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Introduction

During the last years considerable research has been done about what it means for a space that each of its subspaces satisfies a certain, in general not hereditary property. This kind of problem was more or less started by A. V. Arhangel'skiǐ in [1]. He proved that a space, each of which subspaces is a Lindelöf p-space is metrizable. H. Bennett and D. J. Lutzer ([2]) proved that a GO-space that is hereditarily an M-space (p-space) is metrizable. We here prove a theorem about decompositions of GO-spaces from which Bennett and Lutzers result can be derived. Also it can be used to prove something about GO-spaces that are hereditarily a Σ-space.

1. The Decomposition Theorem

First we give some terminology. A subset D of a topological space X is called *discrete in* X if it is closed in X and carries as a subspace the discrete topology. A subset is said to be a *σ-discrete (in* X*)* if it is the union of countably many discrete (in X) sets. For a GO-space $X = (X, \leq, \tau)$ we define

$$E(X) := \{x \in X \mid [x,\rightarrow) \in \tau \text{ or } (\leftarrow, x] \in \tau\}$$

If X is metrizable than obviously E(X) is σ-discrete, since X has a σ-discrete base.

For more terminology on GO-spaces we refer to [4]. Finally we state here a well-known result about decompositions of GO-spaces, which will be used several times.

Proposition 1.1. *If* $X = (X, \leq, \tau)$ *is a* GO-*space and* $\mathcal{D}$ *is an equivalence relation on* X *such that the decomposition space* $X/\mathcal{D}$ *consists of convex closed sets then the triple* $(X/\mathcal{D}, \leq, \delta)$ *is a* GO-*space, where* $\leq$ *is the obvious order on* $X/\mathcal{D}$ *and* δ *is the quotient topology on* $X/\mathcal{D}$.

Theorem 1.2. *Let* $X = (X, \leq, \tau)$ *be a* GO-*space and* $\mathcal{D}$ *an equivalence relation on* X *with convex equivalence classes, such that*

(i) $X/\mathcal{D}$ *is metrizable*

(ii) each equivalence class of $\mathcal{D}$ *has a* G_δ*-diagonal.*

If each subspace of X *is a* p-*space (resp.* M-*space, resp.* Σ-*space) then* X *is metrizable.*

Proof. Denote the quotient space $X/\mathcal{D}$ by dX and let $d\colon X \to dX$ be the quotient map. Whenever $x \in X$ we shall denote $d^{-1}(d(x))$ by $\tilde{x}$. Define the following subsets of X:

$$K: = \{x \in X \mid \tilde{x} = \{x\}\}$$

$$L: = \{x \in X \mid x \text{ is left endpoint of } \tilde{x}, \text{ and } |\tilde{x}| > 1\}$$

$$R: = \{x \in X \mid x \text{ is right endpoint of } \tilde{x}, \text{ and } |\tilde{x}| > 1\}.$$

Moreover

$$A: = L \cup \{x \in R \mid \tilde{x} \text{ has no left endpoint}\}.$$

$$B: = R \cup \{x \in R \mid \tilde{x} \text{ has no right endpoint}\}.$$

The set $V: = \{y \in dX \mid y \text{ is not isolated}\}$ is closed in dX and hence a G_δ-set. Let $O(n)$ $(n=1,2,\ldots)$ be open sets in dX such that $V = \bigcap_{n=1}^{\infty} O(n)$ and $O(n+1) \subset O(n)$, and put $U(n)$: $= d^{-1}[O(n)]$. Now observe that if Z is a subset of X such that $d|Z$ is one-to-one then Z has a G_δ-diagonal, and hence is metrizable, since a GO-space with a G_δ-diagonal is paracompact and hence metrizable if its is a p-space or M-space, by the Okuyama-Borges theorem ([6] or [3]). Also, by [5] a paracompact Σ-space with a G_δ-diagonal is a σ-space, and hence metrizable if it is a GO-space. This implies that $K \cup A$ and $K \cup B$ are metrizable. Clearly A (resp. B) is contained in $E(K \cup A)$ (resp. $E(K \cup B)$) so A (B) is σ-discrete in $K \cup A$ ($K \cup B$ respectively). Consequently A can be written as $\cup_{n=1}^{\infty} A(n)$ where $A(n+1) \supset A(n)$, and for each $x \in K \cup A$ and $n \in \mathbf{N}$ there exists an open (in X) convex neighbourhood $O(x,n)$ of x such that

$$O(x,n) \cap (A(n)\setminus\{x\}) = \emptyset$$

and B can be written as $\cup_{n=1}^{\infty} B(n)$ where $B(n+1) \supset B(n)$ and for each $x \in K \cup B$ and $n \in \mathbf{N}$ there exists an open (in X) convex neighbourhood $U(x,n)$ of x with

$$U(x,n) \cap (B(n)\setminus\{x\}) = \emptyset.$$

We may suppose that if x' belongs to $O(x,n)$ ($U(x,n)$ resp.) and $d(x') \neq d(x)$ then $\tilde{x}'$ is contained in $O(x,n)$ ($U(x,n)$ respectively) for there are at most two points $y \neq d(x)$ such that $d^{-1}(y)$ meets $O(x,n)$ but is not contained in it. Subtracting $d^{-1}(y)$ from $O(x,n)$ for those y, we obtain a set with all the required properties. The same applies to $U(x,n)$. We will now prove that X has a G_δ-diagonal too, from which it follows in all cases that X is metrizable. Let $(\mathcal{V}(n))_{n=1}^{\infty}$ be a sequence of open covers of dX, such that $\cap_{n=1}^{\infty} St(y,\mathcal{V}(n)) = \{y\}$ for each $y \in dX$, and for each $n \in \mathbf{N}$, $y \in dX$ let $W(d^{-1}(y),n)$ be an open neighbourhood of $d^{-1}(y)$ in X that is mapped by d into some element of $\mathcal{V}(n)$, with the additional property that $W(d^{-1}(y),n+1) \subset W(d^{-1}(y),n)$.

Furthermore, for each $y \in dX$ let $(\mathcal{W}_y(n))_{n=1}^{\infty}$ be a sequence of open (in $d^{-1}(y)$) covers of $d^{-1}(y)$ such that $\cap_{n=1}^{\infty} St(x,\mathcal{W}_y(n)) = \{x\}$ for each $x \in d^{-1}(y)$. For each $n \in \mathbb{N}$, $x \in d^{-1}(y)$ choose an open (in $d^{-1}(y)$) neighbourhood $W_y(x,n)$ of x, contained in some element of $\mathcal{W}_y(n)$ such that $W_y(x,n+1) \subset W_y(x,n)$ and such that $W_y(x,n)$ contains no endpoints of $d^{-1}(y)$ except possibly x itself. In particular, this implies that $W_y(x,n)$ is open in X if x is an interior point of $\tilde{x}$.

Now for $x \in X$, $n \in \mathbb{N}$ define $W(x,n)$ as follows:

- if $x \in Int(\tilde{x})$ then $W(x,n):= W_{d(x)}(x,n)$.
- if $x \notin Int(\tilde{x})$ then we have the following possibilities:

(i) $x \in K$

$$W(x,n):= U(x,n) \cap O(x,n) \cap U(n) \cap W(d^{-1}(d(x)),n).$$

(ii) $x \in L$

$$W(x,n):= [(O(x,n) \cap (\leftarrow,x]) \cup W_{d(x)}(x,n)] \cap U(n) \cap W(d^{-1}(d(x)),n).$$

(iii) $x \in R$

$$W(x,n):= [(U(x,n) \cap [x,\rightarrow)) \cup W_{d(x)}(x,n)] \cap U(n) \cap W(d^{-1}(d(x)),n).$$

Observe that in all cases $W(x,n) \cap \tilde{x} = W_{d(x)}(x,n)$ (*)

Put

$$\mathcal{W}(n):= \{W(x,n) \mid x \in X\} \quad (n = 1,2,\ldots)$$

then each $\mathcal{W}(n)$ is an open cover of X. We shall prove that $\cap_{n=1}^{\infty} St(x,\mathcal{W}(n)) = \{x\}$ for each $x \in X$. To this end fix distinct points x_1 and x_2 in X. We claim that there exists a natural number n, depending only on x_1 and x_2 such that each $W(x,n)$ misses either x_1 or x_2.

Let x be an arbitrary element of X. We have the following possible cases:

(I) $d(x_1) \neq d(x_2)$.

Take n such that $d(x_2) \notin St(d(x_1), \mathcal{V}(n))$. Since $W(x,n)$ is contained in $W(d^{-1}(d(x)),n)$ and hence is mapped into some element of $\mathcal{V}(n)$, which cannot contain both $d(x_1)$ and $d(x_2)$ either x_1 or x_2 does not belong to $W(x,n)$.

(II) $d(x_1) = d(x_2)$ (Clearly x_1 and x_2 do not belong to K).

a) $d(x_1)$ is an isolated point of dX.

Take n such that $d(x_1)$ does not belong to $O(n)$ and $x_2 \notin St(x_1, \mathcal{W}_{d(x_1)}(n))$. If $d(x) = d(x_1)$ then (*) and the condition $x_2 \notin St(x_1, \mathcal{W}_{d(x_1)}(n))$ imply that $W(x,n)$ does not contain both x_1 and x_2. If $d(x) \neq d(x_1)$ then $W(x,n) \cap \tilde{x}_1 = \emptyset$ if $W(x,n)$ is contained in $\tilde{x}$, and if $W(x,n)$ is not contained in $\tilde{x}$ then $W(x,n) \subset U(n)$ because $d(x)$ is not isolated; so $W(x,n) \cap \tilde{x}_1$ is empty too.

b) $d(x_1)$ is not an isolated point of dX.

We have three possible subcases

1) $\tilde{x}_1$ has a left endpoint l and no right endpoint. Consequently $l \in A \cap B$.

Take n such that $l \in A(n) \cap B(n)$ and $x_2 \notin St(x_1, \mathcal{W}_{d(x_1)}(n))$. If $d(x) = d(x_1)$ then again (*) implies that $W(x,n)$ misses either x_1 or x_2; if $d(x) \neq d(x_1)$ then $W(x,n) \cap \tilde{x}_1 = \emptyset$ if $W(x,n)$ is contained in $\tilde{x}$. If $W(x,n) \setminus \tilde{x}$ is non-empty then $x \in K \cup A$ or $x \in K \cup B$, and hence $U(x,n)$ (or $O(x,n)$ respectively) is defined and contains $W(x,n) \setminus \tilde{x}$. Consequently, $W(x,n)$ misses l, and hence by the extra condition on $O(x,n)$ (or $U(x,n)$) it also misses x_1.

2) $\tilde{x}_1$ has a right endpoint r and no left endpoint. The argument for this case is completely analogous to that for the preceding case.

3) $\tilde{x}_1$ has a left endpoint l and right endpoint r. Fix n such that $l \in A(n)$, $r \in B(n)$ and $x_2 \notin St(x_1, \mathcal{W}_{d(x_1)}(n))$. Assume that $d(x) \neq d(x_1)$ and that $W(x,n)$ is not contained in $\tilde{x}$ (Else argue as under 1)). Then $W(x,n)\backslash\tilde{x}$ is contained in either $O(x,n)$ or $U(x,n)$, so it misses either l or r, and hence $W(x,n) \cap \tilde{x}_1 = \emptyset$. It follows that in all possible cases x_2 does not belong to $St(x_1, \mathcal{W}(n))$ for some n. So $\bigcap_{n=1}^{\infty} St(x, \mathcal{W}(n)) = \{x\}$ for each $x \in X$ which proves the theorem.

Note. In this proof p-, M-, or Σ-space can of course be replaced by any other property that together with the existence of a G_δ-diagonal implies metrizability in a GO-space, as is clear from the proof. However, the condition cannot be dropped altogether; the lexicographic ordered square L is an example of a non-metrizable space while the equivalence relation $\mathcal{D}$ on L, defined by

$$(x,y)\mathcal{D}(x',y') \iff x = y \qquad ((x,y),(x',y') \in L)$$

satisfies (i) and (ii) in Theorem 1.2.

2. Applications

First we prove the following theorem.

Theorem 1.2. *Let* $X = (X, \leq, \tau)$ *be a* GO-*space such that* $X = A \cup B$ *where* A *and* B *are dense, metrizable subspaces of* X. *Then* X *is metrizable.*

Proof. We claim that a σ-discrete (in A) subset of A is σ-discrete in X. To prove this take a discrete (in A) subset F of A. Since X is hereditarily collectionwise normal, there exists for each $x \in F$ an open (in X) convex neighbourhood U(x) of X such that $U(x) \cap U(x') = \emptyset$ if $x \neq x'$. Since B is dense in X, $\{U(x) \cap B \mid x \in F\}$ is a disjoint collection of non-empty open convex subsets of B. It follows from ([4], Theorem 2.4.5) that this collection can be written as $\cup_{n=1}^{\infty}\mathcal{O}(n)$, where each $\mathcal{O}(n)$ is a discrete collection in B. Now put

$$F(n): = \{x \in F \mid U(x) \cap B \in \mathcal{O}(n)\} \quad (n = 1,2,\ldots).$$

Then each F(n) is discrete in X and $F = \cup_{n=1}^{\infty}F(n)$.

Hence each discrete subset of A is σ-discrete in X, and the same holds for a σ-discrete (in A) subset of A. Of course an analogous statement is true for a σ-discrete (in B) subset of B.

It follows that X has a σ-discrete dense subset, since A has one; moreover $E(X) \subset E(A) \cup E(B)$, and since both E(A) and E(B) are σ-discrete in X, E(X) is σ-discrete in X. Hence, X is metrizable by ([4], Theorem 3.1).

We are now ready to prove Bennett and Lutzers theorem of [2] with the help of Theorem 1.2. Let X be a GO-space that is hereditarily a p-space or an M-space. It was proven in [2] that X is paracompact. Define an equivalence relation $\mathcal{G}_X$ on X by

$$x\mathcal{G}_X y \iff \text{the closed interval between } x \text{ and } y \text{ is compact } (x,y \in X).$$

By ([7], Theorem 2.1.3) the decomposition space $X/\mathcal{G}_X$ is metrizable. Hence, to show that X is metrizable it suffices to show that each equivalence class is metrizable; and to show that an equivalence class G is metrizable, we only have to prove, by paracompactness of X, that for $x,y \in G (x < y)$ the interval $[x,y]$ is metrizable.

Now fix $C = [x,y] \subset X$ such that $[x,y]$ is compact. We define another equivalence relation $\sim$ on C by

$$x \sim y \Longleftrightarrow \text{the closed interval between } x \text{ and } y \text{ is metrizable } (x,y \in C).$$

Clearly, each equivalence class is metrizable, so we have to prove that $C' := C/\sim$ is metrizable. Note that C' is also a hereditary p-space since the quotient mapping is perfect.

Now observe that C' is a compact, connected GO-space, since it cannot have neighbours by the definition of $\sim$. Suppose that C' consists of more than one point; then it is easy to define two disjoint dense subsets P and Q of C' such that $P \cup Q = C'$. Then P and Q are p-spaces; hence the quotient spaces $P/\mathcal{G}_P$ $(\simeq P)$ and $Q/\mathcal{G}_Q (\simeq Q)$ are metrizable by ([7], Theorem 2.1.3). Consequently C' is metrizable by Theorem 2.1, and we are done.

Another application of Theorem 1.2 lies in the field of generalized ordered Σ-spaces. A Σ-*network* for a space X is a σ-locally finite closed cover $\mathcal{F} = \cup_{n=1}^{\infty} \mathcal{F}(n)$ of X (where each $\mathcal{F}(n)$ is locally finite) with the following properties:

(i) $C(x) := \cap \{F | x \in F \in \mathcal{F}\}$ is countably compact

(ii) If U is an open set containing C(x) then there exists an $F \in \mathcal{F}$ such that $C(x) \subset F \subset U \quad (x \in X)$

A space that admits a Σ-network is called a Σ-space (Nagami, [5]).

In [7] we proved the following fact about generalized ordered Σ-spaces: Define for a GO-space X an equivalence relation $\mathcal{L}$ in the following way:

$$x\mathcal{L}y \iff \text{the closed interval between } x \text{ and } y \text{ is a Lindelöf-space } (x,y \in X),$$

and let $lX := X/\mathcal{L}$ be the quotient space. Again, lX is a GO-space, and we have

Theorem 2.2. *Let* $X = (X,\leq,\tau)$ *be a paracompact* GO-*space. Then* X *is a* Σ-*space* $\iff$ lX *is metrizable and each* $L \in X/\mathcal{L}$ *has a* Σ-*network.*

Now the following facts are known about GO-spaces that are hereditarily Σ-spaces (see [7]):

1) Let X be a GO-space that is a hereditary Σ-space. Then X is hereditarily paracompact. (Note that for instance the ordinal space ω_1 is not a hereditary Σ-space; a bi-stationary set in ω_1 is not a Σ-space).

2) Let X be a GO-space that is both a Σ-space and hereditarily paracompact. Then X is first countable.

Corollary. If X *is a hereditarily* Σ-*space then* X *is first countable.*

Furthermore, we state the following theorem, without proof.

Theorem 2.3. ([7], *Theorem* 4.1.3) *Let* $X = (X,\leq,\tau)$ *be a perfectly normal* GO-*space. Then*

$$X \text{ is a } \Sigma\text{-space} \iff X \text{ is an M-space.}$$

It is an unsolved problem whether a GO-space that is a hereditary Σ-space is metrizable. However, with the help of the facts stated above, we are able to prove that the following conjectures are equivalent.

Conjecture I: *Each* GO-*space that is a hereditary* Σ-*space is metrizable.*

Conjecture II: *Each Lindelöf* GO-*space that is a hereditary* Σ-*space is hereditarily Lindelöf.*

That Conjecture II follows if Conjecture I holds, is trivial, so suppose that the second conjecture is true, and let X be a GO-space that is a hereditary Σ-space. By Theorem 2.2. the quotient space lX is metrizable. Hence, by Theorem 1.2. it is sufficient to prove that each L of the decomposition $X/\mathcal{L}$ is metrizable. By paracompactness of X we only have to prove this for each subset $L' \subset L$ with two endpoints. Now let $L' = [a,b]$ be a subset of some $L \in X/\mathcal{L}$. Then L' is Lindelöf by the definition of $\mathcal{L}$, and hence, by Conjecture II it is hereditarily Lindelöf. Consequently, L' is perfectly normal so L' is an M-space by Theorem 2.3. Since the same applies to each subset of L', it follows from the Bennett-Lutzer Theorem that L' is metrizable. Consequently, Conjecture I holds.

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