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Research Announcement:

THE TOPOLOGICAL STRUCTURE OF THE TANGENT AND COTANGENT BUNDLES ON THE LONG LINE

by

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The tangent bundle is, according to M. Spivak, "the true beginning of the study of differentiable manifolds" [3]. Given any differentiable n-manifold M, metrizable or otherwise, there is a differentiable 2n-manifold TM associated with it, called the tangent bundle of M. There are various constructions of TM, but all are equivalent as vector bundles over M [3, Chapter 3].

Using the tangent bundle, one constructs the cotangent bundle T*M [3, Chapter 4], which is also a differentiable 2n-manifold and, like TM, a vector bundle over M. If M is metrizable, the tangent and cotangent bundles are equivalent [3, Corollary 9-5]. The converse is also true [3, p. A-21].

This announcement deals with the tangent bundles on the long line L and the (open) long ray L^+. Aside from the real line and the circle, these are the only connected Hausdorff 1-manifolds [3, Appendix A]. The definition and notation will be as in [3], where L^+ is $\Omega \times [0,1) - \{<0,0>\}$ with the lexicographic order topology, except that we denote ordered pairs by $<, >$ instead of $(,)$, and points of the form $<\alpha, 0>$ will be denoted simply α when there is no danger of confusion.

The long line and long ray are nonmetrizable, normal, countably compact, differentiable 1-manifolds. The intrinsic importance of non-metrizable manifolds is a matter of some controversy, but the paucity of 1-manifolds has made L and

L^+ pedagogically useful, in bringing out the peculiarities of certain constructions; their tangent and cotangent bundles can serve a similar purpose.

It is still an unsolved problem whether all differentiable structures on L and L^+ are equivalent, but many results (including those in this paper) are common to all such structures. For example every bounded subspace of TL+, TL, T*L+ and T*L is metrizable, simply because that is true of L and L^+. ("Subspace" is always meant here in a purely topological sense. There are natural projections (designated π) from TM to M and T*M to M for any manifold M. A subspace of TL, etc. is *bounded* if it is contained in $\pi^{-1}[-\alpha,\alpha]$ for some countable ordinal α.)

Theorem 1. *Any collection of countably many closed, unbounded subspaces of* TL^+ *has nonempty intersection.*

In contrast, $L^+ \times \{r\}$ is a copy of the long ray in $L^+ \times \mathbf{R}$ for each $r \in \mathbf{R}$, and similarly for $L \times \{r\}$.

Corollary. The spaces TL *and* TL^+ *are collectionwise normal, countably paracompact, and* ω_1*-compact.*

A topological space is ω_1-*compact* if every closed discrete subspace is countable. Topological terms not defined here may be found in [1].

The key to proving most of the results announced here is the construction of a space $(Z,\mathcal{J})$ homeomorphic to TL^+, which has $L^+ \times \mathbf{R}$ as its underlying *set*. This also aids in forming a rough picture of TL^+. For each limit ordinal λ, the relative topology on $[\lambda, \lambda + \omega) \times \mathbf{R}$ is the usual (product) topology, as is the relative topology on $(0,\omega)$.

In completing the definition of $\mathcal{J}$, we are guided by an atlas of charts (x_λ, U_λ) on L^+, λ a limit ordinal, such that $[\lambda, \lambda + \omega) \subset U_\lambda$, and satisfying the following condition. Suppose $\{p_n : n \in \omega\}$ is an increasing sequence in L^+, $p_n \in [\lambda_n, \lambda_n + \omega)$, converging to a limit ordinal λ. [We do not require the λ_n's to be distinct.] Then $D(x_\lambda \circ x_{\lambda_n}^{-1})(x_{\lambda_n}(p_n))$ converges to 0. Now, using the definition in [3, Theorem 3-1], we make $\langle p, s\rangle \in L^+ \times \mathbf{R} = Z$ correspond to the equivalence class of (x_λ, s) in $\pi^{-1}(p)$, where λ is the unique zero-or-limit ordinal such that $p \in [\lambda, \lambda + \omega)$. We then have:

Lemma 2. *If* $\{\langle p_n, r_n\rangle : n \in \omega\}$ *is a sequence in* $(Z, \mathcal{J})$ *such that* $\{p_n\}$ *is an increasing sequence in* L^+ *converging to a limit ordinal* λ, *and the sequence* $\{r_n\}$ *is bounded, then* $\langle p_n, r_n\rangle \to \langle \lambda, 0\rangle$.

Theorem 1 now follows from the well known, and easily proven, fact that the intersection of countably many closed unbounded subsets of Ω is again such a subset.

The natural correspondences between the subspace $L^+ \times \{0\}$ of Z, the space L^+, and the set of zero vectors of TL^+, are all homeomorphisms. When the set of zero vectors of TL^+ is removed, the space falls into a "positive half" T^+ and a negative half T^-, both of which are open, simply connected submanifolds of TL^+.

Definition. A topological space X is *collectionwise Hausdorff* if for every closed discrete subspace D of X, there exists a collection $\mathcal{U}$ of disjoint open subsets of X, each of which meets D in exactly one point, such that $D \subset \cup\mathcal{U}$.

Theorem 3. *With notation as above,* T^+ *is a developable, simply connected 2-manifold which is neither normal, nor countably paracompact, nor collectionwise Hausdorff, but does have the property that every separable subspace is metrizable.*

Besides Lemma 2, the following is used in proving the "negative" results about T^+:

Lemma 4. ("The Pressing Down Lemma." For a proof, cf. [2].) *Let* S *be a stationary subset of* Ω. *Let* $f: S \to \Omega$ *be such that* $f(\alpha) < \alpha$ *for each* $\alpha \in S$. *Then there is a stationary* $T \subset S$ *and an ordinal* β *such that* $f(\alpha) = \beta$ *for each* $\alpha \in T$.

The results of Theorem 3, except for developability, carry over from T^+ to the cotangent bundle T^*L^+, which can be pictured in the following way. Turn each of T^+ and T^- "upside down," gluing them back to the zero vectors this way. In fact:

Theorem 5. *Let* Φ *be the map from the space of nonzero vectors of* TL^+ *to those of* T^*L^+, *such that the image of each vector* v *is the unique linear functional* φ *such that* $\varphi(v) = 1$. *Then* Φ *is a diffeomorphism.*

Theorem 6. *The space* T^*L *is none of the following: normal, countably paracompact, collectionwise Hausdorff, developable.*

These same descriptions, with L in place of L^+, and "decreasing" in place of "increasing" where appropriate, hold for the spaces TL and T*L. If we identify v with $\Phi(v)$ for each nonzero vector v, the images of the zero vectors in the resulting identification space Y give us two disjoint closed copies of the long line in Y which cannot be put into disjoint open subsets, because of Theorem 1.

Theorem 7. *The space* Y *is a non-normal, countably compact, differentiable 2-manifold. If* N *is a foliation of* Y, *then every component of* N *is metrizable.*

This theorem is of special interest to general topologists, since it is only recently that a first countable, countably compact, non-normal space has been constructed without using the continuum hypothesis [4].

There are many other countably compact, non-normal 2-manifolds that one can construct by piecing together spaces homeomorphic to T^+, L^+, S^1, $I = [0,1]$, and the product space $L^+ \times \mathbf{R}$. Some are simply connected and/or have the property that every continuous real-valued function is constant outside a compact set. Some can be foliated, others can not.

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