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OPEN PROBLEMS IN INFINITEDIMENSIONAL TOPOLOGY

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List of Problems by Area

I. Introduction

This problem set is a substantial revision of one prepared in 1975 by R. D. Anderson, D. W. Curtis, G. Kozlowski, and R. M. Schori, which appeared as an appendix to *Lectures on Hilbert Cube Manifolds* by T. A. Chapman (CBMS Regional

Conference Series in Mathematics, Number 28, Providence, R.I., 1976). That list, and its predecessors (Mathematisch Centrum Report ZW1/71 and Mathematical Centre Tract 52(1974), 141-175, both published in Amsterdam, as well as earlier privately circulated lists) were put together following special conferences organized by R. D. Anderson in Ithaca (1969), Baton Rouge (1969), Oberwolfach (1970), Baton Rouge (1973) and Athens (Georgia) (1975). The latest of those conferences was held in Athens (Ohio) in 1979, and the present version is the result of that meeting. At the meeting every problem in the previous edition was discussed, and many new problems were presented.

The editor received written contributions from: R. D. Anderson, C. Bessaga, B. Brechner, Z. Čerin, T. Chapman, D. Curtis, T. Dobrowolski, A. Fathi, S. Ferry, R. Geoghegan, H. Hastings, R. Heisey, G. Kozlowski, V. Liem, W. Nowell, L. Rubin, R. Schori, W. Terry, H. Toruńczyk, J. Walsh and J. West. Readers who want more information on problems may wish to consult one of those people. (Addresses are in AMS-MAA Combined Membership List, except: Bessaga, University of Warsaw; Čerin, University of Zagreb; Dobrowolski, University of Warsaw; Fathi, University of Paris SUD-Orsay; Kozlowski, Auburn University; Toruńczyk, Polish Academy of Sciences, Warsaw.) The editor acknowledges his debt to all those named, as well as to many others who took part in discussions at the conference. He repeats the customary but necessary warning that a few of the problems may be inadequately worded, or trivial, or already solved.

The reader who is not familiar with infinite-dimensional

(ID) topology may be struck by the variety of material he finds in this problem set. He will find ideas from modern geometric topology of (finite-dimensional) manifolds, from algebraic K-theory, from classical point-set topology, from shape theory and from that part of functional analysis which is sometimes called the geometry of Banach spaces. Yet the subject is much more than a collection of results from those areas. Infinite-dimensional topology has its own flavor and its own unity. The flavor is found at its purest in the peculiar topological properties of the Hilbert Cube. The unity comes from the fact that the subject's connections with the scattered areas mentioned above can be seen in retrospect to be applications and developments of the same fundamental ideas.

This is a problem set, not an exposition of a subject. Thus it contains no bibliography, and basic knowledge of infinite-dimensional topology is assumed (less in some sections than in others). The two most convenient expositions of the basic material are Chapman's lecture notes (mentioned already) and the book *Selected Topics in Infinite-Dimensional Topology* by C. Bessaga and A. Pełczyński, PWN Warsaw 1975. These books approach the subject with quite different aims. For the reader seeking connections with the geometric topology of manifolds, Chapman's book is the natural starting place. The reader interested in connections with analysis will find them treated in some detail by Bessaga and Pełczyński. However the subject is developing so fast that neither book covers the full range. This problem set is an attempt to outline the frontier; it is assumed that a serious reader will contact workers in the field, perhaps somebody mentioned in the particular section of interest.

In previous editions, the introduction has included a list of what the editors considered to be the major recent developments in the subject. We would prefer not to make such a list, though much of the recent work is mentioned in the various sections. However, it should be said that the development (in 1977) which led Anderson to call for a new version of the problem set was Toruńczyk's remarkable topological characterizations of the Hilbert Cube, of Hilbert Space, and of the manifolds modelled on them. Before stating Toruńczyk's theorems, we will write down some notation and terminology which will be used throughout the problem set.

Q denotes the Hilbert Cube, i.e., the countably infinite cartesian product of copies of the closed interval [-1,1]. s denotes the countably infinite product of copies of the real line $\mathbb{R}$. It is known (see Section LS) that every separable infinite-dimensional Banach space is homeomorphic to s; in particular this is true of the familiar Hilbert space ℓ_2 of square-summable sequences. From the topological point of view s (because of its product topology) is easier to use than its Banach space homeomorphs. Therefore we speak of s and s-manifolds throughout. Of course topologically there is no difference between an s-manifold and an ℓ_2-manifold.

In general, if F is a space, an F-*manifold* is a paracompact Hausdorff space each point of which has a neighborhood homeomorphic to an open subset of F. Q-manifolds are locally compact; s-manifolds are not.

The symbol $\cong$ denotes homeomorphism.

An *absolute neighborhood retract* (ANR) is a metrizable space X such that whenever X is embedded as a closed subset of a metrizable space Y, then X is a retract of some neighborhood of X in Y. If X is always a retract of Y then X is an *absolute retract* (AR). (Note that the AR's are precisely the contractible ANR's.) In some sections the context indicates that we are dealing exclusively with separable locally compact spaces, and then the terms ANR and AR tacitly contain these properties.

An ANR X has the *disjoint* n-*cube property* if any two maps $I^n \to X$ can be arbitrarily closely approximated by maps whose images are disjoint. When n = 2 this is often called the *disjoint disk property* (DDP).

Theorem (H. Toruńczyk). *A separable, locally compact* ANR *is a* Q-*manifold if and only if it has the disjoint* n-*cube property for all* n.

It is remarkable that this theorem was proved only a few months before J. Cannon isolated the DDP as the crucial property needed to solve the Double Suspension Conjecture. Indeed it is conjectured that, *for* $n \geq 5$, *a separable locally compact finite-dimensional* ANR, X, *is an* n-*manifold if and only if* $H_*(X, X\setminus\{x\}; \mathbf{Z}) = H_*(\mathbf{R}^n, \mathbf{R}^n\setminus\{0\}; \mathbf{Z})$ *for all* $x \in X$ *and* X *has the* DDP.

Toruńczyk has also characterized s-manifolds, (and, indeed, F-manifolds for any Fréchet space F) in topological terms. Here is his characterization of s:

Theorem (H. Toruńczyk). *A complete separable* AR*,* X*, is homeomorphic to* s *if and only if for any map* f: $\mathbf{N} \times Q \to X$ *and any open cover* $\mathcal{U}$ *of* X *there is a map* g: $\mathbf{N} \times Q \to X$*,* $\mathcal{U}$*-close to* f*, such that* $\{g(\{n\} \times Q) \mid n \in \mathbf{N}\}$ *is a discrete collection of sets.*

Readers with a sense of classical topology will be aware that the problems of topologically characterizing the Hilbert Cube, Hilbert Space and Euclidean Space were considered by an earlier generation to be principal goals.

II CE Images of ANR's and Q-Manifolds

In this section all ANR's are understood to be locally compact. A map f: X → Y is *cell-like* (or CE) if it is proper and each set $f^{-1}(y)$ has the shape of a point (i.e. when $f^{-1}(y)$ is embedded in an ANR, it can be contracted to a point within any of its neighborhoods).

CE maps *between* ANR's are of great importance. They are the fine homotopy equivalences. They are the hereditary shape equivalences (defined below). When simple homotopy theory is extended to ANR's the elementary expansions and collapses are defined to be CE maps between ANR's. CE maps *between* Q-*manifolds* are precisely those maps which can be approximated by homeomorphisms. Very often a CE map arises as the quotient map of an upper semi-continuous cell-like decomposition of some given ANR, X; then the quotient map f is CE but it is not always the case that the quotient space Y is an ANR; Taylor's example (discussed below) is a case in which X = Q while Y does not have the shape of any ANR.

The broad questions on which there is work to be done are these:

Question A. Given that X is an ANR and f is CE, under what conditions is Y an ANR?

Question M. Given that X is a Q-manifold and f is CE, under what conditions is Y a Q-manifold?

In discussing Question A, we first discuss conditions on Y and f.

Kozlowski defines an *hereditary shape equivalence* to be a proper map f: X → Y such that $f|: f^{-1}B \to B$ is a shape equivalence for every closed subset B of Y, and has shown:

(1) A CE map f: X → Y, with X an ANR, is an hereditary shape equivalence if and only if Y is an ANR. His Vietoris theorems then imply that Y is an ANR in the following cases:

(1a) Y is a countable union of closed finite-dimensional subspaces;

(1b) Y is compact and countable-dimensional;

(1c) the nondegeneracy set $\{y \in Y: f^{-1}(y)$ is not a point$\}$ of f is finite dimensional (or more generally, is contained in a subset of Y having large inductive transfinite dimension).

Kozlowski defines a map f: X → Y between compact metric spaces to be *approximately right invertible* (ARI) if for every $\varepsilon > 0$ there is a map g_ε: Y → X such that $f \circ g_\varepsilon$ is uniformly within ε of the identity map. He has a theorem that if X is an ANR and if f is CE and ARI, then f is an hereditary shape equivalence (and hence Y is an ANR).

These remarks provide background for the first three problems:

(CE1) Let $f: Q \to Y$ be a surjection with each point-inverse a copy of Q. Is Y an AR? It is readily seen, using Edwards' Q-factor theorem and a cone construction, that this is equivalent to a question of Borsuk: If X is a compact ANR and $f: X \to Y$ has AR's for point-inverses, is Y an ANR?

(CE2) Let $f: X \to Y$ be such that X is a compact ANR, and f is ARI. Is Y an ANR? It can easily be shown that Y is movable, but it is not even clear that Y is an FANR.

(CE3) Call a map $f: X \to Y$ between compact metric spaces *refinable* if for every $\varepsilon > 0$ there is a map $f_\varepsilon: X \to Y$ uniformly within ε of f, such that diam $f_\varepsilon^{-1}(y) < \varepsilon$ for all $y \in Y$. If X is an ANR, is Y an ANR? Ford and Kozlowski have shown that the answer is yes if X is finite-dimensional and Y is LC^1. Refinable maps preserve the property of being movable, but it is unknown whether they preserve the property of being an FANR.

The other aspect of Question A concerns conditions on X, specifically the condition that X be finite-dimensional.

(CE4) Let $f: B^n \to Y$ be a CE map. Is Y an AR? This is equivalent to the question: Is the CE image of a finite-dimensional compactum finite dimensional? (The procedure, given a CE map $f: X \to Y$ with dim $X < \infty$, is to embed X in some B^n and consider the quotient map $F: B^n \to B^n \cup_f Y$. If $B^n \cup_f Y$ is an AR, then F and f are hereditary shape equivalences, and these do not raise dimension. The converse follows from (1a) above.

Even the following special case is of interest.

(CE5) If it is further assumed in (CE4) that the non-degenerate point-inverses of f are arcs, is Y an AR?

Some of these questions originated in part from the study of decompositions of manifolds. The situation at present is that one has complete information regarding the homotopy groups but no information about homotopy.

(CE6) If X is B^n or $\mathbf{R}^n$ and f: X → Y is a cell-like map, is Y contractible?

(CE7) Let f: M^{2n+1} → Y be a CE map, where M is a manifold without boundary, and Y is finite-dimensional (equivalently Y is an AR of dimension $\leq$ 2n+1). If Y has the Disjoint Disk Property, prove *directly* that Y has the Disjoint n-cube Property. R. D. Edwards has recently proved that under the given hypotheses Y is a (2n+1)-manifold, so it has the Disjoint n-cube Property by general position. We are asking for a proof by-passing Edwards' theorem. (This problem should be considered in conjunction with D5 and CE9.)

Further discussion of Problem (CE4) will be given in Section D on dimension theory.

We now turn to Question M. As explained in the Introduction, Toruńczyk has characterized those ANR's which are Q-manifolds. So if one is dealing with the CE image of a Q-manifold, the first test should be: is it an ANR and does it have the Disjoint n-cube Property for all n? In particular, Toruńczyk's theorem implies that if X is a Q-manifold and Y is an ANR and if the singular (the *singular* *set* is $\{x: f^{-1}f(x) \neq x\}$) set lies in the countable union of Z-sets in X, then Y is a Q-manifold.

Among the examples which must be faced are the following. If a wild arc (Wong) or a cut slice in Q is shrunk to a point the quotient space is not a Q-manifold. Even if one only looks at cases in which each set $f^{-1}(y) \subset X$ is a Z-set (X is now assumed to be a Q-manifold) there are counter-examples. A modification of Eaton's argument for the existence of dog-bone decompositions for higher dimensional Euclidean spaces shows that there is a dog-bone decomposition of Q, i.e., a surjection $f: Q \to Y$ such that Y is not Q, each nondegenerate point-inverse is a Z-set arc, and the nondegeneracy set of f is a Cantor set in Y. In these cases by (1c) above Y is an AR, but even this information is not obtained in general, for Taylor's example gives a CE map $f: X \to Q$ onto the Hilbert cube which is not a shape equivalence; and considering X as a Z-set of Q and taking the adjunction $F: Q \to Q \cup_f Q = Y$ produces a CE map of Q onto a non-AR.

(CE8) Suppose $f: Q \to Y$ is a CE map onto an AR. Is $Y \cong Q$, if (a) the collection of nondegenerate point-inverses is null (i.e., there are only finitely many such sets of diameter $> \varepsilon$ for every $\varepsilon > 0$), and/or if (b) the closure in Y of the nondegeneracy set is zero-dimensional?

(CE9) Let $f: Q \to Y$ be a CE map with Y an AR. Suppose $Y \times F \cong Q$ for some finite-dimensional compactum F. Is $Y \times I^n \cong Q$ for some n? And can it be that $Y \times I^n \cong Q$ for some n, while $Y \times I^2 \not\cong Q$? Or $Y \times I \not\cong Q$? Toruńczyk has characterized those Y such that $Y \times I \cong Q$. This problem is also interesting for cellular decompositions of Q which we now define.

A compact subset A of a Q-manifold is *cellular* if $A = \bigcap_n K_n$ where each $K_n \cong Q$, Bd $K_n \cong Q$, Bd K_n is a Z-set in K_n and $A \subset \text{int } K_n$. A u.s.c. decomposition of a Q-manifold is *cellular* if each element is.

(CE10) Suppose f: Q → Y is a CE map onto an AR. If the induced decomposition is cellular and the non-degeneracy set is countable is $Y \cong Q$? Is $Y \times I \cong Q$? Probably it is better to ask for a counterexample, but none is known even when the non-degeneracy set is allowed to be finite-dimensional.

(CE11) Suppose f: Q → Y is a CE map onto an AR. If the non-degeneracy set is zero-dimensional and if $Y \cong Q$, is the decomposition cellular? Same question for finite-dimensional non-degeneracy set.

III. Dimension Theory

We only pose problems in dimension theory which have a direct bearing on the geometrical understanding of ANR's and Q-manifolds. The principal question, posed as Problem (CE4) above, is old and well-known:

Question D. If f: X → Y is a CE-map with X compact and finite-dimensional, then is Y finite dimensional?

It is known that if dim Y > dim X, then (i) dim Y = ∞, and (ii) Y cannot be an ANR, and (iii) dim X ≥ 2, and (iv) X is not a 2-dimensional ANR, and (v) every finite-dimensional subset of Y has dimension ≤ dim X.

(i), (iii), and (v) follow from classical facts from the theory of cohomological dimension since the Vietoris Theorem implies that the cohomological dimension of Y is ≤ dim X.

(ii) follows from work of Kozlowski.

Kozlowski observed that (iv) follows from (iii) and the result of K. Sieklucki that an n-dimensional ANR does not contain uncountably many pairwise disjoint closed n-dimensional subsets (see Sieklucki's paper in Bull. Acad. Polon. Sci., Ser. Sci. math., atsr. et phys., 10 (1962)).

CE maps do not raise cohomological dimension; and R. D. Edwards has shown that if there exists Y with infinite dimension but finite cohomological dimension, then Y is the CE image of a finite-dimensional compactum. Hence Question D is equivalent to one of Aleksandrov's problems:

(D1) Is there an infinite-dimensional compact metric space with finite cohomological dimension? (We refer to integer coefficients.)

Edwards (and, independently, D. Henderson, 1966, unpublished) can show that there exists such a compactum (as described in (D1)) if the answer to the next question is positive.

(D2) Do there exist positive integers n and p_i and maps $f_i: S^{n+p_i} \to S^n$ such that, in the following sequence, every finite composition is essential

$$S^n \xleftarrow{f_1} S^{n+p_1} \xleftarrow{\Sigma^{p_1} f_2} S^{n+p_1+p_2} \longleftarrow \cdots ?$$

Question (D2) is recognizable to homotopy theorists. It is related to a question asked by M. Barratt in the Proceedings of the 1976 Summer Institute at Stanford (see p. 252 of Part 2 of Proc. of Symposia in Pure Mathematics, vol. 32, A.M.S., Providence, Rhode Island, 1978). If Barratt's conjecture is true, it would follow that it is impossible to construct the sequence required in (D2) so that all compositions remain essential in stable homotopy. An unstable example might still be possible, however.

More generally, it would be useful to have a homotopy theoretic problem precisely equivalent to (D1) and (CE4). It would also be useful to have information on:

(D3) Classify Taylor examples. In other words, what kinds of compacta can occur as CE images of Q?

A plausible method of attacking Question D had to be discarded when Walsh showed the existence of an infinite-dimensional compactum all of whose finite-dimensional subsets are zero-dimensional (compare with (v) above). But there are related questions of independent interest.

(D4) Does every infinite-dimensional compact ANR contain n-dimensional closed subsets for each n?

(D5) Let X be a compact AR such that for every finite-dimensional compact subset A of X and every open set U in X, $H_*(U,U\setminus A) = 0$. If X has the Disjoint Disk Property, then does X have the Disjoint n-cube Property for all n? Compare (CE7).

It is apparently unknown whether such an X contains n-dimensional closed subsets.

The next problem is also known as Aleksandrov's Problem:

(D6) Does there exist an infinite-dimensional compactum which is neither countable dimensional nor strongly infinite-dimensional?

Assuming the continuum hypothesis, R. Pol has constructed a non-compact separable metric space which is neither countable-dimensional nor strongly infinite-dimensional.

In connection with (D6), if Y is the image of a dimension-raising CE map, then Y is not countable-dimensional. It is not known whether such a Y (if such exists) can be strongly infinite-dimensional.

IV. Shapes of Compacta in Q, and Ends of Q-Manifolds

There has been interplay between shape theory and I-D topology ever since Chapman proved his Complement Theorems. The first of these says that two Z-sets in Q have the same shape if and only if their complements are homeomorphic. The second says that there is an isomorphism between the shape category of Z-sets in Q and the weak proper homotopy category of complements of Z-sets in Q: an isomorphism under which each Z-set goes to its complement. Shape theory is also involved in the study of CE maps, in the study of ends of non-compact Q-manifolds, and in the study of quotient spaces of ANR's.

We restate two problems which have been discussed in Sections CE and D:

(SC1) Is it true that cell-like maps do not raise dimension? This question is equivalent to the following: is a cell-like map defined on a finite-dimensional space a shape equivalence?

(SC2) If f: X → Y is a CE map, whose nondegeneracy set is countable-dimensional, is f a shape equivalence?

Ferry has recently shown that if M and N are homotopy equivalent compact Q-manifolds, there exist a compactum (≡ compact metric space) X and CE maps $M \xleftarrow{f} X \xrightarrow{g} N$. Thus the existence of such maps does not guarantee that M and N are homeomorphic. On the other hand, Chapman has shown that if M and N are compact Q-manifolds for which there exist a compactum X and CE maps $M \xrightarrow{f} X \xleftarrow{g} N$, then there is a homeomorphism h: M → N.

(SC3) Can h be chosen so that gh is close to f?

(SC4) If X and Y are shape equivalent UV^1-compacta, does there exist a finite diagram $X = X_0 \longleftrightarrow X_1 \longleftrightarrow \cdots \longleftrightarrow X_n = Y$ in which $X_i \longleftrightarrow X_{i+1}$ is an hereditary shape equivalence either from X_i to X_{i+1} or from X_{i+1} to X_i?

A recent example of Ferry shows that the condition UV^1 is necessary in (SC4). There is no passage via CE maps from the circle to the "circle with spiral approaching it."

A connected compactum X is an *ANR divisor* (Hyman) if P/X is an ANR for some (equivalently any) embedding of X in some (equivalently any) ANR P. Dydak has characterized the ANR divisors of *finite shape dimension* (nearly 1-movable and stable pro-homology):

(SC5) Characterize the ANR divisors (no restriction on shape dimension). Is the property of being an ANR divisor invariant under shape domination?

A slight generalization of (SC5) is

(SC6) When is the one-point compactification, $\hat{Y}$, of a one-ended locally compact ANR, Y, an ANR?

Concerning (SC6) we remark first that it is a generalization of (SC5) since P/X is the one-point compactification of P\X; and secondly that Dydak's characterization in the finite-dimensional case of (SC5) generalizes to (SC6): let Y be finite-dimensional; then $\hat{Y}$ is an ANR if and only if the end of Y is nearly 1-movable and has stable pro-homology. The interesting problem, then, is the infinite-dimensional case. Dydak has an example of an ANR divisor which has infinite shape-dimension, and is therefore not an FANR ($\equiv$ shape dominated by complex). A special case of (SC6) (infinite-dimensional case) turns up in the problem of classifying compact Lie group actions on Q (see Section GA).

This would be a reasonable place to discuss questions about adding a compactum (as a Z-set, for example) at the end of a non-compact ANR to make a compact ANR. However, we have chosen to include those matters in Section QM. We turn, therefore, to wilder compactifications.

A compactum X in a space Y is a *shape* Z-*set* if for every neighborhood U of X there is a homotopy $h_t: Y \to Y$

such that h_0 is the identity, h_t is the identity outside U, $h_t(Y \setminus X) \subset Y \setminus X$ and $h_1(Y) \subset Y \setminus X$.

(SC7) Is there a version of the Chapman-Siebenmann theory of ends of Q-manifolds in which a manifold is compactified by the addition of a shape Z-set? Which manifolds admit such a compactification? Ferry has an example which does not have a Z-set compactification in the sense of Chapman-Siebenmann; its finiteness (σ_∞) obstruction at ∞ is non-zero; but it does have a non-locally-connected shape Z-set compactification.

(SC8) Are there versions of the Chapman Complement Theorems for shape Z-sets in Q? Compare Venema's finite-dimensional version in the trivial range, where ILC compacta seem to be analogous to shape Z-sets.

A positive answer to the next question would imply that every FANR is a pointed FANR (an important question in shape theory):

(SC9) Let X be an FANR Z-set in Q, and let h be a homeomorphism of X which is homotopic to the identity map in any neighborhood of X. Is there a nested basic sequence of compact Q-manifold neighborhoods M_i of X and a homeomorphism H of Q extending h, such that $H(M_i) = M_i$ for all i? (The question may be interesting for non-FANR Z-sets also.)

We now come to the question of whether every shape equivalence is a strong shape equivalence. The strong shape category of compacta was first formulated by D. A. Edwards and H. M. Hastings (Springer Lecture Notes, vol. 542, p. 231). It has antecedents in the work of Christie (1944) and Quigley (1970). For another formulation see Dydak-Segal (Dissertationes Mathematicae). In I-D terms, the *strong shape category* has compact Z-sets in Q as objects, and proper homotopy classes of proper maps from $Q\setminus X$ to $Q\setminus Y$ as morphisms from X to Y. (For comparison, morphisms in the (ordinary) *shape category* can be thought of as weak proper homotopy classes of proper maps from $Q\setminus X$ to $Q\setminus Y$, by Chapman's Complement Theorem).

(SC10) With the above notation, let $f: Q\setminus X \to Q\setminus Y$ be a proper map which is a weak proper homotopy equivalence. Is f a proper homotopy equivalence (i.e., an isomorphism in strong shape theory)? (SC10) contains two quite separate questions only one of which is connected with ID topology. As posed, it is not clear that f even induces an isomorphism on π_1 of the ends; to avoid this we state a version of (SC10) for which the answer would be known (and positive) if Q were replaced by a suitable finite-dimensional space. Notation is as in (SC10):

(SC11) Let X and Y be connected, let base rays be chosen for the ends of $Q\setminus X$ and $Q\setminus Y$, and let $f: Q\setminus X \to Q\setminus Y$ be a proper base-ray-preserving map which is invertible in base-ray-preserving weak proper homotopy theory. Is f a proper homotopy equivalence?

Many reformulations of (SC10) and (SC11) in shape theoretic terms are known (see Dydak-Segal, Springer Lecture Notes, vol. 688, p. 141). We state one version to indicate the flavor:

(SC12) Let i: $X \to Y$ be an embedding of one compactum in another. Suppose i is a shape equivalence. Is it true that whenever f,g: $Y \to P$ are two maps into an ANR P with $f|X = g|X$, then f and g are homotopic rel X?

Another question involving strong shape theory has connections with I-D topology (Ferry):

(SC13) Can one choose representatives in the shape classes of all UV^1 compacta, so that on the full subcategory generated by those representatives, strong shape coincides with homotopy theory?

Local contractibility is not a well-understood property for infinite-dimensional spaces. The next problem is proposed with that in mind:

(SC14) If $(X,*)$ is a pointed connected compactum for which pro-$\pi_i(X,*)$ is stable for all i, is X shape equivalent to a locally contractible compactum? Answers are known when X is finite-dimensional (Edwards-Geoghegan, Ferry); the interest is in the infinite-dimensional case. Generally, it might be useful to study locally contractible spaces, and find new examples.

Finally we pose a problem which is the surviving version of Borsuk's question (solved by West) on whether compact ANR's have finite homotopy type:

(SC16) Let X be compact, locally connected and homotopy dominated by a finite complex. Must X have the homotopy type of a finite complex? Ferry has non-locally-connected counterexamples, and he believes a more interesting hypothesis in (SC16) would be "locally 1-connected."

V. Topology of Q-Manifolds

Chapman's CBMS Lecture Notes provide an exposition of the basic structure and classification of Q-manifolds. Compact Q-manifolds are classified up to homeomorphism by simple homotopy type. The new problems suggested for this edition by Chapman, Ferry and others have close connections with well-known questions in finite-dimensional geometric topology and algebraic K-theory.

(QM1) Let M be a compact Q-manifold and let $f: M \to M$ be a map such that f^2 is homotopic to the identity map. When is f homotopic to an involution (i.e., a homeomorphism g with $g^2 = 1$)? Analogous low dimensional questions have been studied by Nielsen, Conner-Raymond, Raymond-Scott, Tollefson and others--generally for $K(\pi,1)$-manifolds. There is a homology condition which must be satisfied by f, but once it holds no counterexamples are known.

(QM2) Let π be a group for which there exist $K(\pi,1)$ compact Q-manifolds M and N. Must M and N be homeomorphic? Farrell and Hsiang have given a positive answer when π is a Bieberbach group (Inv. Math 45 (1978)) and their proof uses a theorem of Ferry on Q-manifolds, so even though (QM2) has a purely algebraic formulation, it may not be absurd to consider it as a geometrical problem.

Ferry's α-approximation theorem says, roughly, that for fine covers α on a Q-manifold N, α-equivalences from other Q-manifolds into N are homotopic (with control) to homeomorphisms. There is interest in a similar theorem for certain kinds of coarse covers α:

(QM3) Let α be an open cover of N, a compact Q-manifold. What conditions on α imply that any α-equivalence $f: M \to N$ is homotopic to a homeomorphism?

(QM4) Let M be a compact Q-manifold, and $\mathcal{U}$ a finite open cover of M by contractible open subsets such that intersections of subcollections of $\mathcal{U}$ are either empty or contractible. Is M homeomorphic to $N(\mathcal{U}) \times Q$? Here $N(\mathcal{U})$ denotes the nerve of $\mathcal{U}$. The α-approximation theorem quoted above shows that the answer is yes for fine covers.

The recent "parametrized end theorem" of F. Quinn gives rise to the next two problems:

(QM5) Is there a Q-manifold version of Quinn's Theorem? i.e., given a Q-manifold M, a compact ANR X, and a proper map $f: M \to X \times [0,\infty)$ under what conditions can we extend f to $\tilde{f}: \tilde{M} \to X \times [0,\infty]$, where $\tilde{M}$ is a compact Q-manifold and $\tilde{M} - M$ is a Z-set in $\tilde{M}$? The case X = point has been dealt with by Chapman-Siebenmann.

A map $f: X \to Y$ is said to be UV^1 if f is onto and each point-inverse is UV^1 in X. The following appears to be the crux of (QM5):

(QM6) Let B be a compact polyhedron and let $\varepsilon > 0$ be given. Does there exist a $\delta > 0$ so that if M, N are compact Q-manifolds and f: $M \to N$, p: $N \to B$ are maps such that (1) p is UV^1 and (2) f is a $p^{-1}(\delta)$-equivalence, then f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism?

It follows from work of Chapman that f must be homotopic to a homeomorphism, but the proof fails to yield the control. Problem (QM6) has an affirmative answer if $N = B \times F$, where F is a compact 1-connected Q-manifold, and p = proj: $B \times F \to B$. Indeed, the answer is affirmative if N is a Q-manifold and p is an approximate fibration with UV^1 fibers.

We have remarked that the case X = point in Problem (QM5) was done by Chapman-Siebenmann (Acta Math 1976). The analogous finite-dimensional problem has not been tackled (as far as we know). One version might read:

(QM7) If Y is a locally compact polyhedron, when can one add a compactum A to Y so that $Y \cup A$ is a compact ANR and A is a Z-set in $Y \cup A$? Siebenmann's thesis handles the case of Y a manifold and A a boundary. Work of Tucker and others on "the missing boundary problem" for 3-manifolds is closer to what we have in mind.

(QM8) If Y is a locally compact ANR such that the Q-manifold $Y \times Q$ can be compactified by adding a compact Z-set A (in which case $(Y \times Q) \cup A$ is necessarily a Q-manifold) is it possible to compactify Y by adding a compact Z-set?

Now we discuss fibrations and locally trivial maps ($\equiv$ bundle projections) between Q-manifolds. The first problem deals with triangulation of locally trivial maps:

(QM9) Let $p: M \to B$ be a locally trivial bundle, where B is a compact polyhedron and the fibers are Q-manifolds. Does there exist a locally compact polyhedron P, a PL map $q: P \to B$, and a fiber-preserving homeomorphism $h: M \to P \times Q$? By recent work of Chapman and Ferry this is true if the fibers are compact. The proof uses Hatcher's work. The method of proof fails for non-compact fibers.

(QM10) If $E \to S^1$ is a locally trivial bundle with fiber F, a noncompact Q-manifold, such that F admits a compactification, when does there exist a locally trivial bundle $\tilde{E} \to S^1$ which contains E as a subbundle and such that each fiber $\tilde{E}_x$ is a compact Q-manifold compactifying E_x?

(QM11) Is every Hurewicz fibration over a compact ANR base, with compact Q-manifold fibers, a locally trivial map?

Some comments on (QM11): if B is finite-dimensional the answer is yes (Chapman-Ferry), and if the total space is multiplied by Q the answer becomes yes (Chapman-Ferry, also R. D. Edwards); if one defines a fibered version of Toruńczyk's disjoint n-disk property in the obvious way, then the presence of this property for all n ensures that the fibration is locally trivial (Toruńczyk-West). Special cases of (QM11) would be interesting: (i) base, total space and fiber homeomorphic to Q; (ii) base $\equiv \{K \subset Q \times Q \mid K$ is compact, convex, containing $\{0\} \times Q\}$, total space $\equiv \{(K,k) \mid k \in K\}$, projection $\equiv (K,k) \longmapsto K$. (ii) proposed by Ščepin.

(QM12) Let $p: M \to B$ be a Hurewicz fibration where M is a compact Q-manifold. Is B an ANR? This should be compared with problems in Section CE. CE maps are not, in general, Hurewicz fibrations, but CE maps between ANR's are approximate fibrations--a concept we now define.

A proper surjection $p: M \to N$ is an *approximate fibration* (Coram-Duvall) provided that given a space X, mappings $g: X \times \{0\} \to M$ and $H: X \times I \to N$ such that $pg = H|X \times \{0\}$, and an open cover $\mathcal{U}$ of N, there exists a mapping $G: X \times I \to M$ such that G extends g, and pG and H are $\mathcal{U}$-close.

(QM13) Let $p: M \to S^2$ be an approximate fibration, where M is a (compact) Q-manifold. Must the fiber have the shape of a finite complex? The answer is no if S^2 is replaced by S^1 (Ferry).

The next four problems concern the improvement of a map either by homotopy or by approximation. Let B be a compact polyhedron for which a triangulation has been chosen. A map $p: M \to B$ is an *h-block bundle map* if there are a space F (the "fiber") and for each simplex σ of B a homotopy equivalence $h_\sigma: p^{-1}\sigma \to F \times \sigma$ such that $h_\sigma|p^{-1}\tau: p^{-1}\tau \to F \times \tau$ is a homotopy equivalence for each face τ of σ (equivalently, h_σ is a homotopy equivalence of n-ads).

(QM14) Let $p: M \to B$ be an approximate fibration with M a compact Q-manifold and B a polyhedron. What is the obstruction to approximating p by an h-block bundle map? Quinn and Chapman know that p can be so approximated when $Wh(F \times T^k) = 0$ for all $0 \leq k \leq \dim B$.

(QM15) Let $p: M \to B$ be an approximate fibration with

M a compact Q-manifold and B a polyhedron. If p is homotopic to a locally trivial map, is p approximable by locally trivial maps? Ferry suggests separating this problem into a concordance part and a concordance-versus-homotopy part. The answer to (QM15) is yes when B is 1-dimensional (Chapman-Ferry).

(QM16) Again let p: M → B be as in (QM15). What is the obstruction to homotoping p to (a) an h-block bundle map and (b) a locally trivial map? When $B = S^1$ Ferry has identified the obstructions: a $\tilde{K}_0$ obstruction in case (a), and a Wh obstruction in case (b). Chapman has results when $B = T^n$.

(QM17) Let f: M → B be a map, with M a compact Q-manifold and B a polyhedron. What is the obstruction to homotoping f to an approximate fibration? Ferry shows there is a Nil obstruction (and no other) when $B = S^1$. Chapman has partial results when $B = T^n$.

We next turn to sub-Q-manifolds of finite codimension. Chapman has shown that there exists a codimension 3 locally flat embedding of $S^3 \times Q$ in $S^3 \times Q$ which has no tubular neighborhood. The basic conjecture is that every locally flat Q-manifold embedding of finite codimension admits an h-block bundle neighborhood and that there are "computable" obstructions to the existence of tubular neighborhoods. Nowell has partial results. Specific problems include the following:

(QM18) Does there exist a Q-manifold pair of codimension greater than two having two non-isotopic tubular neighborhoods?

(QM19) Does there exist a Q-manifold pair having open tubular neighborhoods (of finite codimension) but no closed subtubes?

Finally we pose a compactification problem for Q-manifolds suggested (Brechner) by the well-known "non-separating plane continua" problem.

(QM20) Let X be a connected point-like compactum in Q such that $X = c\ell(\mathrm{int}(X))$ and $Q\setminus X$ is also point-like. Is there a compactification Y of $Q\setminus X$ such that $Y\setminus(Q\setminus X)$ is a Q-manifold and every homeomorphism $h: (Q,X) \to (Q,X)$ induces a homeomorphism $k: Y \to Y$ with $k = h$ on $Q\setminus X$?

VI. Topology of Non-Locally Compact Manifolds

The book by Bessaga and Pełczyński includes the basic structure and classification theorems. Typical models for non-locally compact manifolds are $s(\cong \ell_2)$, ℓ_2^f and Σ in the separable case, as well as non-separable Hilbert spaces and pre-Hilbert spaces (Anderson, Bessaga, Chapman, Henderson, Pełczyński, Schori, West). In the non-metrizable case the models are $\mathbb{R}^\infty$ and Q^∞ (Heisey). In all these cases, the manifolds are classified up to homeomorphism by homotopy type, and are triangulable (i.e., homeomorphic to polyhedron × model).

These theorems were inspired by similar theorems proved for C^∞ Banach manifolds during the late 1960's (Burghelea, Eells, Elworthy, Kuiper, Moulis).

There has been much recent interest in characterizing such manifolds, and finding them "in nature" (Sections TC and N) but the recent activity on the topology of such manifolds does not match the enormous interest in Q-manifolds. Many of the questions asked in Section QM (on Q-manifolds) have analogues for s. Usually the answers are easier; properties of an s-manifold often match those of (compact Q-manifolds) $\times$ [0,1). K-theoretic obstructions are not present.

We first deal with finite codimensional s-manifolds in s-manifolds. Chapman's example of a codimension 3 locally flat embedding of $S^3 \times Q$ in $S^3 \times Q$ which has no tubular neighborhood also works when Q is replaced by s. Questions analogous to (QM18) and (QM19) are sensible (the reader is left to state them). Other problems along these lines are:

(NLC1) Let $M \subset N$ be s-manifolds and let $R \subset M$. Suppose M has local codimension 1 at each point of $M \setminus R$. Does M have local codimension 1 at points of R when R is (a) a single point or (b) compact or (c) a Z-set in both M and N? Kuiper has given an example in codimension 2 where R is a single point or an n-cell or a copy of ℓ_2, such that M does not have local codimension 2 on R.

(NLC2) Same as (NLC1) for codimension greater than 2.

Other problems which survive from earlier editions are:

(NLC3) For M a separable C^∞ ℓ_2-manifold, can every homeomorphism of M onto itself be approximated by diffeomorphisms? Burghelea and Henderson have proved that such homeomorphisms are isotopic to diffeomorphisms.

(NLC4) Let M and K be s-manifolds with $K \subset M$ and K a Z-set in M. Then K may be considered as a "boundary" of M, i.e., for any $p \in K$ there exists an open set U in M with $p \in U$ and a homeomorphism h of U onto $s \times (0,1]$ such that $h(K \cap U) = s \times \{1\}$. Under what conditions on the pair (M,K) does there exist an embedding h of M in s such that the topological boundary of h(M) is h(K)? Sakai has good results on this question (e.g., it is sufficient that K contains a deformation retract of M, but not that K be a retract of M).

(NLC5) Let $\xi: E \to B$ be a fiber bundle over a paracompact space B with fiber F an s-manifold. Suppose K is a closed subset of E such that $K \cap \xi^{-1}(b)$ is a Z-set in each $\xi^{-1}(b)$. Is there a fiber-preserving homeomorphism of $E \setminus K$ onto E? The answer is yes when B is a polyhedron.

The most important non-metrizable models for infinite-dimensional manifolds are $\mathbf{R}^\infty \equiv \text{dir lim } \mathbf{R}^n$ and $Q^\infty \equiv \text{dir lim } Q^n$. $\mathbf{R}^\infty$ is well-known from algebraic topology, and Q^∞ occurs in functional analysis: any separable, reflexive infinite-dimensional Banach space endowed with its bounded weak topology is homeomorphic to Q^∞. Heisey has shown that the classification of $\mathbf{R}^\infty$- and Q^∞-manifolds is similar to that of s-manifolds (homotopy equivalence implies homeomorphism, etc.).

(NLC6) Does every homeomorphism between Z-sets in $\mathbf{R}^\infty$ or Q^∞ extend to an ambient homeomorphism? If so, is there an appropriate analogue of the Anderson-McCharen Z-set unknotting theorems for $\mathbf{R}^\infty$- and Q^∞-manifolds?

(NLC7) Are countable unions of Z-sets strongly negligible in $\mathbf{R}^\infty$- and Q^∞-manifolds?

(NLC8) Is there an analogue of Ferry's α-approximation theorem (see Section QM) for $\mathbf{R}^\infty$-manifolds?

At the 1970 International Congress in Nice (see p. 265, vol. 2 of those Proceedings), Palais suggested that some of the naturally arising smooth Banach manifolds of sections had the property that the transition maps in a suitably chosen atlas were not only diffeomorphisms but also homeomorphisms with respect to the bounded weak* topology (b* topology) of the Banach space model. Hence he suggested studying manifolds with two topologies: precisely, if B is a Banach space, a (C^p, b^*) *manifold* modelled on B is one in which the transitions are C^p diffeomorphisms and b*-homeomorphisms (see Heisey, Trans. A.M.S. 206 (1975)). *Note that a separable infinite-dimensional Banach space* B *endowed with its* b* *topology is homeomorphic to* Q^∞. When B is reflexive, the b* topology is the same as the bounded weak topology.

(NLC9) Are (C^p, b^*) manifolds stable? Do they embed as open subsets of their model? Are they classified by homotopy type? The above discussion shows that the answer to all these questions is yes if one looks at the C^p structure or the b* structure alone. The problem is to handle the two simultaneously.

VII. Topological Characterizations of Infinite-Dimensional Manifolds

We mean theorems whose hypotheses on a space X are topological, and whose conclusions say that X is a manifold with some specified infinite-dimensional model (Q,s, etc.). Such questions naturally break into two parts: when is X an ANR? and when is the ANR X a manifold? Questions on characterizing ANR's are dealt with in Section ANR.

Once X is known to be an ANR Toruńczyk's recent topological characterizations of Q-manifolds (among locally compact ANR's), of s-manifolds (among complete separable ANR's) and of manifolds modelled on various Hilbert spaces (among complete ANR's) have made obsolete many of the characterization questions posed in the last edition of this problem set. However, some interesting problems remain.

(TC1) Let G be a complete metrizable topological group which is an ANR. Is G a manifold modelled on some Fréchet space? In particular, if G is separable non-locally compact, is G an s-manifold? It is known (Fathi-Visetti, Toruńczyk, Montgomery, etc.) that a separable locally compact ANR group is a Lie group.

Toruńczyk's characterization of non-separable Fréchet spaces would be improved if the answer to the next question were known:

(TC2) If $X \times s \cong H$ for a non-separable Hilbert space H, is $X \cong H$?

Toruńczyk's characterization of s does not quite answer the next question:

(TC3) Is $X \cong s$ if X is a complete separable AR such that each compact subset is a Z-set?

(TC4) Let X be a topologically complete separable metric space.

(i) If X is an ANR, $Y \subset X$ is dense in X, and Y is an s-manifold, under what conditions can we conclude that X is an s-manifold? Toruńczyk has proved this result in the case that $X \setminus Y$ is a Z-set in X (recall that Z-sets are closed and thus Y is open in X).

(ii) Let M be an s-manifold, and suppose that $X \subset M$ is the closure of an open set Y. Under what conditions can we conclude that X is an s-manifold?

Henderson has observed relative to (i) that if Z-sets are strongly negligible in X and if $X \setminus Y$ is a countable union of Z-sets, then $X \cong Y$. However, it seems difficult to verify these conditions in many naturally arising cases. A theorem of Sakai is relevant: he gives conditions under which an s-manifold pair (X, X') with X' a Z-set can be embedded in s with X' as bicollared boundary.

The best understood models for incomplete separable infinite-dimensional manifolds are $\ell_2^f \equiv \{\langle x_i \rangle \in \ell_2 \mid \text{only finitely many } x_i \neq 0\}$ and Σ, the linear span in ℓ_2 of the Hilbert Cube $\{\langle x_i \rangle \in \ell_2 \mid -2^{-i} \leq x_i \leq 2^{-i}\}$. ℓ_2^f is an fd cap set in ℓ_2 and Σ is a cap set. Interesting examples of ℓ_2^f- and Σ-manifolds occur "in nature" (e.g., the space of PL homeomorphisms of a compact PL manifold, with compact open topology, is an ℓ_2^f-manifold) but there is still no topological characterization:

(TC5) Characterize ℓ_2^f-manifolds and Σ-manifolds topologically. Mogilski has partial results.

Note that, in (TC5), we are asking for an intrinsic characterization; if the space comes suitably embedded in a completion we have characterizations--namely: the completion should be an s-manifold (characterized by Toruńczyk) and the space should be an fdcap set, or a cap set, in the completion (Anderson, Bessaga-Pełczyński). Note that Σ is homeomorphic to rint Q, and is sometimes so designated in the literature.

(TC6) If G is a locally contractible separable metric topological group which is the countable union of compact finite-dimensional subsets and not locally compact, then is G an ℓ_2^f-manifold? The hypotheses imply that G is an ANR (Haver).

(TC7) Under what conditions on the inverse sequence $\{X_n;f_n\}$, where each X_n is a compact AR and each f_n is a CE map, is the inverse limit X homeomorphic to Q? Clearly (see Chapman's notes) $X \times Q \cong Q$.

(TC8) Under what conditions on the direct sequence $\{X_n;f_n\}$, where each X_n is an ANR and each f_n is an embedding, is the direct limit an $\mathbb{R}^\infty$- or Q^∞-manifold? (See section NLC for a discussion of $\mathbb{R}^\infty$ and Q^∞.)

Problems on homogeneous spaces are listed in Section ANR.

VIII. Group Actions on Infinite-Dimensional Manifolds

The article by Berstein and West in the Proceedings of the 1976 Summer Institute at Stanford (pp. 373-391 of Part 1 of Proc. of Symposia in Pure Mathematics, vol. 32, A.M.S. Providence, Rhode Island 1978) contains an excellent exposition of the problem of classifying based free compact Lie group actions on Q. The reader unfamiliar with the problem should consider the following special case, an answer to which would be a major breakthrough in I.D. topology:

(GA1) Let $h: Q \to Q$ be an involution with unique fixed point. Must h be topologically conjugate to the "standard" involution, multiplication by -1?

The reader should next read the Berstein-West article, where instead of an involution ($\mathbb{Z}_2$-action) the transformation group is any compact Lie group. But it must be repeated that the depth of the problem is already present in the $\mathbb{Z}_2$ case. Let $Q_0 = Q \setminus \{0\}$. Assume the involution h in (GA1) fixes 0, and consider the orbit space M of $h|Q_0$. M is a Q-manifold $K(\mathbb{Z}_2, 1)$. The main results on standardness of h are: (i) h is standard if and only if the end of M is movable (Wong-Berstein-West); (ii) infinite products of involutions on finite-dimensional AR's are standard (Berstein-West); (iii) h is standard if and only if the one-point compactification of M is an AR (Wong-West). Compare (iii) with the discussion of ANR divisors in Section SC. (i) and (ii) are known for all compact Lie groups--see the Berstein-West article, (iii) is known for finite groups and tori. An example of D. Edwards and Hastings (Springer Notes, vol. 542,

p. 203) shows that in the more general setting of pro-homotopy theory the question analogous to (GA1) has a negative answer.

(GA2) Same problem as (GA1) but with h having any finite period.

Here is the homotopy theoretic problem equivalent to (GA1) (see Berstein-West, p. 388); that is: there exists a non-standard involution as in (GA1) if and only if the answer to the next question is yes:

(GA3) Does there exist a sequence

$$(E_1, \widetilde{\mathbf{R}P}^\infty) \xleftarrow{\tilde{f}_1} \cdots \longleftarrow (E_{i-1}, \widetilde{\mathbf{R}P}^\infty) \xleftarrow{\tilde{f}_{i-1}} (E_i, \widetilde{\mathbf{R}P}^\infty) \longleftarrow \cdots$$

of principal $\mathbf{Z}_2$-bundles of CW complexes and bundle maps, each the identity on $\widetilde{\mathbf{R}P}^\infty$, such that, if $f_{i-1}\colon (B_i, \mathbf{R}P^\infty) \to (B_{i-1}, \mathbf{R}P^\infty)$ is the induced map on orbit spaces, then (i) each $(E_i, \widetilde{\mathbf{R}P}^\infty)$ is relatively finite; (ii) each $(B_i, \mathbf{R}P^\infty)$ is relatively 1-connected; (iii) each $\tilde{f}_i$ is null homotopic and (iv) each finite composition of f_i's is essential (as maps of pairs)?

(GA4) Let the compact Lie group G act semifreely on Q in two ways such that the fixed point sets are identical. If the orbit spaces are ANR's are the actions conjugate? The case of one fixed point is handled by West-Wong for certain G.

Recent results of S. Ferry and H. Toruńczyk show that the group of homeomorphisms of Q is an s-manifold. The constructions of West yield a based semi-free action of any compact Lie group, G, on Q. If G is a compact Lie group acting on Q, let $\mathrm{Homeo}_\alpha(Q)$ be the collection of all α-equivariant homeomorphisms of Q. If G is a finite cyclic group and α is a standard action, then Liem has shown $\mathrm{Homeo}_\alpha(Q)$ is locally contractible.

(GA5) If α is a standard action of a finite cyclic group, G, is $\mathrm{Homeo}_\alpha(Q)$ an s-manifold? Conversely, if $\mathrm{Homeo}_\alpha(Q)$ is an s-manifold is α standard?

(GA6) What is the structure of $\mathrm{Homeo}_\alpha(Q)$ for arbitrary actions of a compact Lie group, G?

An action α of a compact group on a Q-manifold, M, is called *factorable*, provided that there is a finite-dimensional manifold (or polyhedron), K, an action α_1 of G on K, an action α_2 of G on Q, and an equivariant homeomorphism from (M,α) on to $(K \times Q, \alpha_1 \times \alpha_2)$. Recent results of Liem have shown that if α is a free action and G is a finite group, then α factors into an action on a finite-dimensional manifold L and the identity on Q. Hence we pose:

(GA7) Under what conditions can a non-free action of a compact group G on a Q-manifold be factored?

If we assume the action α to be semi-free, then (GA7) may be reduced to:

(GA8) Let α be a semi-free action of a finite group G on Q, whose fixed point set, F, is a Z-set in Q and homeomorphic to a copy of Q. When is the action equivalent to the product $\sigma \times \mathrm{Id}_F$, where σ is the standard action of G on

Q? Alternatively, what happens if F is the n-cell, I^n?

Here are examples that can be factored.

1. The action $\sigma \times Id_{[0,1]}$ induces naturally a semi-free action $\tilde{\sigma}$ on $Q \cong Q \times [0,1]/\{Q \times 1\}$ whose fixed point set is $\{0\} \times [0,1]$. This action $\tilde{\sigma}$ is equivalent to $\sigma \times Id_{[0,1]}$.

2. Let K be an AR and L an AR which is a CE-decomposition of K via the CE map $K \xrightarrow{\varphi} L$. The action $\sigma \times Id_K$ induces naturally a semi-free action $\tilde{\sigma}$ on $(Q \times K)/\sim$, where $(0,x) \sim (0,y)$ iff $\varphi(x) = \varphi(y)$, whose fixed point set is L. It can be shown that $\tilde{\sigma}$ is equivalent to $\sigma \times Id_L$.

Now, on the product $Q \times [0,1]^n$, we consider fiber-preserving semi-free actions of a finite group G whose fixed point set is $\{0\} \times [0,1]^n$.

Liem has shown that if the restricted action on each fiber $Q \times t$ $(t \in [0,1]^n)$ is equivalent to the standard action σ, then the given action is actually equivalent to the product $\sigma \times Id_{[0,1]^n}$. This result induces a generalization of Wong's result: Given two based semi-free actions α, β on Q, then we can join α and β by a family of based semi-free actions to obtain a fiber-preserving semi-free action of G (finite group) on $Q \times [0,1]$ such that the natural map from the orbit space $(Q_0 \times [0,1])/\sim$ to $[0,1]$ is a Hurewicz fibration, where $Q_0 = Q - \{0\}$. A similar conclusion holds true for every parametrized family of based semi-free actions of a finite group over any n-cell$_{[0,1]^n}$.

(GA9) What can be said about fiber-preserving semi-free actions on $Q \times Q$?

(GA10) Suppose an exotic based semi-free action α on Q of a finite group G exists, and suppose given a level-preserving semi-free action of G on $Q \times [0,1]^n$ such that the restriction on $Q \times t$ ($t \in [0,1]^n$) is equivalent to α, is the given action equivalent to $\alpha \times Id_{[0,1]^n}$.?

IX. Characterizations of Infinite-Dimensional ANR's

Incentives for finding characterizations of ANR's are provided by Edwards' and Toruńczyk's results that products of ANR's with appropriate standard I-D spaces are infinite-dimensional manifolds, and by Toruńczyk's theorems characterizing infinite-dimensional manifolds among ANR's.

The following results, which give sufficient conditions for a space to be an ANR, have recently been useful.

(i) (*Haver*) If X is a locally contractible metric space that can be written as a countable union of finite-dimensional compacta then X is an ANR.

(ii) (*Toruńczyk*) X is an ANR iff there is a space E such that $X \times E$ has a basis $\mathcal{B}$ of open sets such that for any finite subcollection $\mathcal{C}$ of $\mathcal{B}$, the intersection $\cap\mathcal{C}$ is path-connected and all its homotopy groups are trivial.

(iii) (*Kozlowski*) Y is an ANR if there is an ANR X and a map $f: X \to Y$ onto a dense subset of Y with the property that for every open cover $\mathcal{V}$ of Y there exist a homotopy $h_t: X \to X$ ($0 \le t \le 1$) and a map $g: Y \to X$ such that $h_0 = id_X$, $h_1 = gf$, and the homotopy is limited by $f^{-1}\mathcal{V}$.

The following questions are inspired by the homeomorphism group problem (see HS).

(ANR1) If a metrizable space has a basis of contractible open neighborhoods, is it an ANR?

(ANR1a) If a topological group has a basis of contractible open neighborhoods, is it an ANR?

(ANR2) If a metrizable space is such that every open subset is homotopically dominated by a CW complex, is it an ANR?

A subset A of X is said to be *locally homotopy negligible* (abbrev. l.h.n.), provided that the inclusion $U\setminus A \to U$ is a weak homotopy equivalence for every open set U in X. Toruńczyk has shown this to be equivalent to his original definition of l.h.n. and has also shown that if X is an ANR and $X\setminus X_0$ is l.h.n., then X_0 is an ANR. Unfortunately, the converse of this last result is false: Taylor's example gives a CE map f: $Q \to Y$ such that Y is not an ANR, although Y is an l.h.n. subset of the mapping cylinder M(f) of f (Lacher, Toruńczyk) and M(f) - Y is an ANR.

According to Kozlowski, a subset A of X is *hazy*, provided the inclusion $U\setminus A \to U$ is a homotopy equivalence for every open subset U of X. He has shown that a map f: $X \to Y$ is a homotopy equivalence over every open subset of Y if and only if f is a fine homotopy equivalence. As a corollary one has that if $X\setminus X_0$ is hazy in X and X_0 is an ANR, then X is an ANR. There seems to be difficulty in verifying that a subset is hazy rather than just l.h.n. In particular, the following questions are open.

(ANR3) Is $X\setminus X_0$ hazy in X when

(a) X is a separable linear space and X_0 is the linear hull of a countable dense subset,

(b) X is the component of the identity in the homeomorphism group H(M) of a closed PL manifold M of dimension ≥ 5 and X_0 consists of all PL-homeomorphisms of M which are in X?

In (ANR3a) and (3b), it is known that $X \setminus X_0$ is l.h.n. We will return to (ANR3) in Sections HS and L.

We have already discussed the question of when the CE image of a locally compact ANR is an ANR (Section CE). That is certainly an aspect of the characterization question being discussed here.

Finally, there are problems on homogeneity. (X is *homogeneous* if there is a homeomorphism carrying any point to any other point.)

(ANR4) Let X be a non-trivial homogeneous contractible compactum. Is X an AR? Is $X \cong Q$?

(ANR5) Let X be a separable contractible homogeneous complete non-locally compact metric space. Is X an ANR? Is $X \cong s$? Compare (HS3).

(ANR6) More generally, when are homogeneous spaces ANR's?

X. The Space of Homeomorphisms of a Manifold

In this section all function spaces are understood to have the compact open topology.

Let M be a compact n-manifold; then H(M) denotes the space of homeomorphisms of M and $H_\partial(M)$ denotes the subspace of $H(M)$ consisting of those h which are the identity on the boundary ∂M (in case $\partial M = \emptyset$, $H_\partial(M) = H(M)$). It is known (Anderson) that the space $H_\partial(I)$ is homeomorphic to s ($\cong \ell_2$).

The following is the problem of greatest current interest involving s-manifolds and is often referred to as the "Homeomorphism Group Problem."

(HS1) For M a compact n-manifold ($n > 2$), is $H_\partial(M)$ an s-manifold?

Obviously $H_\partial(M)$ is a complete separable metric space. Toruńczyk's theorem that any complete separable metric ANR multiplied by s is an s-manifold, plus Geoghegan's theorem that $H_\partial(M) \times s \cong H_\partial(M)$ for a large class of spaces M (including manifolds, Q-manifolds and polyhedra) reduce (HS1) to:

(HS2) For M a compact n-manifold ($n > 2$) is $H_\partial(M)$ an ANR?

The version of (HS2) in which $n = 2$ was answered positively by Luke-Mason. The version in which M is a Q-manifold was answered positively by Ferry and by Toruńczyk. In both cases, the theorems cited above imply that $H_\partial(M)$ is an s-manifold.

Haver has given the following reduction of $H_\partial(M)$ being an ANR to the problem of showing $H_\partial(B^n)$ is an AR: For a given compact n-manifold M obtain a cover of M by n-cells B_i^n ($1 \le i \le p$); by Edwards and Kirby there is an open neighborhood N of the identity such that any $h \in N$ can be written as the composition $h = h_p \dots h_1$, where $h_i \in H_\partial(B_i^n)$, and the assignment $h \to (h_p, \dots, h_1)$ from N into $P = \Pi_{i=1}^p H_\partial(B_i^n)$ defines a map φ: N → P; clearly composition defines a map of an open neighborhood G of φN into N, which establishes $\varphi N \cong N$ as a retract of G; thus, by Hanner's theorem, if $H_\partial(B^n)$ is an AR, $H_\partial(M^n)$ is an ANR.

Consequently, (HS1) has been reduced to the following.

(HS3) Is $H_\partial(B^n)$ $(n > 2)$ an AR?

Except possibly when n = 4 or 5, a further reduction can be made. $PLH_\partial(B^n)$, the subspace consisting of PL homeomorphisms, is obviously locally contractible (Alexander Trick). So Connell's engulfing lemma (or low dimensional arguments) plus theorems of Geoghegan and Haver imply that $PLH_\partial(B^n)$ is an ANR and has locally homotopy negligible complement in $H_\partial(B^n)$ when $n \neq 4$ or 5. This, together with Kozlowski's theorem on hazy sets (Section ANR) and the Whitehead Theorem (used on each open set) reduces (HS1) to:

(HS4) For $n \neq 4$ or 5, is every open subset of $H_\partial(B^n)$ homotopically dominated by a CW complex?

A variation on the above argument gives a similar reduction for 5-dimensional PL manifolds M without boundary.

We should point out that the question analogous to (HS1) for spaces of PL homeomorphisms is solved. By combining theorems of the above-named authors and of Toruńczyk and Keesling-Wilson, $PLH_\partial(M)$ is an ℓ_2^f-manifold when M is a compact PL manifold.

Haver has studied $\overline{H}(M)$, the closure of H(M) in the space of mappings of a compact manifold M. He has shown that $\overline{H}(M)\setminus H(M)$ is a countable union of Z-sets in $\overline{H}(M)$ and, hence, it follows that if $\overline{H}(M)$ is an s-manifold, so is H(M).

(HS5) Is there a continuous map $\overline{H}(M) \to H(M)$, arbitrarily close to the identity map, whose image lies in H(M)? Note that for all dimensions except perhaps 4, $\overline{H}(M)$ is the space of CE maps (Armentrout, Siebenmann, Chapman).

(HS6) Is $\overline{H}(M)$ an ANR? If so it is an s-manifold (Toruńczyk, Geoghegan-Henderson). This question is open for Q-manifolds, too.

XI. Linear Spaces

Infinite-dimensional topology originated with problems posed by Fréchet and Banach on the topological (as distinct from the joint linear and topological) structure of linear spaces. Outstanding results include: (i) every compact convex infinite-dimensional subset of a Fréchet space is homeomorphic to Q (Keller-Klee), and (ii) every Fréchet space is homeomorphic to a Hilbert space (Anderson, Bessaga-Pełczyński, Kadec, Toruńczyk).

For readers unaccustomed to such matters, we define some terms. A *linear topological space*, X, is a real vector space carrying a topology with respect to which addition and scalar multiplication are continuous. An *invariant metric* for X is a metric d compatible with the topology of X such that d(x,y) = d(x - y,0). A norm (we assume this term is known) is a special kind of invariant metric. If a linear topological space is metrizable then it admits an invariant metric, and if it admits a complete metric, then all its invariant metrics are complete. Hence it is sensible to define a [*complete*] *linear metric space* to be a linear topological space which is [completely] metrizable. A *normed linear space* is a linear topological space which admits a norm. A complete normed linear space is a *Banach space.*

A linear topological space X is *locally convex* if 0 has a basic system of convex neighborhoods. Every normed linear space is locally convex, but there are familiar complete linear metric spaces which are not locally convex (e.g., L_p-spaces $0 < p < 1$). A locally convex complete linear metric space is a *Fréchet space* (s is an example of a Fréchet space which admits no norm). As we have said, all separable infinite-dimensional Fréchet spaces are homeomorphic to ℓ_2, hence to s.

A complete linear metric space is called an F-*space*. The topology of non-locally convex F-spaces (necessarily infinite-dimensional) is mysterious and gives rise to many of the problems in this section. [For more terminology, see the Bessaga-Pełczyński book]. The problems in this section have been proposed mainly by Bessaga, Dobrowolski, Terry and Toruńczyk.

(LS1) Is every F-space an AR? What about admissible F-spaces (i.e., ones such that for every compact convex K, the identity map of K can be uniformly approximated by maps of K into finite-dimensional linear subspaces)?

(LS2) Let X be an F-space with invariant metric d.

Let $\tilde{X}$ be the set of all functions $\lambda: X\setminus\{0\} \to \mathbf{R}$ such that $\rho(\lambda,0) \equiv \sum_{x\in X\setminus\{0\}} d(\lambda(x)x,0) < \infty$. Assume w.l.o.g. that $d(tx,0)$ is strictly increasing in t. Regard $\tilde{X}$ as an F-space with invariant metric ρ. Define u: $\tilde{X} \to X$ by $u(\lambda) = \sum_{x\in X\setminus\{0\}} \lambda(x)\cdot x$. Does u admit a continuous cross section?

(LS2) is motivated by Terry's construction of co-universal F-spaces of any given weight. $\tilde{X}$ is homeomorphic to a Hilbert space, so a positive answer to (LS2) would imply that X is an AR.

(LS3) Is every infinite-dimensional F-space X homeomorphic to $X \times s$? to $X \times Q$? to $X \times \mathbf{R}$?

(LS4) Are compacta negligible in infinite-dimensional F-spaces? Do homeomorphisms between compacta in an infinite-dimensional F-space X extend to homeomorphisms of X? Compare (NLC6) and (NLC7).

(LS5) Does every infinite-dimensional F-space contain an fdcap set?

(LS6) Let K be a convex subset of an F-space X. Is K a retract of X? Is K an AR? What if K is compact? Or closed?

(LS7) Does every compact convex subset of an F-space have the fixed point property? J. W. Roberts (Studia Math 60 (1977)) has shown that compact convex subsets of L_p spaces ($p < 1$) can fail to have extreme points. Are these counterexamples to (LS6) and (LS7)?

(LS8) For each $\varepsilon > 0$ does there exist an open cover G of ℓ_1 such that for each point p the sum of the diameters of the elements of G containing p is less than ε? If the answer is yes, Terry has a new way of recognizing infinite-dimensional ANR's.

(LS9) Is every convex subset of a Banach space (more generally an F-space) homeomorphic to a convex subset of a Hilbert space?

(LS10) Is every closed convex subset of a Hilbert space (more generally, of a Banach space) either locally compact or homeomorphic to a Hilbert space?

(LS11) Is every I-D separable normed space homeomorphic to some pre-Hilbert space, i.e., to a linear subspace (not necessarily closed) of a Hilbert space?

(LS12) Let X be an I-D separable pre-Hilbert space. Is $X \times R \cong X$? $X \times X \cong X$? $X_f^{\omega} \cong X$ or $X^{\omega} \cong X$? The answers are probably negative for the added condition of uniform homeomorphisms.

(LS13) If a σ-compact separable normed space E contains a topological copy Q' of Q, is E homeomorphic to $\{x \in \ell_2: \Sigma i^2 \cdot x_i^2 < \infty\}$? Note that the closed convex hull of Q' need not be compact.

(LS14) Let E be a locally convex linear metric space and let X be a noncomplete retract of E. Is $X \times E^{\omega} \cong E^{\omega}$? It is known by Toruńczyk that $X \times E^{\omega} \times \ell_2^f \cong E^{\omega} \times \ell_2^f$ and that if X is complete, then $X \times E^{\omega} \cong E^{\omega}$.

(LS15) Let X be a Banach space, GL(X) its general linear group, $||\cdot||$ the induced norm on GL(X) and w the topology of pointwise convergence on GL(X). Is the "identity map"

$$(GL(X), ||\cdot||) \to (GL(X), w)$$

a homotopy equivalence? It is conceivable that Wong's technique can be used to prove contractibility of (GL(X),w) for "infinitely divisible" spaces X.

XII. Infinite-Dimensional Manifolds in Nature

The development of infinite-dimensional topology is tied to the possibility of using it to prove theorems outside the subject. The first major example of this was the proof that for analysts who wish to classify Banach spaces there is nothing interesting at the purely topological level (see the introduction to Section LS); a negative result, but one of great importance. The second example, also negative, was the proof that for global analysts interested in Banach manifolds of maps, the differential topology of those manifolds is no richer than their homotopy theory (see the introduction to Section NLC). The third example was positive: that the algebraic K-theoretic invariants of polyhedra (e.g., Whitehead torsion, Wall's finiteness obstruction) are not lost when those polyhedra are turned into Q-manifolds by taking their cartesian product with Q, whereas many of the irrelevant finite-dimensional complications of those spaces disappear, making previously intractable problems tractable (see Section QM).

Other, less dramatic, contacts between I-D topology and the rest of mathematics exist; they have motivated every section of this problem set.

Here we consider problems on where infinite-dimensional manifolds are found "in nature," i.e., naturally occurring in mathematics. Some occurrences have already been handled in other sections: (complete separable ANR) × s is an s-manifold (Toruńczyk); (locally compact ANR) × Q is a Q-manifold (R. Edwards); homeomorphism groups of manifolds *may be* manifolds (Section HS).

For emphasis we repeat (HS1):

(N1) Let M be a compact n-manifold, n > 2. Is $H_\partial(M)$ an s-manifold? Reductions are given in Section HS.

The space of maps from a compact polyhedron to a locally compact polyhedron is, in general, an s-manifold (Eells, Geoghegan) and the subspace of PL maps is generally an ℓ_2^f-manifold (Geoghegan). But there may be interest in finding Q-manifold function spaces. Some have been found (Geoghegan, Jones), but better examples are probably available using Toruńczyk's characterization of Q-manifolds. Here is a representative problem:

(N2) Let (X,d) and (Y,ρ) be metric spaces, X being compact. Under what conditions is the space Lip(X,Y) of 1-Lipschitz maps a Q-manifold? (f is 1-*Lipschitz* if $\rho(f(p), f(q)) \le d(p,q)$ for all p,q.)

In a different direction we have:

(N3) Is the space of all $\mathbb{Z}_2$-actions on a compact

Q-manifold an s-manifold? Is it even LC^0? Ferry has positive answers for the space of *free* $\mathbf{Z}_2$-actions. Compare (GA5) and (GA6).

Another source of examples is the hyperspace. If X is a metric space 2^X denotes the hyperspace of non-empty compact subsets, and C(X) the subspace whose points are connected compacta; topologize by the Hausdorff metric. Combining work of West, Schori and Curtis one can prove that $2^X \cong Q$ if and only if X is a non-trivial Peano continuum ($\equiv$ compact, connected, locally connected metric space), and $C(X) \cong Q$ if and only if the Peano continuum X is non-trivial and contains no free arcs.

Further results on various subspaces of 2^X, where X is a nondegenerate Peano continuum have been obtained. In particular, for $A, A_1 \ldots, A_n \in 2^X$, the containment hyperspace $2^X_A = \{F \in 2^X : F \supset A\}$ is homeomorphic to Q if and only if $A \neq X$, while the intersection hyperspace $2^X(A_1 \ldots, A_n) = \{F \in 2^X : F \cap A_i \neq 0 \text{ for each } i\}$ is always homeomorphic to Q. Also, for every compact connected polyhedron K, there exists a hyperspace $2^K_{sst} \subset 2^K$ of "small" subsets of K such that $2^K_{sst} \cong K \times Q$ (Curtis-Schori).

For any set of primes P, let $\mathbf{Z}_P$ denote the localization of $\mathbf{Z}$ at P. Toruńczyk and West have found naturally occurring hyperspace models of $K(\mathbf{Z}_P, 2)$; they let X be the circle, S^1, and let α be the action by translation of S^1 on 2^X: the orbit space with the obvious copy of S^1 removed is a Q-manifold $K(\mathbf{Q}, 2)$ and the required $K(\mathbf{Z}_P, 2)$'s occur as sub-manifolds in a natural way.

(N4) Does a similar procedure work when G is any compact Lie group; specifically, does it yield interesting models of localizations of BG?

Other problems on hyperspaces are:

(N5) Let ℓ^∞ be the nonseparable Banach space of bounded real sequences, and let ~ be the equivalence relation in 2^{ℓ^∞} of isometry between compact metric spaces. Is the quotient space $2^{\ell^\infty}/\sim \,\cong \ell^2$? D. Edwards has obtained some basic properties of $2^{\ell^\infty}/\sim$ in a paper which claims some connection with ideas of the physicist J. A. Wheeler.

(N6) Let $F(X) \subset 2^X$ consist of finite subsets of X. If X is a compact connected polyhedron is F(X) an fdcap set in 2^X? Curtis has a positive answer when X = I.

(N7) Is F(Q) a cap set in 2^Q? X must be locally infinite-dimensional for F(X) to be a cap set (Curtis).

(N8) If $X \subset \mathbb{R}^2$ is a 2-cell containing no singular segments is the hyperspace of compact convex sets, cc(X), homeomorphic to Q? Theorems on cc(X) have been obtained by Curtis, Nadler, J. Quinn and Schori.

XIII. Topological Dynamics

In the last edition of this problem set two problems were posed concerning flows on Q-manifolds. One has been solved by Mañé who has proved that no infinite-dimensional compactum admits an expansive flow. The other has been solved independently by Fathi-Herman and Glasner-Weiss (work of Katok in 1972 probably implies a solution also). Oxtoby and Prasad have written on measure theory in Q.

Appendix

The problem set was compiled in April 1979. Since then comments and updatings have been received. This Appendix is being added in November 1979.

I. Introduction: The finite-dimensional conjecture following Toruńczyk's theorem is claimed by F. Quinn.

II. CE: Ancel improves Kozlowski's Theorem (1). Daverman has a new wild Cantor set in Q from which he constructs a non-CE map $Q \to Q$ whose point-inverses are acyclic finite-dimensional polyhedra, and whose non-degeneracy set is a Cantor set. (CE 1) answered negatively by van Mill; further refinements by Kozlowski-van Mill-Walsh; see below. (CE 2) solved by Kozlowski and Toruńczyk when Y is LC^1.

III. D: There is no dimension raising CE map on a 3-manifold (Kozlowski-Walsh). If there is a dimension raising CE map, then there is a CE map $f: X \to Y$ and an integer n, such that every point inverse is an AR of dimension $\leq n$, $\dim Y = \infty$, and f is not a hereditary shape equivalence (Kozlowski-van Mill-Walsh).

IV. SC: Concerning (SC 7) and (SC 8) see Appendix 3 of Chapman-Siebenmann (Acta Math. 1977) which was added in proof to that paper.

V. QM: See work by Väisälä on Lipschitz theory of Q-manifolds. (QM 9): See new relevant paper by Chapman.

VII. TC: Daverman-Walsh refine Toruńczyk's characterization of Q-manifolds. (TC 9) has been answered by Heisey-Toruńczyk.

VIII. GA: See Liem: Notices AMS (1979) page A-532.

X. HS: Questions on spaces of Lipschitz homeomorphisms which appear in a previous version of this problem set (Mathematical Center Tract 52 (1974), 141-175) may still be interesting.

XI. LS: The following comments on LS come from Bessaga and Dobrowolski:

(LS 1) Klee extended Leray-Schauder theory to admissible F-spaces.

(LS 2) is open even in the locally convex case, because it is not clear that ker u is locally convex: Michael's selection theorem requires this. (LS 2) is solved in the normed case.

(LS 4) On first part: Dobrowolski and Riley have a positive answer when X has a strictly weaker Hausdorff linear topology. Kalton has conjectured that s is the only ID F-space which fails to have a strictly weaker Hausdorff topology. On second part: Dobrowolski has a positive answer for finite-dimensional compacta.

(LS 5) The question should be asked for cap sets. The usual (Hamel basis) fdcap set works in any F-space. When the F-space is an AR, Mazur's Lemma yields a cap set.

(LS 6) K must be closed if it is to be a retract of X. If the closure of K is an AR, so is K.

(LS 7) What Roberts shows is that there exists an F-space containing a compact convex subset which has no extreme point.

(LS 9) Bessaga can show that locally compact convex sets can be affinely embedded in a Hilbert space.

XII. N: Colvin has new Q-manifold function spaces.

XIII. TD: See Keynes and Sears, Notices AMS (1979) page A-561.