TOPOLOGY PROCEEDINGS
Volume 4, 1979
Pages 239–263

http://topology.auburn.edu/tp/

Research Announcement:

A SUMMARY OF RESULTS ON ORDER-CAUCHY COMPLETIONS OF RINGS AND VECTOR LATTICES OF CONTINUOUS FUNCTIONS

by

M. Henriksen

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

Research Announcement

A SUMMARY OF RESULTS ON ORDER-CAUCHY COMPLETIONS OF RINGS AND VECTOR LATTICES OF CONTINUOUS FUNCTIONS

M. Henriksen

This paper is a summary of joint research by F. Dashiell, A. Hager and the present author. Proofs are largely omitted. A complete version will appear in the Canadian Journal of Mathematics. It is devoted to a study of sequential order-Cauchy convergence and the associated completion in vector lattices of continuous functions. Such a completion for lattices C(X) is related to certain topological properties of the space X and to ring properties of C(X). The appropriate topological condition on the space X equivalent to this type of completeness for the lattice C(X) was first identified for compact spaces X in [D]. This condition is that every dense cozero set S in X should be C*-embedded in X (that is, all bounded continuous functions on S extend to X). We call Tychonoff spaces X with this property <u>quasi</u>-F <u>spaces</u> (since they generalize the F-spaces of [GH]).

In Section 1, the notion of a completion with respect to sequential order convergence is first described in the setting of a commutative lattice group G. A sequence $\{g_n\}$ in G is said to be o-<u>Cauchy</u> if there exists a decreasing sequence $\{u_n\}$ with $\wedge u_n = 0$ in G and $|g_n - g_{n+p}| \leq u_n$ for all n,p. If there exist such a sequence $\{u_n\}$ and a $g \in G$ with $|g_n - g| \leq u_n$, then $\{g_n\}$ o-<u>converges</u> <u>to</u> g. G is o-<u>Cauchy</u> <u>complete</u> if each o-Cauchy sequence o-converges to some $g \in G$. We give an abstract characterization of this completion and show how it applies to vector lattices and to certain lattice-ordered rings (including function rings) which satisfy a mild continuity condition for the multiplication.

In Section 2, the discussion of Section 1 is specialized to the lattice-ordered algebra C(X) of all continuous real-valued functions on a Tychonoff space X. In Theorem 2.1, the o-Cauchy completion of C(X) is described as the algebra of all bounded continuous functions defined on some countable intersection of dense cozero sets in βX (the domain depending on the function).

In Section 3, the description of the o-Cauchy completion of C(X) and of the subalgebra C*(X) of bounded functions is made more explicit. It is described in terms of the uniform completion of certain algebras of functions defined on dense cozero subsets of X or of βX (see Corollary 3.5). It is shown in Theorem 3.7 that for any Tychonoff space X, C(X) is o-Cauchy complete if and only if X is a quasi-F space (as defined above). For every space X, the o-Cauchy completion of C*(X) takes the form C(K(X)) for a certain compact space K(X) (which is necessarily a quasi-F-space). We show how to construct K(X) as the inverse limit space of {βS: S is a dense cozero set in X}, as well as two other equivalent inverse limit constructions. An example is given of a C(X) whose o-Cauchy completion is not a C(Y).

In Section 4, the above mentioned space K(X), for compact X, is characterized by the property of being a quasi-F space admitting a continuous irreducible surjection onto X which is minimal in a certain natural sense. Accordingly, we call K(X) the minimal quasi-F cover of X. This is similar to the description of Gleason's minimal projective cover G(X) for a compact X[Gℓ] as being the only extremally disconnected space admitting a continuous irreducible surjection onto X. We show that, for an arbitrary X, the o-Cauchy completion of C(X) coincides with the Dedekind completion if and only if K(X) = G(βX), and this is true whenever every dense open subset of X contains dense cozero set.

In Section 5, we study quasi-F spaces *per se* and characterize them in terms of the ring C(X). If βX is zero-dimensional, then X is a quasi-F space if and only if every non-divisor of zero in C(X) is a multiple of its absolute value, but the sufficiency can fail if X is not strongly zero-dimensional. A σ-compact space is a quasi-F-space if and only if each of its dense Baire sets is C*-embedded. First countable quasi-F spaces are discrete. Every Tychonoff space is a closed subspace of some quasi-F space. We conclude with some results on products of quasi-F spaces.

1. Order-Cauchy Completions of ℓ-Groups

The term "ℓ-group" will be used to denote a commutative lattice-ordered group G(+, ∨, ∧), where, as usual a ∨ b, respectively, a ∧ b denote the least upper bound and the greatest lower bound of a and b. We let $a^+ = a \vee 0$, $a^- = (-a) \vee 0$ and $|a| = a^+ + a^-$. We will write $g_n \downarrow$ if the sequence $\{g_n\}$ is decreasing; if in addition $\wedge g_n = g$, we will write $g_n \downarrow g$. Increasing sequences are handled similarly. An embedding $G \hookrightarrow H$ of G into an ℓ-group H is called σ-regular if it preserves all existing countable suprema and infima in G; that is if $g_n \downarrow 0$ in G implies $g_n \downarrow 0$ in H.

1.1. *Definitions.* Suppose G is an ℓ-group, $\{g_n\}$ is a sequence in G, and $g \in G$.

(a) The sequence $\{g_n\}$ order-converges (or o-converges) to g, written $g_n \xrightarrow{o}$ or o-lim $g_n = g$, if $|g_n - g| \le u_n$, for $n = 1,2,3,\cdots$, for some $u_n \downarrow 0$ in G. (Such limits are unique.)

(b) The sequence $\{g_n\}$ is order-Cauchy (or o-Cauchy) if, for some $u_n \downarrow 0$ in G, $|g_n - g_{n+p}| \le u_n$ for all n,p.

(c) G is called order-Cauchy complete (or o-Cauchy complete) if each o-Cauchy sequence in G o-converges to a limit in G.

We are interested in constructing a minimal "completion" of ℓ-groups G with respect to o-Cauchy sequences. Our applications to follow are concerned with richer structure (i.e., $G = C(X)$), and it is pertinent to ascertain what algebraic structure is preserved by this completion process. Accordingly, we take the following as our definition of completion.

1.2. *Definition.* Let L denote any subcategory of ℓ-groups (e.g., ℓ-groups, vector lattices, ℓ-rings, ℓ-algebras). For G in L, an o-Cauchy completion of G (in L) is an H in L together with an L-embedding $G \hookrightarrow H$ satisfying:

(a) H is o-Cauchy complete;

(b) G is σ-regular in H; and

(c) for each $h \in H$ there exist sequences $\{g_n\}$, $\{u_n\}$ in

G with $u_n \downarrow 0$ and $|g_n - h| \leq u_n$, $n = 1,2,\cdots$.

Such an H is called *essentially unique* (*in* L) if, for every H' which is an o-Cauchy completion of G in L, there is an L-isomorphism from H onto H' which restricts to the identity on G.

We record below (1.3 and 1.5) two lemmas due to Papangelou which are used in the construction of a completion, in the proof of its uniqueness, and in subsequent material.

1.3. *Lemma.* [P, 2.10]. *A sequence* $\{g_n\}$ *in an* ℓ*-group* G *is* o*-Cauchy if and only if there exist sequences* $\{u_n\}$, $\{v_n\}$ *in* G *such that* $u_n \leq g_n < v_n$ *for all* n, $\{u_n\}$ *is increasing,* $\{v_n\}$ *is decreasing, and* $\wedge(v_n - u_n) = 0$ *in* G.

The following is immediate.

1.4. *Corollary. The* ℓ*-group* G *is* o*-Cauchy complete if and only if for every increasing sequence* $\{u_n\}$ *in* G *sitting below a decreasing sequence* $\{v_n\}$ *with* $\wedge(v_n - u_n) = 0$, *there exists* $g \in G$ *with* $u_n \leq g \leq v_n$ *for all* n *(and hence* $g = \vee u_n = \wedge v_n$*).*

Given ℓ-groups G and H and an ℓ-group embedding $G \hookrightarrow H$, let G_1^H consist of all $h \in H$ for which there exist sequences $\{g_n\}$, $\{u_n\}$ in G such that $u_n \downarrow 0$ in G and $|g_n - h| \leq u_n$.

1.5. *Lemma.* [P, 3.3]. *Suppose* G *is* σ*-regular in* H *and* $\{v_n\}$ *is a decreasing sequence in* G_1^H *with* $v_n \downarrow v$ *for some* $v \in H$. *Then there exists a decreasing sequence* $\{u_n\}$ *in* G *with* $u_n \geq v_n$ *for all* n *and* $u_n \downarrow v$. *The corresponding* *statement for increasing sequences also holds.*

By "*ℓ-ring*" we mean an ℓ-group G with a multiplication making G into a ring satisfying $xy \geq 0$ whenever $x \geq 0$ and $y \geq 0$ in G (see [F] or [BKW]). In order to construct an o-Cauchy completion for ℓ-rings G, it seems necessary to assume some kind of order continuity for the multiplication in G, for example:

(*) *If* $u_n \downarrow 0$ *in* G *and* $h \geq 0$ *then* $hu_n \downarrow 0$ *and* $u_n h \downarrow 0$.

1.6. *Lemma. Suppose* G *is an ℓ-ring satisfying* (*).

(a) If $g_n \xrightarrow{o} g$ *and* $h_n \xrightarrow{o} h$, *then* $g_n h_n \xrightarrow{o} gh$.

(b) If H *is an ℓ-ring and* G *is embedded as a σ-regular sub-ℓ-ring of* H, *then* G_1^H *is a σ-regular sub-ℓ-ring of* H, *and* G_1^H *satisfies* (*).

We can now state the main theorem of this section.

1.7. *Theorem. Suppose* G *is an ℓ-group (resp. vector lattice, ℓ-ring satisfying* (*), *ℓ-algebra satisfying* (*)*). Then* G *has an essentially unique* o*-Cauchy completion* H *among ℓ-groups (resp. vector lattices, ℓ-rings, ℓ-algebras). Moreover,* H *is minimal in the sense that if* H' *is an* o*-Cauchy complete ℓ-group and* $\phi: G \longrightarrow H'$ *is a σ-regular ℓ-group embedding, then there is a unique order-preserving* $\tilde{\phi}: H \longrightarrow H'$ *extending* ϕ, *and* $\tilde{\phi}(H) = \phi(G)_1^{H'}$. *The map* $\tilde{\phi}$ *is necessarily an ℓ-group embedding, and if in addition* $\phi: G \longrightarrow H$ *is an embedding of vector lattices (ℓ-rings, ℓ-algebras), then so is* $\tilde{\phi}$.

An ℓ-ring is called an f-*ring* if $g \wedge h = 0$ and $f \geq 0$

imply $fg \wedge h = gh \wedge h = 0$, or equivalently if it is a subdirect sum of totally ordered rings [F]. The following is due independently to Bernau [B, p. 622] and Johnson [J].

1.8. *Lemma. Every Archimedean* f-*ring* G *satisfies property* (*).

1.9. *Corollary. Every lattice-ordered ring (respectively, lattice-ordered-algebra) of real-valued functions on some set (with pointwise operations) has an essentially unique* o-*Cauchy completion in* ℓ-*rings (respectively, in* ℓ-*algebras).*

We close this section with two remarks.

(i) It can be shown if $G = Ba_1\,[0,1]$ and $H = Ba_2\,[0,1]$ denote, respectively, the functions of Baire class 1 and 2 on [0,1], then each of G and H are o-Cauchy complete, and the natural embedding of G into H is σ-regular. Hence condition defin tion (c) of Definition 1.2 cannot be replaced by the requirement that each h in H be the o-limit <u>in</u> H of a sequence in G.

(ii) Condition (*) is not always necessary for the existence of an order-Cauchy completion of an ℓ-ring. For example if G = R[x] is the ring of polynomials with real coefficients lexicographically ordered by terms of highest degree, then G is o-Cauchy complete but fails to satisfy (*).

We do not know of any necessary and sufficient condition on an ℓ-ring to guarantee that multiplication is preserved under the embedding described in Theorem 1.7.

2. The o-Cauchy Completion of C(X)

We now specialize the discussion of §1 to the ℓ-algebra C(X) of all continuous real-valued functions on the Tychonoff space X (equipped with pointwise operations). The sub-ℓ-algebra of bounded functions is denoted C*(X). In this section and the next we describe the o-Cauchy completion of C(X) (see 1.9) in several ways as ℓ-algebras of functions, and for compact X we obtain in fact a C(K) for a certain compact space K.

Some terminology: For f: X $\longrightarrow$ R, the *cozero set* of f is coz f = $\{x | f(x) \neq 0\}$ and the *zeroset* is Z(f) = X - coz f. In a topological space X, a *cozero set* is a set coz f for some f $\in$ C(X). For X compact Hausdorff (or just normal), the cozero sets are exactly the open F_σ's. βX will denote the Stone-Čech compactification of X. For its properties, see [GJ, Chapter 6].

The method of construction employed here is quite similar to the method of [FGL, §2.4 and §4.1]. We first recall the generalities. Suppose $\mathcal{J}$ is a filter base of dense subsets of a topological space X, i.e., $\mathcal{J}$ is a family of dense, nonempty subsets of X closed under finite intersections. Consider the set of all functions f $\in$ C(S) for some S $\in \mathcal{J}$, and identify f $\in$ C(S) with g $\in$ C(T) if and only if f = g on S $\cap$ T. Denote the set of all equivalence classes by C[$\mathcal{J}$], and let C*[$\mathcal{J}$] denote all the equivalence classes containing bounded functions. Alternatively, observe that {C(S): S $\in \mathcal{J}$} or {C*(S): S $\in \mathcal{J}$} form directed systems, where S $\supset$ T in $\mathcal{J}$ yields the bonding homomorphism f $\longrightarrow$ f|T for f $\in$ C(S) (or f $\in$ C*(S)). Then C[$\mathcal{J}$] and C*[$\mathcal{J}$] are the direct limits $\varinjlim \{C(S): S \in \mathcal{F}\}$ and $\varinjlim \{C^*(S): S \in \mathcal{F}\}$. One easily checks that $C[\mathcal{F}]$ and $C^*[\mathcal{F}]$ are ℓ-algebras under the operations canonically induced by the $C(S)$. Furthermore, each $C(S)$ or $C^*(S)$ for $S \in \mathcal{F}$ is isomorphically embedded as an ℓ-algebra into $C[\mathcal{F}]$ or $C^*[\mathcal{F}]$, since each S is dense. In particular, if $X \in \mathcal{F}$, then $C(X)$ and $C^*(X)$ are sub-ℓ-algebras of $C[\mathcal{F}]$ and $C^*[\mathcal{F}]$.

As a notational convenience, we shall write $f \in C[\mathcal{F}]$ if $f \in C(S)$ for some $S \in \mathcal{F}$, thus ignoring the distinction between equivalence classes and representatives. In this case, we write $S = \text{dom } f$.

If $\mathcal{F}$ is a filter base of dense sets in βX and $\mathcal{F}$ contains all the dense cozero sets of βX, then there is a natural embedding of $C(X)$ into $C[\mathcal{F}]$, as follows. Each $f \in C(X)$ has a unique Stone-Čech extension $\beta f: \beta X \longrightarrow R \cup \{\infty\}$ (the one-point compactification of R), and if $\text{fin}(f) = (\beta f)^{-1}(R)$ then $\text{fin}(f) \in \mathcal{F}$ and $(\beta f)|\text{fin}(f) \in C(\text{fin}(f))$. This provides the canonical ℓ-algebra embedding $C(X) \longrightarrow C[\mathcal{F}]$. Moreover, this embedding induces an embedding $C^*(X) \longrightarrow C^*[\mathcal{F}]$.

If $\mathcal{F}$ is a filter base of dense sets in βX containing all the dense cozero sets in βX, there is an ℓ-algebra intermediate between $C^*[\mathcal{F}]$ and $C[\mathcal{F}]$ which is central to our subject. This is defined to be

$$C^\#[\mathcal{F},X] = \{h \in C[\mathcal{F}]: |h| \leq f \text{ for some } f \in C(X)\},$$

where we have assumed $C(X) \subset C[\mathcal{F}]$ by the above canonical embedding. Thus, $C^*[\mathcal{F}] \subset C^\#[\mathcal{F},X] \subset C[\mathcal{F}]$, and in case X is compact, then $C^*[\mathcal{F}] = C^\#[\mathcal{F},X]$. In the following, the dependence of $C^\#[\mathcal{F},X]$ on the space X will be implicitly understood, and we shall for convenience suppress explicit mention of X and write simply $C^{\#}[\mathcal{J}]$.

The results of [FGL] deal primarily with the case where $\mathcal{J}$ is taken to be either the family of dense open sets or the family of dense G_δ sets in X (for the latter, X is assumed compact, and closure under finite intersections follows from the Baire category theorem). It turns out that the structure required for the present purposes is obtained by taking for $\mathcal{J}$ either the family of dense cozero sets or the family of countable intersections of dense cozero sets. These families are denoted $\mathcal{C}(X)$ and $\mathcal{C}_\delta(X)$, respectively. Some of the results here are exactly analogous to the corresponding results in [FGL], but the proofs are different, apparently of necessity.

The main result of this section now follows. It is analogous to the representation of the Dedekind MacNeille completion (by cuts) of C(X) as $C^{\#}[\mathcal{G}_\delta]$, where $\mathcal{G}_\delta$ is the class of all dense G_δ-sets in βX (see [FGL], 4.11 and 4.6]).

2.1. *Theorem. The* o-*Cauchy completion of* C(X) *(as an* ℓ*-algebra) is* $C^{\#}[\mathcal{C}_\delta(\beta X)]$.

Some preliminary facts are needed to outline the proof. Recall that a subgroup G of an ℓ-group H is called *order-convex* if $o \leq h \leq g$ and $g \in G$ imply $h \in G$.

2.2. *Lemma. Any order-convex sub-*ℓ*-group* G *of an* o-*Cauchy complete* ℓ*-group* H *is itself* o-*Cauchy complete.*

2.3. A subspace S of a space X is called z-*embedded* if whenever Z is a zero-set in S, then $Z = Z' \cap S$ for some zero-set Z' in X. Since every $S \in \mathcal{C}_\delta(\beta X)$ is a Baire set in βX and is therefore Lindelöf [CN, p. 77], and a Lindelöf subspace is always z-embedded [CN, p. 79], each $S \in \mathcal{C}_\delta(\beta X)$ is z-embedded in every superspace.

The following approximation property characterizes z-embedded subspaces. See [H] or [BH] for a proof.

2.4. *Lemma.* S *is* z-*embedded in* X *if and only if given* $h \in C(S)$ *and* $\varepsilon > 0$ *there exist a cozero set* T *in* X *with* $S \subset T$ *and* $g \in C(T)$ *such that* $|h(x) - g(x)| < \varepsilon$ *for* $x \in S$.

By using 2.4, we can establish:

2.5. *Lemma. Suppose* S *is* z-*embedded in* X, $h \in C(S)$, *and there exists* $f \in C(X)$ *such that* $|h(x)| \leq f(x)$ *for all* $x \in S$. *Then there exist sequences* $\{u_n\}$ *and* $\{v_n\}$ *in* C(X) *such that* $u_1 \leq u_2 \leq \cdots \leq v_2 \leq v_1$, *and for each* $x \in S$

$$h(x) = \sup u_n(x) = \inf v_n(x).$$

Proof of 2.1. (Outline) We need to show that the ℓ-algebras $G = C(X)$ and $H = C^{\#}[\mathcal{C}_\delta(\beta X)]$ satisfy the three conditions of Definition 1.2.

To prove conditions (a) and (b) of Definition 1.2, we show first the following:

(†) If $S \in \mathcal{C}_\delta(\beta X)$, $w_n \downarrow 0$ in C(S), and $T = \{x \in S: w_n(x) \longrightarrow 0\}$, then $T \in \mathcal{C}_\delta(\beta X)$.

To show (a) that H is o-Cauchy complete, it suffices by Lemma 2.2 to show that $C[\mathcal{C}_\delta(\beta X)]$ is o-Cauchy complete with the aid of Corollary 1.4 and (†). To show (b), we use (†) again, and to obtain (b), use is made of Lemma 2.5.

2.6. *Corollary. The* o-*Cauchy completion of* $C^*(X)$ *is* $C^*[\mathcal{C}_\delta(\beta X)]$. *In particular, for compact* X, *the* o-*Cauchy completion of* $C(X)$ *is* $C[\mathcal{C}_\delta(X)]$.

3. More on the o-Cauchy Completion of C(X)

In order to amplify the description of the o-Cauchy completion of C(X) given in 2.1, we need to study the relationship between the ℓ-algebra $C[\mathcal{J}]$ for various filter-bases $\mathcal{J}$ of dense sets in X or in βX. We will be specifically concerned with $\mathcal{C}(X)$, $\mathcal{C}(\beta X)$, and $\mathcal{C}_\delta(\beta X)$.

Observe first that $C[\mathcal{C}(\beta X)]$ is embedded as a sub-ℓ-algebra of $C[\mathcal{C}(X)]$ by restriction: if $S \in \mathcal{C}(\beta X)$ and $f \in C(S)$ then $S \cap X \in \mathcal{C}(X)$ and $f|(S \cap X) \in C[\mathcal{C}(X)]$. By abuse of notation we write $C[\mathcal{C}(\beta X)] \subset C[\mathcal{C}(X)]$. This relation is in fact an equality, as the following lemma will show. The essence of this result is contained in [FGL, 3.8]. Recall that a subspace S of X is C^*-*embedded* if every $f \in C^*(S)$ extends to some $\bar{f} \in C^*(X)$.

3.1. *Lemma. Let* X *be a* C^*-*embedded subspace of a Tychonoff space* Y. *Every continuous function on a cozero subset of* X *extends continuously to a cozero subset of* Y.

3.2. *Corollary.*

$$C[\mathcal{C}(\beta X)] = C[\mathcal{C}(X)].$$
$$C^{\#}[\mathcal{C}(\beta X)] = C^{\#}[\mathcal{C}(X)].$$
$$C^*[\mathcal{C}(\beta X)] = C^*[\mathcal{C}(X)].$$

If $\mathcal{J}$ is a filter base of dense sets in X, $C[\mathcal{J}]$ has a natural metric topology: the topology of uniform convergence, in which a sequence $\{f_n\}$ converges to f if and only if for each $\varepsilon > 0$, eventually $|f_n - f| \leq \varepsilon \cdot 1$ in the lattice $C[\mathcal{J}]$.

The following lemma is [FGL, 4.5].

3.3. *Lemma. If* $\mathcal{J}$ *is closed under countable intersections then* $C[\mathcal{J}]$ *is uniformly complete.*

3.4. *Proposition.* $C[\mathcal{C}(\beta X)]$ *is uniformly dense in* $C[\mathcal{C}_\delta(\beta X)]$, *so that* $C[\mathcal{C}_\delta(\beta X)]$ *is the uniform completion of* $C[\mathcal{C}(X)]$ *(or of* $C[\mathcal{C}(\beta X)]$*).*

3.5. *Corollary. The* o-*Cauchy completion of* $C(X)$ *(i.e.,* $C^{\#}[\mathcal{C}_\delta(\beta X)]$*) is the uniform completion of* $C^{\#}[\mathcal{C}(X)] = C^{\#}[\mathcal{C}(\beta X)]$. *The* o-*Cauchy completion of* $C^*(X)$ *(i.e.,* $C^*[\mathcal{C}_\delta(\beta X)]$*) is the uniform completion of* $C^*[\mathcal{C}(X)] = C^*[\mathcal{C}(\beta X)]$.

The analogues of 3.2 and 3.4 for dense open sets are proved in [FGL].

Recall that X is an F-*space* if each of its cozero sets is C*-embedded (See [GJ, 14.25]).

3.6. *Definition.* A Tychonoff space is called a quasi-F-space if each dense cozero set is C*-embedded.

The following result was originally proved in [D] for the case of compact X by a rather more direct argument. An extensive description of quasi-F spaces is given in Section 5.

3.7. *Theorem. For an arbitrary Tychonoff space* X, $C(X)$ *is* o-*Cauchy complete if and only if* X *is a quasi*-F *space.*

We recall some generalities about direct and inverse limits. Let $\{K_a\}$ be any inverse system of compact Hausdorff spaces with respect to surjections $\pi_a^b\colon K_b \longrightarrow K_a$ for $a \leq b$. Then $\{C(K_a)\}$ is a direct system of ℓ-algebras with respect to the embeddings $f_a \longrightarrow f_b = f_a \circ \pi_a^b$, $a \leq b$. The *inverse limit space* $K = \varprojlim_a K_a$ is a compact Hausdorff space and the direct limit $A = \varprojlim_a C(K_a)$ is an ℓ-algebra.

3.8. *Theorem.* [FGL, 6.8]. *The ℓ-algebra* $A = \varprojlim_a C(K_a)$ *is isomorphic with a uniformly dense sub-ℓ-algebra of* $C(K)$, *where* $K = \varprojlim_a K_a$, *and* K *is the maximal ideal space of* A.

For a Tychonoff space X, we consider the directed systems $\mathcal{C}(X)$ of dense cozero sets in X and $\mathcal{C}_\delta(\beta X)$ of dense countable interesections of cozero sets in βX. The system $\{\beta S\colon S \in \mathcal{C}(X)\}$ is an inverse system of compact spaces with respect to the surjections $\pi_T^A\colon \beta S \longrightarrow \beta T$ which extend the inclusions $S \subset T$. Similarly, $\{\beta S\colon S \in \mathcal{C}_\delta(\beta X)\}$ is an inverse system of compact spaces. We now define the inverse limit spaces

$$K(X) = \varprojlim \{\beta S\colon S \in \mathcal{C}(X)\}$$

and

$$K_\delta(X) = \varprojlim \{\beta S\colon S \in \mathcal{C}_\delta(\beta X)\}.$$

Since $K(X)$ is a certain subset of $\prod\{\beta S\colon S \in \mathcal{C}(X)\}$, there exists a natural, continuous surjection and projection $\pi_X\colon K(X) \longrightarrow \beta X$. This induces a natural embedding of $C(\beta X)$ into $C(K(X))$ by $f \longrightarrow f \circ \pi_X$. Since $C^*(X)$ is isomorphic with $C(\beta X)$, $C^*(X)$ is naturally embedded as a sub-ℓ-algebra of $C(K(X))$.

3.9. *Theorem.* *(a) The spaces* $K(X)$, $K(\beta X)$, *and* $K_\delta(X)$ *are all homeomorphic and are quasi*-F *spaces.*

(b) The natural embedding $C^*(X) \longrightarrow C(K(X))$ *is a realization of the* o-*Cauchy completion of* $C^*(X)$ *as the space* $C(K(X))$.

In contrast to 3.9, if X fails to be compact, the o-Cauchy completion of C(X) need not be a C(Y). Such an example may be constructed with the aid of Proposition 4.6 below (which gives a sufficient condition for K(X) to coincide with the Gleason cover) and enables us to modify an example given in [MJ] of a space X such that the Dedekind-MacNeille completion of C(X) is not a C(Y).

4. The Quasi-F Cover

Next, we examine some of the properties of the pair $(K(X), \pi_X)$, which we shall call the *minimal quasi*-F *cover* of X, where π_X: $K(X) \longrightarrow X$ is the canonical projection (see 4.3).

Recall that a map π: $X \longrightarrow Y$ is *irreducible* if X is the only closed subspace of X whose image under π is all of Y. A subset G of an ℓ-group H is *order-dense* if for each nonzero $h \geq 0$ in H there exists a nonzero $g \in G$ with $0 \leq g \leq h$. The following lemma appears in [We, p. 17].

4.1. *Lemma.* *If* X *and* Y *are compact then a map* π: $X \longrightarrow Y$ *is irreducible if and only if the dual embedding* π°: $C(Y) \longrightarrow C(X)$ *has an order-dense image in* C(X).

4.2. *Definition.* A minimal quasi-F cover for a compact space X is a pair (K,π) such that:

(a) K is a compact quasi-F space;

(b) $\pi: K \longrightarrow X$ is a continuous irreducible surjection;

(c) if (K_1, π_1) is a pair satisfying (a) and (b) then there exists a continuous surjection $\tau: K_1 \longrightarrow K$ such that $\pi_1 = \pi \circ \tau$.

4.3. *Theorem. If* X *is compact, then* $(K(X), \pi_X)$ *is a minimal quasi-*F*-cover which is unique in the sense that if* (K, π) *is a minimal quasi-*F*-cover, then there exists a unique homeomorphism* $\tau: K \longrightarrow K(X)$ *such that* $\pi = \pi_X \circ \tau$.

4.4. *Remarks.* As continuous surjection $\pi: K \longrightarrow X$ is called *strongly irreducible* if for every cozero set $V \subset K$, there is a cozero set $W \subset X$ such that $\pi^{-1}[W]$ is dense in V. F. Dashiell has shown that the projection map $\pi: K(X) \longrightarrow X$ is strongly irreducible if X is compact and that a strongly irreducible map of a compact space onto a quasi-F-space is a homeomorphism.

In as yet unpublished work, Charles Neville has determined some classes of mappings for which quasi-F-spaces become projective in the sense of [Gℓ].

For any ℓ-group G, the o-Cauchy completion and the Dedekind-MacNeille completion by cuts are the same if and only if the o-Cauchy completion (as given in Theorem 1.7) is Dedekind complete.

4.5. *Proposition. Let* X *be a Tychonoff space. The following are equivalent:*

(1) The o*-Cauchy completion of* C(X) *is Dedekind complete.*

(2) The o*-Cauchy completion of* C*(X) *is Dedekind complete.*

(3) K(X) *is extremally disconnected.*

*(4) The minimal quasi-*F *cover of* βX *is the same as Gleason's minimal projective cover of* βX.

4.6. *Proposition. If* X *is a Tychonoff space and every dense open set of* X *contains a dense cozero set of* X, *then* K(X) *is extremally disconnected, and the* o-*Cauchy completion of* C(X) *is Dedekind complete.*

Recall from [CHN] that a space X is called *weakly Lindelöf* if each of its open covers contain a countable subfamily whose union is dense in X.

4.7. *Corollary. If* X *satisfies any one of the conditions:*

(1) X *is perfectly normal (in particular if* X *is metrizable);*

(2) X *has the countable chain condition;*

(3) every dense (open) subset of X *is weakly Lindelöf; then* K(X) *is extremally disconnected and the* o-*Cauchy completion of* C(X) *is the Dedekind-MacNeille completion.*

Note that (2) implies (3).

4.8. *Corollary. If* X *is a quasi-*F *space in which every dense open subset contains a dense cozero set (in particular, if any of the conditions of* 4.7 *hold), then* X *is extremally disconnected.*

4.8 also follows immediately from the definition of quasi-F-sapces, since X is extremally disconnected whenever every dense open set is C*-embedded.

5. Characterizations of Quasi-F-Spaces

In this section, quasi-F-spaces are characterized in a number of ways both topologically and in terms of the ring of continuous real-valued functions on the space. These characterizations are used in a number of ways; in particular to study when a finite product of quasi-F-spaces is a quasi-F-space.

Recall that an element r of a commutative ring A is called *regular* if ra = 0 for a $\in$ A implies that a = 0. An ideal of A is called *regular* if it contains a regular element. Note that an r $\in$ C(X) is regular if and only if coz(r) is dense in X.

If A and A' are lattice-ordered and ϕ: A $\longrightarrow$ A' is a ring homomorphism that preserves the partial ordering on A, then we call the kernel of ϕ an *order-convex ideal* of A. If ϕ also preserves the lattice operations of A, we call its kernel an *ℓ-ideal* of A. It is well-known that a ring ideal I is order-convex [resp. an ℓ-ideal] if and only if $0 \le a \le b$ [resp. $|a| \le |b|$ and b $\in$ I imply that a $\in$ I [F]. (In [GJ] our order-convex ideals are called convex ideals, and our ℓ-ideals are called absolutely convex ideals.)

5.1. *Theorem. If* X *is a Tychonoff space, then the following are equivalent.*

(a) X *is a quasi-*F *space.*

(b) Every dense z*-embedded subspace of* X *is* C**-embedded.*

(c) Whenever f *and* r *are elements of* C(X) *such that* $|f| \le |r|$ *and* r *is regular, then* f *is a multiple of* r.

(d) Every regular ideal of C(X) *is order-convex.*

(e) Every regular ideal of C(X) *is an* ℓ*-ideal.*

(f) Every finitely generated regular ideal of C(X) *(with generators* $f_1, \cdots, f_n$*) is principal (with generator* $|f_1| + \cdots + |f_n|$*).*

(f') Every regular ideal of C(X) *with two nonnegative generators is principal.*

(g) C(X) *is* o-*Cauchy complete as a vector lattice.*

(h) βX *is a quasi*-F-*space.*

Furthermore, an equivalent condition is obtained if C(X) *is replaced by* C*(X) *in any of the preceding conditions.*

Suppose X is a topological space. The members of the σ-field of subsets of X generated by the cozero sets of X are called *Baire sets.*

5.2. *Corollary. Consider the following properties of a Tychonoff space* X.

(a) Every dense Baire set in X *is* C*-*embedded.*

(b) X *is a quasi*-F-*space.*

(c) Every dense Lindelöf subspace of X *is* C*-*embedded.*

Then (a) implies (b), (b) implies (c), and if X *is* σ-*compact then (a), (b), and (c) are equivalent.*

We call a space X *strongly zero-dimensional* if βX has a base for its topology consisting of sets that are closed (and open). In [He], L. Heider showed that X is strongly zero-dimensional if and only if each of its zerosets is a countable intersection of open and closed sets.

5.3. *Lemma. Suppose* X *is strongly zero-dimensional.*

(a) *Every* z-*embedded subspace of* X *is strongly zero-dimensional.*

(b) *If* Z_1 *and* Z_2 *are disjoint zerosets of* X, *then there is open and closed set* U *in* X *such that* $Z_1 \subset U$ *and* $Z_2 \subset X\setminus U$.

Next, some known properties of F-spaces are generalized.

5.4. *Theorem. Consider the following conditions of a Tychonoff space* X.

(a) X *is a quasi*-F-*space.*

(b) *If* $f \in C(X)$ *is regular, then there is a* $k \in C(X)$ *such that* $f = k|f|$.

(c) *If* $f \in C(X)$ *is regular, then* pos f *and* neg f *are completely separated.*

Then (a) implies (b), (b) and (c) are equivalent, and if X *is strongly zero-dimensional, then (a), (b), and (c) are equivalent.*

5.5. *Proposition. If* X *is a quasi*-F-*space, and* $x \in X$ *has a countable base of neighborhoods, then* x *is an isolated point. In particular, any quasi*-F-*space satisfying the first axiom of countability is discrete.*

Next we give an example to show that neither the assumption that X is strongly zero-dimensional in Theorem 5.4 nor the assumption of σ-compactness in Corollary 5.2 can be deleted. First we describe a way of constructing certain kinds of topological spaces.

Let D denote an uncountable discrete space and let $\alpha D = D \cup \{\infty\}$ denote its one-point compactification. Suppose

Y is a subspace of a Tychonoff space X, let $\Delta = \Delta(Y,X) = (X \times \alpha D)\setminus\{(p,q): p \notin Y \text{ and } q \neq \infty\}$, and refine the product topology on Δ by letting any point whose second coordinate is not ∞ be isolated. Then Δ is said to be *the space obtained by attaching a copy of* αD *to each point of* Y.

5.6. *Example.* A Tychonoff space satisfying (c) of Theorem 5.4 that is not a quasi-F-space.

$\Delta_1 = \Delta([0,1), [0,1])$, the space obtained by attaching a copy of αD to the closed unit interval [0,1] at each point of [0,1) is such a space.

Note also that Δ_1 contains no dense Lindelöf subspace, so the implication (c) implies (b) of Corollary 5.2 need not hold if the hypothesis of σ-compactness is deleted.

Recall from [GJ, Chapter 14] that a Tychonoff space X is called a P-*space* if every zeroset of X is open and from [L], that X is called an *almost*-P-*space* if each of its zerosets has a nonempty interior. Clearly every almost-P-space is a quasi-F-space. If X is any noncompact realcompact sapce, then $\Delta(X,\beta X)$ is a quasi-F-space that is not an almost-P-space. A space with this latter property is called a *proper quasi*-F-*space*.

A closed subspace of a quasi-F-space need not be a quasi-F-space. In fact, since X is a closed subspace of $\Delta(X,X)$ we have:

5.7. *Proposition. Every Tychonoff space* X *is homeomorphic to a closed subspace of an almost*-P-*space.*

X is called an F'-*space* if for every $f \in C(X)$, pos f and neg f have disjoint closures. Every normal F'-space is an F-space, but there are F'-spaces that are not F-spaces [GH, 8.14] and [CHN]. In [CHN, Theorem 1.1] it is shown that X is an F'-space if and only if every cozero set in X is C*-embedded in its closure.

5.8. *Proposition. Consider the following properties of a Tychonoff space* X.

(a) X *is an* F'-*space.*

(b) The closure of any cozero set of X *is a quasi*-F-*space.*

(c) X *is an* F-*space.*

(d) Every closed subset of X *is a quasi*-F-*space.*

Properties (a) and (b) are equivalent. If X *is normal, then (a), (b), (c), and (d) are equivalent.*

In [K, Example 3], Carl Kohls gives an example of an (extremally disconnected) F-space X with a closed subspace Y that is not an F'-sapce; indeed, Y is not a quasi-F-space.

Next we consider conditions under which the property of being a quasi-F-space is preserved under finite products.

5.9. *Proposition. Suppose* X_1 *and* X_2 *are Tychonoff spaces.*

(a) $X_1 \times X_2$ *is an almost*-P-*space if and only if both* X_1 *and* X_2 *are almost*-P-*spaces.*

(b) If $X_1 \times X_2$ *is a quasi*-F-*space, then so are* X_1 *and* X_2.

(c) If X_1 *and* X_2 *are strongly zero-dimensional and* $X_1 \times X_2$ *is a quasi*-F-*space, then* X_1 *or* X_2 *is an*

almost P-*space.*

(d) If $X_1 \times X_2$ *is a quasi-*F*-space and* X_2 *is a compact proper quasi-*F*-space, then* X_1 *is a* P*-space.*

5.10. *Corollary. The product of two infinite compact spaces is never a proper quasi-*F*-space.*

We do not know if the requirements that X_1 and X_2 be strongly zero-dimensional in the statement of Proposition 5.9(c), or the requirement that X_2 be compact in the statement of Proposition 5.9(d) are necessary.

In [N, Theorem 6.5], S. Negrepontis shows that X is a P-space if and only if $X \times \beta X$ is an F-space. An analog of this result follows.

5.11. *Corollary. For any Tychonoff space* X, $X \times \beta X$ *is a quasi-*F*-space if and only if* X *is a* P*-space or* βX *is an almost-*P*-space.*

In [G], an example of an extremally disconnected and a P-space whose product is not an F-space is given. By modifying Gillman's argument, it can be shown that this latter product is not even a quasi-F-space.

The problem of determining exactly when a product of two spaces is a quasi-F-space seems to be at least as complicated as the corresponding one for F-spaces. See [CHN].

References

[BKW] A. Bigard, K. Keimel, and S. Wolfenstein, *Groupes et Anneaux Reticulés*, Lecture Notes in Mathematics 608, Springer-Verlag, New York, 1977.

[B] S. J. Bernau, *Unique representation of Archimedean lattice groups and normal Archimedean lattice rings*, Proc. London Math Soc. (3) 15 (1965), 599-631.

[BH] R. Blair and A. Hager, *Extensions of zero-sets and of real-valued functions*, Math. Zeit. 136 (1974), 41-52.

[CN] W. Comfort and S. Negrepontis, *Continuous Pseudometrics*, Lecture Notes in Pure and Applied Mathematics 14, Marcel Dekker Inc., New York, 1975.

[CHN] W. Comfort, N. Hindman, and S. Negrepontis, F-*spaces and their product with* P-*spaces*, Pacific J. Math 28 (1969), 489-502.

[D] F. Dashiell, *Non-weakly compact operators on semi-complete* C(S) *lattices, with applications to Baire classes*, Trans. Amer. Math. Soc. (to appear).

[F] L. Fuchs, *Partially Ordered Algebraic Systems*, Pergammon Press, London, 1963.

[FGL] N. J. Fine, L. Gillman, and J. Lambek, *Rings of Quotients of Rings of Functions*, McGill University Press, Montreal, 1965.

[G] L. Gillman, *A* P-*space and an extremally disconnected space whose product is not an* F-*space*, Archiv. der Math. 11 (1960), 53-55.

[Gℓ] A. Gleason, *Projective topological spaces*, Ill. J. Math. 2 (1958), 482-489.

[GH] L. Gillman and M. Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. 82 (1956), 366-391.

[GJ] L. Gillman and M. Jerison, *Rings of Continuous Functions*, D. Van Nostrand Co. Inc., Princeton, N.J., 1960.

[H] A. W. Hager, *On inverse-closed subalgebras of* C(X), Proc. London Math. Soc. 19 (1969), 233-257.

[He] L. Heider, *Compactifications of dimension zero*, Proc. Amer. Math. Soc. 10 (1959), 377-384.

[J] D. G. Johnson, *Completion of an Archimedean* f-*ring*, J. London Math. Soc. 40 (1965), 493-496.

[K] C. Kohls, *Hereditary properties of some special spaces*, Arch. Math. (Basel) 1 (1961), 129-133.

[L] R. Levy, *Almost-P-spaces*, Can. J. Math. 29 (1977), 284-288.

[LZ] W. Luxemberg and A. Zaanen, *Riesz Spaces*, Amsterdam, 1971.

[MJ] J. E. Mack and D. G. Johnson, *The Dedekind completion of* C(X), Pac. J. Math. 20 (1967), 231-243.

[N] S. Negrepontis, *On the product of* F-*spaces*, Trans. Amer. Math. Soc. 136 (1969), 339-346.

[P] F. Papangelou, *Order convergence and topological completion of commutative lattice-groups*, Math. Annalen 155 (1964), 81-107.

[We] E. Weinberg, *Higher degrees of distributivity in lattices of continuous functions*, Thesis, Purdue University, 1961.

Harvey Mudd College

Claremont, California 91711

and

Wesleyan University

Middletown, Connecticut 06457