Vietoris-Begle mapping theorem that the
image of a CE dimension raising map would be infinite dimen-
sional and would have finite cohomological dimension.

In [12] R. D. Edwards indicates why Questions 1 and 2
are equivalent. Moreover, he outlines a program to find a
counterexample in terms of an infinite collection of map-
pings between spheres whose finite compositions are all
essential. Finally, Kozlowski has shown that the problem
of CE dimension raising maps can be formulated in shape
theory as follows.

Question 3. If X is a finite dimensional compactum
and f: X $\to$ Y is a CE map, then is f a shape equivalence?

4. Strong Shape and Some Geometric Problems

Another category, called the strong shape category, is
also closely related to geometric problems. There has been
considerable interest in the strong shape category because

it behaves better with respect to pairs and extensions that
the shape category. Another reason for this interest is
that if every shape equivalence is a strong shape equiva-
lence, then every ANSR is a pointed ANSR. The exact rela-
tionship between the equivalence of these categories is
unresolved at the moment (see [9]). R. Geoghegan [14] has
given a detailed account of the pointed versus the unpointed
case and its importance is shape theory. D. A. Edwards and
H. M. Hastings [11] have given a categorical treatment of
strong shape theory. The objects in their category are
compact Z-sets in the Hilbert cube Q and the morphism
between such compacta X and Y are proper homotopy classes
of maps from Q - X to Q - Y. Most likely they were influ-
enced in this approach by T. A. Chapman's [5] description
of the shape category as one whose objects are complements
of compact Z-sets in Q and whose morphisms are weak proper
homotopy classes of proper maps. A. Calder and H. M.
Hastings [4] have recently announced a purely categorical
description of the strong shape category.

   Dydak and Segal in [9] describe the strong shape cate-
gory as one whose objects are compacta X and whose morphisms
(strong shape maps) from X to Y are natural transformations
from $\Pi_{\mathrm{CTel}Y}$ to $\Pi_{\mathrm{CTel}X}$ where $\Pi_{\mathrm{CTel}(X,A)}: W_p \to Ens$ is the
functor sending a pair of ANR's (P,R) to the set of mor-
phisms in the proper homotopy category from the contractible
telescope CTel(X,A) to (P,R). The category obtained is iso-
morphic to the one previously described by Edwards and
Hastings. However, Dydak and Segal concentrate on finding
geometric conditions for a map to induce a strong shape

equivalence.   They use proper homotopy theory to obtain
geometric information about the strong shape category.   For
example, they are able to reduce the question of whether a
morphism is a strong shape equivalence to verifying whether
other maps are shape equivalences.   This takes various
forms, two of which follow.

   Theorem 6.   Let f: X → Y *be a map of compacta.   Then* f
induces a strong shape equivalence iff for each compactum
Z *containing* X *the natural projection* p: Z → Z ∪$_f$ Y *induces*
a shape equivalence.

   Theorem 7.   Let f: X → Y *be a map of compacta.   Then* f
induces a strong shape equivalence iff f: X → Y *and*
$\hat{f}$: $\hat{M}$(f) → Y *induce shape equivalences.*

   In the above $\hat{M}$(f) denotes the double mapping cylinder
of f: X → Y which is defined as the adjunction space

$$\hat{M}(f) = (X \times [-1,1]) \cup_\phi (Y \times \{-1,1\})$$

where

$$\phi = f \times 1: X \times \{-1,1\} → Y \times \{-1,1\}.$$

The image of (u,t) under the quotient map q: (X × [-1,1]) +
(Y × {-1,1}) → $\hat{M}$(f) is denoted by [u,t].   The map $\hat{f}$: $\hat{M}$(f) → Y
is defined by $\hat{f}$[u,t] = f(u), if (u,t) ∈ X × [-1,1] and by
$\hat{f}$[u,t] = u, if (u,t) ∈ Y × {-1,1}.   Theorem 7 is obtained
from Theorem 6 using a technique due to G. Kozlowski [16].

   Dydak and Segal first prove a strong shape version of
the Fox theorem.   Then the question of whether a strong
shape morphism is a strong shape equivalence is reduced to
the consideration of simpler cases, namely, what happens

with maps induced by inclusions. As an application of their results they are able to determine when various classes of mappings induce strong shape equivalences.

Theorem 8. *A hereditary shape equivalence of compacta induces a strong shape equivalence.*

This theorem follows from Theorem 6 since Kozlowski has shown that any extension of a hereditary shape equivalence is a shape equivalence.

Theorem 9. *If* f: X → Y *is a* CE *map and* ddim X, ddim Y < ∞, *then* f *induces a strong shape equivalence.*

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