THE PARACOMPACTNESS OF PREPARACOMPACT SPACES

by

S. W. Davis and J. C. Smith

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1. Introduction

In this paper we present an extension of the work begun in [S_1] and [D_1]. We introduce the notion of isoparacompactness which is that all closed preparacompact subsets are paracompact. In [S_1], it is shown that in the class of q-spaces many of the popular weak covering properties imply isoparacompactness. In [D_1], similar results are proved for the class of Lob-spaces. It is our purpose here to study the property of isoparacompactness itself. For convenience, we assume that all spaces are at least T_1. We repeat now a few relevant definitions.

Definition 1.1 [Br]. A T_2 space X is called *preparacompact* (respectively, $\aleph$-preparacompact) if each open cover of X has an open refinement $H = \{H_\alpha: \alpha \in A\}$ such that if $B \subset A$ is infinite (respectively, uncountable) and if $p_\beta \in H_\beta$ and $q_\beta \in H_\beta$ for each $\beta \in B$ with $p_\alpha \neq p_\beta$ and $q_\alpha \neq q_\beta$ for $\alpha \neq \beta$, then the set $Q = \{q_\beta: \beta \in B\}$ has a limit point iff the set $P = \{p_\beta: \beta \in B\}$ has a limit point.

The collection H in 1.1 will be called a ppc-collection (respectively, $\aleph$-ppc-collection). The terms "σ-ppc-collection" and "σ-$\aleph$-ppc-collection" have the obvious meanings, namely, countable unions of these types of collections.

Definition 1.2. A space X is called *isoparacompact* if

each closed preparacompact subset of X is paracompact.

Definition 1.3 [D_1]. A space X is called an *Lob-space*
if for each x ∈ X there is an open neighborhood base U_x
which is linearly ordered by reverse subset inclusion.

Definition 1.4 [M_1]. A space X is called a q-*space* if
at each point x ∈ X there is a sequence ⟨N(x,n): n ∈ ω⟩ of
neighborhoods of x such that if x_n ∈ N(x,n) for each n ∈ ω,
then ⟨x_n: n ∈ ω ⟩ has a cluster point.

Definition 1.5 [N]. A space X is called a *quasi-k-space*
if a subset F ⊂ X is closed in X iff F ∩ C is closed in C
for every countably compact C ⊂ X.

In this paper we will use the assumption "X is an Lob-
space" fairly often. In every instance, this may be replaced
by "X is a q-space" with the result remaining valid. Occa-
sionally, "X is a quasi-k-space" may be used. The interested
reader is referred to [S_4]. Michael has proved that all
q-spaces are quasi-k-spaces [M_3], but there is no subclass
relationship between the Lob-spaces and either the q-spaces
or the quasi-k-spaces as can be seen from examples 3.7 and
3.8 of [D_1].

The following theorem summarizes the results from [S_1]
and [D_1] regarding isoparacompactness.

Theorem 1.6. *If X is a regular Lob-space, then each of*
the following will imply that X is isoparacompact:

  1. X *is* θ-*refinable.*

  2. X *is* δθ-*refinable.*

3. X *is weakly* $\overline{\theta}$*-refinable.*

4. X *is weakly* $\overline{\theta\theta}$*-refinable.*

5. *Every closed subset of* X *is irreducible.*

6. *Every open cover of* X *has a* σ*-weakly discrete*
    refinement.

In large part, the interest in this condition stems
from the fact that all of these covering properties also
imply isocompactness [Ba]. In fact, we shall see that all
isoparacompact spaces are isocompact. We remind the reader
that a space is *isocompact* if each of its closed countably
compact subsets is compact.

2. Properties of Isoparacompact Spaces

We open this section with the theorem to which we
alluded in the preceding paragraph.

Theorem 2.1. Every isoparacompact space is isocompact.

Proof. If X is isoparacompact and A is a closed counta-
bly compact subset of X, then A is preparacompact. Hence A
is paracompact and countably compact, so A is compact.
Thus X is isocompact.

We now present a group of preservation theorems. Inter-
woven in this presentation of the preservation of isopara-
compactness will be the presentation of certain preservation
theorems for preparacompactness. These have not been studied
previously and are, of course, fundamental to the study of
isoparacompactness.

In order to prove that isoparacompactness is preserved
by perfect mappings, we need that preparacompactness is

inversely preserved by perfect mappings. Unfortunately,
this is not true.

    Example 2.2.   There is a space X which is prepara-
compact, but X × (ω + 1) is not preparacompact.

    Proof. We construct X by what we think of as levels.
That is X = $\cup_{n \in \omega} L_n$ where L_n is defined as follows:

    Let L_0 be the discrete space $2^{\omega 1} \times \{0\}$. Suppose n ∈ ω
and L_n is defined. Choose $\{A_{\alpha,n} : \alpha < 2^{\omega 1}\}$ a maximal almost
disjoint collection of subsets of cardinality ω_1 contained
in L_n. [By "almost disjoint" we mean $A_{\alpha,n} \cap A_{\beta,n}$ is countable
if α ≠ β.] The set L_{n+1} will be the set $2^{\omega 1} \times \{n + 1\}$, and
a subset U of $\cup_{k \leq n+1} L_k$ will be open provided U ∩ ($\cup_{k \leq n} L_k$) is
open and if (α,n + 1) ∈ U ∩ L_{n+1} then $|A_{\alpha,n} \setminus U| \leq \omega$.

    The space X = $\cup_{n \in \omega} L_n$. Every uncountable subset of X
has a limit point, but all countable subsets are closed and
discrete. Hence X is preparacompact, $\aleph_1$-compact, and not
metacompact. To see that X × (ω + 1) is not preparacompact,
we use the following lemma.

    *Lemma. If Z is a non-metacompact space such that at
each point x ∈ Z, on every one-to-one sequence of neighbor-
hoods of x there is a one-to-one choice function with closed
discrete range, then Z × (ω + 1) is not preparacompact.*

    Proof of Lemma. Let $\mathcal{U}$ be an open cover of Z with no
point finite refinement. $\mathcal{U}* = \{U \times (\omega + 1) : U \in \mathcal{U}\}$ is an
open cover of Z × (ω + 1). Suppose $\mathcal{V}$ is an open refinement
of $\mathcal{U}*$. Choose x ∈ Z such that ord((x,ω),$\mathcal{V}$) $\geq$ ω. Choose
$\{V_n : n \in \omega\} \subset \mathcal{V}$ such that $\pi_1(V_n) \neq \pi_1(V_m)$ for n ≠ m and

$(x,\omega) \in \cap_{n\in\omega}V_n$, and for $n \in \omega$ choose V_n' open in Z and $k_n \in \omega$ with $(x,\omega) \in V_n' \times (k_n,\omega] \subseteq V_n$. For each $n \in \omega$, let $p_n = (x,k_n + n)$. Now we have $(x,\omega) \in \overline{\{p_n: n \in \omega\}} \setminus \{p_n: n \in \omega\}$. For each $n \in \omega$, choose $x_n \in V_n'$ such that $\{x_n: n \in \omega\}$ is an infinite closed discrete set, and let $q_n = (x_n,\omega)$. Now $\{q_n: n \in \omega\}$ is closed and discrete, so V is not a ppc-collection. Hence U^* has no open ppc-refinement.

We next prove that X is Hausdorff. It is easy to see that it suffices to show that if D and E are disjoint sets on level L_n each with cardinality less than or equal to ω_1, then we can get disjoint weak neighborhoods of D and E on level L_{n-1}. Suppose $D = \{d_\alpha: \alpha < \omega_1\}$ and $E = \{e_\beta: \beta < \omega_1\}$. Let $\{A_x: x \in L_n\}$ be the maximal almost disjoint collection in L_{n-1} which defines the weak base on L_n. Let $U_{d_0} = A_{d_0}$ and $V_{e_0} = A_{e_0}\setminus U_{d_0}$. For $0 < \alpha < \omega_1$, let $U_{d_\alpha} = A_{d_\alpha}\setminus\cup_{\beta<\alpha}V_{e_\beta}$ and $V_{e_\alpha} = A_{e_\alpha}\setminus\cup_{\beta\leq\alpha}U_{d_\beta}$. Now $U = \cup_{\alpha<\omega_1}U_{d_\alpha}$ and $V = \cup_{\alpha<\omega_1}V_{e_\alpha}$ do the job.

This example is clearly weakly θ-refinable. Thus it is only the second published example of Hausdorff weakly θ-refinable $\aleph_1$-compact space which is not Lindelöf [vDW]. It bears a strong similarity to the T_1 example of Wicke [W]. This example also satisfies property L of Bacon [Ba] and is $\aleph_1$-compact, but is not Lindelöf. This answers question 3.3 of $[D_2]$. It also, of course, is an isocompact space which is not isoparacompact.

Failing to get the perfect preimage theorem, we settle for the following consolation prize which does have some

interesting consequences.

     We first recall the following lemma which is proved

in [D₁]. We shall need only the case for collections of

finite sets in our applications here.

     Lemma 2.3. *Suppose $\mathcal{G}$ is a collection of subsets of a*

space X, x $\in$ X, and there is a linearly ordered base at x.

If x $\in$ $\overline{\cup\mathcal{G}}$, then either there exists G $\in$ $\mathcal{G}$ with x $\in$ $\overline{G}$ or there

exists $\mathcal{G}'$ $\subset$ $\mathcal{G}$ and a choice function y on $\mathcal{G}'$ with

x $\in$ $\overline{\{y(G): G \in \mathcal{G}'\}}$.

     There is also a somewhat weaker version of this lemma

which holds in quasi-k-spaces (and hence q-spaces) which

will allow us the results we want.[*]

     [*]*Lemma* 2.3.1. *If X is a quasi-k-space and $\mathcal{G}$ is a collec-*

tion of subsets which is not closure-preserving, then there

is a subset $\mathcal{G}'$ $\subset$ $\mathcal{G}$ and a choice function y on $\mathcal{G}'$ such that

$\{y(G): G \in \mathcal{G}'\}$ has limit points.

     Interestingly, the full strength of 2.3 cannot be ob-

tained for q-spaces, i.e. we cannot specify what the limit

points will be. We are indebted to D. K. Burke for the fol-

lowing example.

     Example 2.4. Let N denote the natural numbers. For

n $\in$ N, we let B_n = {k $\in$ N: $\frac{n(n-1)}{2}$ < k $\leqq$ $\frac{n(n+1)}{2}$}. Note

that B_1 = {1}, B_2 = {2,3}, B_3 = {4,5,6}, $\cdots$, and $\cup_{n=1}^{\infty} B_n$ = N.

We define a filter t on N by U $\in$ t iff there exists k $\in$ N

such that $|B_n \backslash U|$ $\leqq$ k for every n $\in$ N. Extend t to an ultra-

filter p on N. Now, in the q-space βN, p is a limit point

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     [*]See note added in proof.

of $N = \bigcup_{n=1}^{\infty} B_n$, but there is no choice function on the set $\{B_n: n \in N\}$ with p in its closure.

Theorem 2.5. *If* Y *is a preparacompact Lob-space and* f: X → Y *is a continuous, closed, finite-to-one mapping, then* X *is preparacompact.*

Proof. Suppose $\mathcal{U}$ is an open cover of X. Well order Y. For each $y \in Y$, choose $\{U(y,1),U(y,2),\cdots,U(y,n(y))\} \subset \mathcal{U}$ such that $f^{-1}(y) \subset \bigcup_{i=1}^{n(y)} U(y,i)$. Choose V_y open in Y with $f^{-1}(y) \subset f^{-1}(V_y) \subset \bigcup_{n=1}^{n(y)} U(y,i)$. Pick an open ppc-refinement $\mathcal{H} = \{H_a: \alpha \in A\}$ of $\{V_y: y \in Y\}$. For each $\alpha \in A$, choose y_α the first element of Y with $H_\alpha \subset V_{y_\alpha}$. Let $G_{\alpha,i} = f^{-1}(H_\alpha) \cap U(y_\alpha,i)$ for $\alpha \in A$, $1 \leq i \leq n(y_\alpha)$. Now $\mathcal{G} = \{G_{\alpha,i}: \alpha \in A, 1 \leq i \leq n(y_\alpha)\}$ is an open refinement of $\mathcal{U}$. We will prove that $\mathcal{G}$ is a ppc-collection. Suppose $p(\alpha,i) \in G_{\alpha,i}$, $q(\alpha,i) \in G_{\alpha,i}$ for $(\alpha,i) \in B$ with $|B| \geq \omega$, and $p(\alpha,i) \neq p(\beta,j)$, $q(\alpha,i) \neq q(\beta,j)$ for $(\alpha,i) \neq (\beta,j)$, and $\{p(\alpha,i): (\alpha,i) \in B\}$ has no limit points, while $\{q(\alpha,i): (\alpha,i) \in B\}$ has a limit point, say x. Let $B' = B \setminus \{(\alpha,i): f(x) = f(q(\alpha,i))\}$. Since X is T_1, and fibers are finite, we have that $|B'| \geq \omega$ and x is a limit point of $\{q(\alpha,i): (\alpha,i) \in B'\}$. Now $f(x) \in \overline{\{f(q(\alpha,i)): (\alpha,i) \in B'\}} \setminus f(q(\alpha,i)): (\alpha,i) \in B'\}$ while $\{f(p(\alpha,i)): (\alpha,i) \in B'\}$ is closed, discrete and infinite. Moreover, $f(p(\alpha,i)) \in H_\alpha$ and $f(q(\alpha,i)) \in H_\alpha$ for each $(\alpha,i) \in B'$. Now using Lemma 2.3 we can select from these sets choice functions which will contradict the ppc condition on $\mathcal{H}$. Hence $\mathcal{G}$ is a ppc-collection, and the result is proved.

Corollary 2.5.1.   If X is isoparacompact, Y is an Lob-space and f: X → Y is a continuous, closed, finite-to-one mapping, then Y is isoparacompact.

Proof.   Suppose A is a closed preparacompact subset of Y. By 2.5, $f^{-1}(A)$ is a closed preparacompact subset of X. Hence $f^{-1}(A)$ is paracompact, and thus so is A = $f(f^{-1}(A))$.

Corollary 2.5.2.   If an Lob-space X is the union of the locally finite closed collection $\{F_\alpha : \alpha \in A\}$ and F_α is iso-paracompact for each $\alpha \in A$, then X is isoparacompact.

Proof.   The natural mapping from the disjoint union of $\{F_\alpha : \alpha \in A\}$ onto X is continuous, closed and finite-to-one. The result now follows from 2.5.

In addition to the "locally finite sum" theorem we have just obtained, we also have the "countable sum" theorem.

Theorem 2.6.   If X is a regular Lob-space and X = $\cup_{n \in \omega} F_n$ where F_n is closed and isoparacompact, then X is isoparacompact.

Proof.   Suppose A is a closed preparacompact subset of X. Since F_n is isoparacompact, we have that A $\cap$ F_n is para-compact for each $n \in \omega$. Thus A = $\cup_{n \in \omega} A \cap F_n$ is a subpara-compact and preparacompact Lob-space and hence A is para-compact.

Corollary 2.6.1.   If A is an F_σ-subset of an isopara-compact Lob-space, then A is isoparacompact.

3. Expandability Conditions

Considerable success has been enjoyed, particularly by Smith [S_2], [S_3], Smith and Krajewski [SK], and Junnila [J],

in using expandability conditions to characterize certain strong separation axioms and covering properties. In this section, we investigate the uses of expandability conditions which are based on ppc-collections.

Definition 3.1. A space X is called (discretely) *ppc-expandable* iff whenever $\{F_\alpha: \alpha \in A\}$ is a (discrete) locally finite collection of closed subsets of X, there is an open ppc-cpllection $\{H_\alpha: \alpha \in A\}$ with $F_\alpha \subset H_\alpha$ for each $\alpha \in A$. [We are assuming a one-to-one indexing of $\{F_\alpha: \alpha \in A\}$.]

The analogous meaning is given to ℵ-ppc-expandable and σ-ℵ-ppc-expandable and their discrete versions.

Theorem 3.2. *Every preparacompact Lob-space is ppc-expandable.*

Proof. Suppose $\mathcal{J} = \{F_\alpha: \alpha \in A\}$ is a locally finite collection of closed subsets of the preparacompact Lob-space X. For each $x \in X$, choose U_x open which meets only finitely many elements of $\mathcal{J}$. The collection $\{U_x: x \in X\}$ is an open cover of X, and we pick an open ppc-refinement $\mathcal{H} = \{H_i: i \in I\}$. For each $\alpha \in A$, we let $G_\alpha = \cup\{H \in \mathcal{H}: H \cap F_\alpha \neq \emptyset\}$. The collection $\{G_\alpha: \alpha \in A\}$ is open, and $F_\alpha \subset G_\alpha$ for each $\alpha \in A$. Suppose $p_\alpha \in G_\alpha$ for each $\alpha \in A' \subset A$ and that $\{p_\alpha: \alpha \in A'\}$ has a limit point. For each $\alpha \in A'$, choose $i_\alpha \in I$ such that $p_\alpha \in H_{i_\alpha}$ and $H_{i_\alpha} \cap F_\alpha \neq \emptyset$. Since each element of $\mathcal{H}$ can meet only finitely many elements of $\mathcal{J}$, we have that $\{i_\alpha: \alpha \in A'\}$ is infinite. Let $B_\alpha = \{p_\beta: i_\beta = i_\alpha\}$ for each $\alpha \in A'' \subseteq A'$, where A" is chosen so that $\alpha_1 \neq \alpha_2$ implies $B_{\alpha_1} \cap B_{\alpha_2} = \emptyset$ and $\cup_{\alpha \in A''}B_\alpha = \{p_\beta: \beta \in A'\}$. By Lemma 2.3 we choose $p_\alpha \in B_\alpha$ for each $\alpha \in A''$ such that A" is

infinite and $\{p_\alpha: \alpha \in A''\}$ has a limit point. Now choose
$q_\alpha \in H_{i_\alpha} \cap F_\alpha$ for each $\alpha \in A''$, and $\{q_\alpha: \alpha \in A''\}$ is closed
and discrete which contradicts the ppc-condition on $\{H_i:$
$i \in I\}$. Thus every choice function on $\{G_\alpha: \alpha \in A\}$ has closed
discrete range, so this is a ppc-expansion of $\mathcal{J}$.

One easily sees that without the Lob-space assumption
the above proof will work if $\mathcal{J}$ is taken to be a discrete
collection. Thus we have the following result.

Theorem 3.3. Every preparacompact space is discretely
ppc-expandable.

We next recall another lemma from $[D_1]$. As was the case
in 2.3, a slightly different version holds for q-spaces and
quasi-k-spaces, $[S_1]$, $[S_4]$. Strangely, unlike 2.3, in this
case we have the stronger result with q-space.

Lemma 3.4. Let X be an Lob-space and let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$
be an $\aleph$-ppc-collection of subsets of X. If there exists a
discrete collection $\{D_\beta: \beta \in B \subset A\}$ of non-empty subsets of X
such that $D_\beta \subset G_\beta$ for each $\beta \in B$, then $\{G_\beta: \beta \in B\}$ is either
countable or closure preserving.

Lemma 3.4.1. Let X be a q-space and let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$
be an $\aleph$-ppc-collection of subsets of X. If there exists a
discrete collection $\{D_\beta: \beta \in B \subset A\}$ of non-empty subsets of
X such that $D_\beta \subset G_\beta$ for each $\beta \in B$, then $\{G_\beta: \beta \in B\}$ is
either countable or locally finite.

Lemma 3.4.2. Let X be a quasi-k-space and let $\mathcal{G} =$
$\{G_\alpha: \alpha \in A\}$ and $\mathcal{H} = \{H_\alpha: \alpha \in A\}$ be $\aleph$-ppc-collections such

that $\overline{G}_\alpha \subset H_\alpha$ *for each* $\alpha \in A$. *If there exists a discrete collection* $\{D_\beta : \beta \in B \subset A\}$ *of non-empty subsets of* X *such that* $D_\beta \subset G_\beta$ *for each* $\beta \in B$, *then* $\{G_\beta : \beta \in B\}$ *is either countable or closure preserving.*

In these lemmas, if "ℵ-ppc" is replaced by "ppc" we have no need to consider countability in our conclusions.

It is also clear that to get our results for the quasi-k-space case a bit of regularity will be needed.

We will now break for a moment with our comments preceding 1.6 and spotlight a theorem of q-spaces. The reason for this is the strength of 3.4.1 as opposed to 3.4, from which the result easily follows:

Theorem 3.5. *If* X *is a ppc-expandable q-space, then* X *is expandable.*

For Lob-spaces, we have only the following:

Theorem 3.5.1. *If* X *is a ppc-expandable Lob-space, then every locally finite collection of closed sets can be expanded to a closure preserving collection of open sets.*

We again use lemma 3.4 to obtain the next pair of theorems. Since the proofs are so similar, we shall do them together.

Theorem 3.6. *If* X *is a regular discretely ppc-expandable Lob-space, then* X *is collectionwise Hausdorff.*

Theorem 3.7. *If* X *is a normal discretely ppc-expandable Lob-space, then* X *is collectionwise normal.*

Proof. Let $\{D_\alpha : \alpha \in A\}$ be a discrete collection of closed sets (for 3.6 these will be singletons). We expand to an open ppc-collection $\{H_\alpha : \alpha \in A\}$. For each $\alpha \in A$, we use normality in 3.7 and regularity in 3.6 to obtain an open set V_α with $D_\alpha \subset V_\alpha$ and $\overline{V}_\alpha \cap (\cup_{\beta \neq \alpha} D_\beta) = \emptyset$. Let $G_\alpha = V_\alpha \cap H_\alpha$ for each $\alpha \in A$. Then $\{G_\alpha : \alpha \in A\}$ is an open ppc-collection and by the lemma is closure preserving. For $\alpha \in A$, let $U_\alpha = G_\alpha \setminus \cup_{\beta \neq \alpha} \overline{G}_\beta$, then $\{U_\alpha : \alpha \in A\}$ separates $\{D_\alpha : \alpha \in A\}$.

We note that in 3.6 and 3.7 we could use "$\aleph$-ppc" in the place of "ppc" since countable collections can be picked apart inductively using just the regularity or normality.

By using the approach presented in §3 of $[S_4]$, one can obtain the following characterization of paracompactness.

Theorem 3.8. *Suppose X is a regular Lob-space. Then X is paracompact iff X is weakly $\overline{\theta}$-refinable and discretely $\aleph$-ppc-expandable.*

This result leads one to wonder if others of the pro-perties listed in 1.6 can be used with a discrete ppc-expandability condition to characterize paracompactness. Certain of these questions remain open, and we shall defer discussion of them to a later section. Our next example answers this question in the negative for irreducibility.

Example 3.9. There is a machine $\mathcal{G}$ such that if X is any space, $\mathcal{G}(X)$ is an irreducible space and X can be embed-ded in $\mathcal{G}(X)$ as a closed subspace. Moreover, $\mathcal{G}$ preserves cardinality (if infinite), character, separation, closed sets being G_δ's, collectionwise normality, collectionwise

Hausdorffness, Lob-space and q-space.

Proof. For any space X, $\mathcal{I}(X)$ is obtained from the product space $X \times (\omega + 1)$ by isolating the points of $X \times \omega$. Clearly, X is homeomorphic to $X \times \{\omega\}$.

We now show that $\mathcal{I}(X)$ is irreducible. Suppose $\mathcal{U}$ is an open cover of $\mathcal{I}(X)$. For each $x \in X$, choose $n_x \in \omega$ and U_x open in X such that $U_x \times [n_x, \omega]$ is contained in some member of $\mathcal{U}$. Let $A_n = \{x : n_x = n\}$. Let $\mathcal{H}_0 = \{(U_x \times [n_x + 1, \omega]) \cup \{(x, n_x)\} : x \in A_0\}$. For $n > 0$, let $\mathcal{H}_n = \{(U_x \times \{[n_x + 1, \omega])$ $\cup \{(x, n_x)\} : x \in A_n$ and $(x, \omega) \notin \cup(U_{k<n}\mathcal{H}_k)\}$. Let $\mathcal{H}_\omega = \{\{x\} : x \in \mathcal{I}(X) \setminus \cup(U_{n \in \omega}\mathcal{H}_n)\}$. Now $\mathcal{G} = U_{n \leq \omega}\mathcal{H}_n$ is an irreducible open refinement of $\mathcal{U}$.

The verifications for the remaining properties are routine, and we omit them. This may not be a new construction. The result that every space can be embedded as a closed subspace of an irreducible space was observed by E. K. van Douwen in 1976. We have not seen the construction he used, and do not know if it has appeared in print. It would seem that van Douwen's construction must have been similar to this one.

Example 3.9.1. $\mathcal{I}(\omega_1)$ is irreducible, collectionwise normal but not paracompact.

Example 3.9.2. Assume Axiom ♠ (which is true in the model L, hence is consistent with ZFC). Let S be deCaux's space [dC]. Then $\mathcal{I}(S)$ is irreducible, weakly θ-refinable, collectionwise normal, and not paracompact.

4. Questions

Question 4.1. Must every weakly θ-refinable,

preparacompact Lob-space be paracompact?

The example Γ of van Douwen and Wicke [vDW] shows that the answer to this is in the negative for $\aleph$-preparacompact. Also, example 2.2 shows that the answer to this is in the negative if the Lob-space assumption is dropped.

Question 4.2. Must every preparacompact Lob-space which satisfies property L [Ba] be paracompact?

Again, example 2.2 answers this in the negative for the non-Lob-space case.

Question 4.3. Must every $\delta\theta$-refinable, ppc-expandable Lob-space be paracompact?

If the expandable condition is replaced by the corresponding covering property, then the answer to 4.3 is in the affirmative, see [Thm. 3.4, D_1]. If the "$\delta\theta$" is replaced by "weakly $\overline{\theta}$," then the answer is again in the affirmative, see 3.8.

Question 4.4. Is isoparacompactness inversely preserved by perfect mappings?

One could obtain a positive answer to this by getting a positive answer to the following.

Question 4.5. Is the perfect image of a preparacompact space necessarily preparacompact?

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Added in Proof: Lemma 2.3.1 is correct as stated for q-spaces.
Jim Boone has pointed out that it is trivially false for
quasi-k-spaces and that in this case the sets $G \in \mathcal{G}'$ must be
closed.