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Research Announcement:

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by

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1. Introduction

In this paper, the interrelationship among prime ends, indecomposable continua, and the fixed point property, is studied. In particular, easy-to-understand proofs of the following theorems are given:

(1) the Cartwright-Littlewood-Bell (C-L-B) theorem, asserting the existence of a fixed point of any homeomorphism of the plane keeping some continuum invariant;

(2) the Rutt theorems asserting (a) for a non-separating plane continuum M with indecomposable boundary, the existence of a prime end E whose impression is all of Bd M; and (b) for a non-separating plane continuum M, if there is a prime end E whose impression is all of Bd M, then Bd M is either indecomposable or the union of two indecomposable continua.

In addition, corollaries of our methods show

(1) that every extendable homeomorphism of the Lakes-of-Wada continuum in the plane must have a fixed point in the composant accessible from the unbounded complementary domain; and

(2) that if M is a non-separating plane continuum such that $M = \overline{\text{Int } M}$ and Int M is a connected, simply connected domain, and if h: M —>> M is a homeomorphism with Bd M *minimal invariant in* Bd M, then there exists a point x ∈ M such that h(x) = x. (Note that it is still an open question as

to whether the hypothesis that Bd M be minimal invariant in Bd M can be removed.)

The C-L-B Theorem, above, was proved for the orientation preserving (OP) case by Cartwright and Littlewood, using prime end theory. The OP hypothesis was later removed by Bell [2]. The proofs are long and complicated.

In this announcement, we outline a simple proof of the hard part of Bell's Theorem, using both prime end theory (See Section 2 of [3]) and a paper by Sieklucki [10].

The major contributions of this paper are the methods of proof, which we believe, may eventually lead to a proof of the fixed point property for non-separating plane continua. Some related questions are raised at the end of this paper. Complete details will appear in a monograph, to be written jointly with Morton Brown.

2. Sieklucki's Work

In [10], Sieklucki obtains the following result: If $f:X \longrightarrow\!\!> X$ is a fixed point free continuous map of the non-separating plane continuum X onto itself, then there exists an indecomposable continuum $Y \subsetneq \mathrm{Bd}\, X$ such that $F(Y) = Y$. The same result was obtained independently by Bell [2] (Bell's work having been done several years earlier), and later by Iliadis [9]. More important than the result, however, is Sieklucki's method of proof. If one examines his proof carefully, one sees that, if fact, Sieklucki develops some prime end theory and proves an important theorem about prime ends: *If* $h:S^2 \longrightarrow\!\!> S^2$ *is a fixed point free homeomorphism taking a non-separating continuum* X *onto itself, then* h *has a fixed prime end* E *of* $S^2 - X$*; and the impression of* E, I(E), *is an invariant indecomposable continuum on* Bd X. It further follows from his proof that if Y represent a minimal invariant indecomposable continuum on Bd X, then Y is a Lake-of-Wada *type* of continuum. That is, there is at least one Lake-of-Wada type of channel leading to Y. The accessible composant(s), however, need not be 1-1 continuous image(s) of the reals.

Thus, let $h: E^2 \longrightarrow\!\!> E^2$ be a fixed point free homeomorphism, with M non-separating and Bd M minimal invariant in Bd M. Let $\phi: (S^2 - M) \longrightarrow\!\!> \text{Ext } B$(the unit disk) be a C-map (conformal map) of prime end theory, and let E with corresponding point $e \in \text{Bd } B$ be the fixed prime end given by Siekluckі's theorem.

In the proof of the above theorem, Sieklucki constructs a set $K \subseteq S^2 - B$ with connected subset K' having limit point e, and such that each point of K' moves *directly back* when viewed by means of $\phi h \phi^{-1}$ on $S^2 - B$. K' has the further property that it intersects each sufficiently "small" simple closed curve in $S^2 - B$ which contains B in its interior.

3. Outline of Our Proof of Bell's Theorem

The homeomorphism h induces a homeomorphism $g: (S^2 - B) \longrightarrow\!\!> (S^2 - B)$, by prime end theory. We will show that if h is an OR homeomorphism of S^2, then the set K' above forces e to represent an "out channel" - that is, motion away from the fixed point $e \in \text{Bd } B$, by the induced homeomorphism on $\overline{S^2 - B}$.

3.1. *Theorem. Let* h *and* M *be as in Section* 2, *and let* $e \in \text{Bd } B$ *correspond to the fixed prime end* E *given by Sieklucki's proof. If* h *is* OR, *then* e *represents either an* *out channel or an in channel.*

Proof. Let $\{Q_i\}$ be a chain of crosscuts defining the prime end E, and suppose that for each i, there exists $j_i > i$ such that the endpoints of j_i are not either both inward moving or both outward moving. Then $g(\phi(Q_{j_i})) \cap \phi(Q_{j_i}) \neq \phi$. Thus $Q_{j_i} \cap h(Q_{j_i}) \neq \phi$. Now since $\{Q_i\}$ is a chain defining E, $\lim Q_i$ is some point $x \in \text{Bd } M$, and therefore $\lim h(Q_i)$ is $h(x) \in \text{Bd } M$. But for an infinite subsequence, $Q_{j_i} \cap h(Q_{j_i}) \neq \phi$, and the diameter of the crosscuts have limit 0. It follows that $h(x) = x$. But this is a contradiction. Thus from some subscript on, each $\phi(Q_i)$ moves outward or inward.

We next show that they must all move in the same direction. Suppose $\phi(Q_i)$ moves inward and $\phi(Q_j)$ moves outward on $\overline{\text{Ext } B}$. Let S be the simple closed curve formed by $\phi(Q_i) \cup \phi(Q_j) \cup$ the small segments A and B between their endpoints on either side of $e \in \text{Bd } B$. Since g will be OR on $\overline{\text{Ext } B}$, and therefore on Bd B, $g|S$ has no fixed points. Further, either $\phi(S) \subseteq \overline{\text{Int } S}$ or $S \subseteq \overline{\text{Int } \phi(S)}$, and in either case, there is a fixed point in Int S. But if Q_i and Q_j are sufficiently small, g has no fixed point in Int S. (The only fixed point of g in Ext B corresponds to ∞.) This is a contradiction.

The theorem follows.

3.2. *Theorem. Let* h *and* M *be as in Section 2, with* h *being* OR. *Let* $e \in \text{Bd } B$ *correspond to the particular fixed prime end* E *obtained by Sieklucki's proof. Then* e *represents an out channel.*

Proof. By Theorem 3.1, e represents either an in channel or an out channel. Let K' be the connected subset of K discussed in Section 2. Let Q be any sufficiently small crosscut of $S^2 - M$ such that $\phi(Q)$ is a small crosscut of Ext B whose endpoints are on alternate sides of the point $e \in$ Bd B and such that $\phi(Q)$ bounds a convex cell in Ext B. (We note that the construction of ϕ (h^{-1} in Sieklucki's notation) guarantees the existence of a chain of crosscuts $\{Q_i\}$ defining E and with this property.)

We have the following diagram:

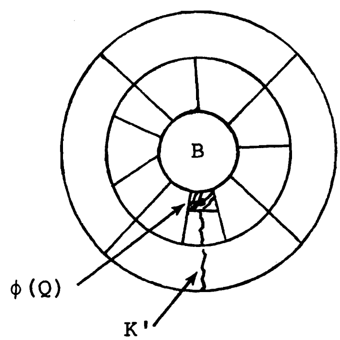

Figure 1

But in Ext B, each point of K' moves *directly back* under the induced homeomorphism g. Thus if $\phi(Q)$ is *any* crosscut in Ext B around e, $g(\phi(Q))$ contains some point outside the small domain cut off by $\phi(Q)$. It follows by the above theorem, that for sufficiently small crosscuts in any chain $\{Q_i\}$ defining E, $\phi(Q_i)$ moves outward, and e represents an out channel.

3.3. *Bell's Theorem. Let* h *be an* OR *homeomorphism of* S^2 *onto itself, keeping the non-separating continuum* M *invariant. We may assume* Bd M *is minimal invariant in* Bd M. *Then there is a point* $x \in M$ *such that* $h(x) = x$.

Proof. If h is fixed point free, then there is an out channel. The existence of an out channel forces the existence of an in channel (the out channel for h^{-1}). One then completes the proof with Bell's argument, showing that the existence of both an out channel and an in channel leads to a contradiction. It follows that h must have a fixed point in M.

4. Questions

(1) Can Sieklucki's proof of the existence of the separating set K', discussed in Section 2, be modified to obtain a similar set K" with the property that each point of K" moves *directly inward* when viewed on Ext B?

If the answer to this question were yes, the techniques of this paper could probably be applied to obtain the fixed point property for arbitrary non-separating plane continua.

(2) Let M be a non-separating plane continuum, $\phi:(S^2 - M) \longrightarrow\!\!> \text{Ext } B$ a C-map. Let $h:S^2 \longrightarrow\!\!> S^2$ be a homeomorphism such that h(M) = M, and let E be a fixed prime end corresponding to the point $e \in \text{Bd } B$ such that each point of I(E) is *principal* and I(E) is indecomposable. Let A_1 and A_2 be 2 endcuts of S^2 - B defining E.

Under what conditions does there exist a homeomorphism $\alpha:\overline{(S^2 - B)} \longrightarrow\!\!> \overline{(S^2 - B)}$ such that $\alpha|\text{Bd } B$ is the identity *and* $\alpha(A_1) = A_2$ *and* $\phi^{-1}\alpha\phi$ extends to a homeomorphism of S^2 which is the identity on M?

(3) Can every hoemomorphism of a non-separating plane continuum M, such that $M = \overline{\text{Int } M}$ and Int M is a connected, simply connected domain, be extended to the plane? If so,

then the f.p.p. holds for M by the C-L-B theorem. (See also the last sentence of the second paragraph of the Introduction.)

(4) (Well-known) If h is an arbitrary homeomorphism of a non-separating plane continuum M. does h have a fixed point (in M)?

(5) (Bing) Does a single Lake-of-Wada continuum have the fixed point property? for homeomorphisms? Does it admit a homeomorphism with exactly one fixed point?

(6) Can every homeomorphism of a single Lake-of-Wada continuum (or Lake-of-Wada type of continuum) be extended to the plane? (Ans: No. This will appear in a forthcoming paper by Brechner and Mayer.)

(7) Can a single Lake-of-Wada (or similar) continuum be imbedded in the plane, so that the complementary "channel" disappears? Can every such continuum be so imbedded?

Remark. Partial answers to (5) and (7) will also appear in the paper mentioned in (6).

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