TOPOLOGY PROCEEDINGS

Volume 4, 1979

Pages 361–370

http://topology.auburn.edu/tp/

EMBEDDING FINITE COVERING SPACES INTO BUNDLES

by

P. F. Duvall and L. S. Husch

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

EMBEDDING FINITE COVERING SPACES INTO BUNDLES

P. F. Duvall and L. S. Husch[1]

1. Introduction

In [H], Hansen shows that if X has the homotopy type of a CW-complex of dimension $\ell \geq 1$ and if $p: \tilde{X} \to X$ is a finite covering map, then there exists an embedding $g: \tilde{X} \to X \times \mathbf{R}^{\ell+1}$ such that $\rho g = p$ where $\rho: X \times \mathbf{R}^{\ell+1} \to X$ is projection and $\mathbf{R}^{\ell+1}$ is Euclidean$(\ell+1)$-space. He shows that this is the best possible result for trivial bundles since when the standard 2-fold covering, $p: S^\ell \to \mathbf{R}P^\ell$, of the real projective ℓ-space is considered, there exists no embedding $g: S^\ell \to \mathbf{R}P^\ell \times \mathbf{R}^\ell$ such that $\rho g = p$. We extend this result as follows.

Theorem 1. *Let* X *be a connected, topological space with the homotopy type of a locally finite simplicial complex of dimension* ℓ. *Let* $p: \tilde{X} \to X$ *be a finite* n-*fold covering map and let* $f: V \to X$ *be an* $\mathbf{R}^k$-*bundle over* X. *If* $k > \ell$, *then there exists an embedding* $g: \tilde{X} \to V$ *such that* $f \circ g = p$.

The proof of Theorem 1 reduces to showing the existence of a cross-section of a certain bundle over X with fiber $C_n(\mathbf{R}^k)$, the configuration space of n unordered points in $\mathbf{R}^k$. As a result, we also obtain the following.

Theorem 2. *Let* X, p, f *be as in Theorem* 1. *If*

[1]Research of both authors was supported in part by an N.S.F. grant.

$k = \ell \geq 4$, *then there exists a unique obstruction in* $H^k(X;\hat{\pi}_{k-1}(C_n(\mathbf{R}^k)))$, *the* k^{th} *cohomology group of* X *with local coefficients (in the sense of* [B]*) whose vanishing is a necessary and sufficient condition that there exists an embedding* $g: \tilde{X} \to V$ *such that* $f \circ g = p$.

In [D-H], we show that if X is a closed orientable manifold of even dimension $k > 2$ and if $f: V \to X$ is an orientable $\mathbf{R}^k$-bundle over X, then the obstruction is the Euler class of the bundle $f: V \to X$. Results of this type have consequences in embedding k-dimensional compacta up to shape in Euclidean 2k-space, $\mathbf{R}^{2k}$.

2. Preliminaries

$f: V \to X$ is an $\mathbf{R}^k$-*bundle* if f is a bundle in the sense of [S]. In particular, f is a locally trivial map. We shall suppress the structure group which could be some subgroup of the group of homeomorphisms of $\mathbf{R}^k$ with the compact-open topology.

Let $F_n(\mathbf{R}^k) = \{(x_1,x_2,\cdots,x_n) \in (\mathbf{R}^k)^n | x_i \neq x_j \text{ for } i \neq j\}$. The symmetric group on n symbols, Σ_n, acts freely on $F_n(\mathbf{R}^k)$ by permutation of coordinates: if $\sigma \in \Sigma_n$, then $\sigma(x_1,x_2,\cdots,x_n) = (x_{\sigma(1)},x_{\sigma(2)},\cdots,x_{\sigma(n)})$. Let $C_n(\mathbf{R}^k) = F_n(\mathbf{R}^k)/\Sigma_n$ be the orbit space of the action.

Two covering spaces $p_i: \tilde{X}_i \to X$ are *equivalent* if there exists a homeomorphism $h: \tilde{X}_1 \to X_2$ such that $p_2 h = p_1$.

Let $f: V \to X$ be an $\mathbf{R}^k$-bundle and let $p: \tilde{X} \to X$ be an n-fold covering of X. Let $f_{(n)}: V_{(n)} \to X$ be the n-fold Whitney sum of f; i.e. consider the n-fold product bundle

$f\times\cdots\times f: V\times\cdots\times V \to X\times\cdots\times X$ and let $f_{(n)}: V_{(n)} \to X$ be the pullback of the latter bundle induced by the diagonal map $X \to X\times\cdots\times X$. $V_{(n)} = \{(x,(v_1,\cdots,v_n)) \in X\times V\times\cdots\times V | f(v_1) = f(v_2) = \cdots = f(v_n) = x\}$. Let $\nabla = \{(x,(v_1,\cdots,v_n)) \in V_{(n)} | v_i = v_j$ for some $i \neq j\}$ and let $F_n(V) = V_{(n)} | \nabla$. It is easily checked that $\alpha = \{f_{(n)} | F_n(V): F_n(V) \to X\}$ is a bundle over X whose fiber is $F_n(\mathbf{R}^k)$.

The symmetric group Σ_n acts freely on $F_n(V)$: if $\sigma \in \Sigma_n$, define σ_* on $F_n(V)$ by $\sigma_*(x,(v_1,\cdots,v_n)) = (x,(v_{\sigma(1)},\cdots,v_{\sigma(n)}))$. Let $C_n(V) = F_n(V)/\Sigma_n$ be the orbit space of this action and let $\tau: F_n(V) \to C_n(V)$ be the natural mapping. Since $\alpha\cdot\sigma_* = \alpha$ for all $\sigma \in \Sigma_n$, we have an induced bundle mapping $\beta: C_n(V) \to X$ whose fiber is $C_n(\mathbf{R}^k)$.

Let $\Sigma'_{n-1} = \{\sigma \in \Sigma_n | \sigma(1) = 1\}$, let $E_n(V) = F_n(V)/\Sigma'_{n-1}$ be the orbit space of this subaction and let $\mu: F_n(V) \to E_n(V)$ be the natural mapping. Let $\gamma: E_n(V) \to X$ be the induced bundle mapping. Note that we have an induced map $\rho: E_n(V) \to C_n(V)$ which is an n-fold covering map [M-Z, p. 235].

We have the following analogue of Proposition 3.1 of [H].

Proposition 3. *There exists an embedding* $g: \tilde{X} \to V$ *such that* $fg = p$ *if and only if there exists a section* $\emptyset: X \to C_n(V)$, *of* β (i.e. $\beta\emptyset = 1_X$ identity on X) *such that the pullback of* $\rho: E_n(V) \to C_n(V)$ *by* $\emptyset$ *is equivalent to* p.

Proof. Let $g: \tilde{X} \to V$ be an embedding such that $fg = p$. Let $x \in X$ and let $p^{-1}(x) = \{x_1,\cdots,x_n\}$. Define $\emptyset(x) = \tau(x,(g(x_1),g(x_2),\cdots,g(x_n)))$; note that $\emptyset$ is well-defined.

We leave to the reader to supply the straightforward argument to show that $\emptyset$ is continuous, and, hence, a section of β.

Let

$$\begin{array}{ccc} E & \xrightarrow{\emptyset'} & E_n(V) \\ \rho' \downarrow & & \downarrow \rho \\ X & \xrightarrow{\emptyset} & C_n(V) \end{array}$$

be the pullback diagram where $E = \{(x,y) \in X \times E_n(V) \mid \emptyset(x) = \rho(y)\}$. Define $h\colon \tilde{X} \to E$ by $h(y) = (p(y), \mu(p(y), (g(y), g(y_2), \cdots, g(y_n))))$ where $p^{-1}p(y) = \{y, y_2, \cdots, y_n\}$. Again it is straightforward to check that h is a continuous map and that the inverse of h is given by

$$h^{-1}(x, \mu(x, (v_1, \cdots, v_n))) = g^{-1}(v_1).$$

Note that $\rho' h(x) = p(x)$; hence, the covering spaces $p\colon \tilde{X} \to X$ and $\rho'\colon E \to X$ are equivalent.

Now suppose that the section $\emptyset\colon X \to C_n(V)$ exists so that there exists a homeomorphism $h\colon \tilde{X} \to E$ with $\rho' h = p$ where $\rho'\colon E \to X$ is the pullback of ρ by $\emptyset$. Define $\lambda\colon E_n(V) \to V$ by $\lambda(\mu(x, (v_1, v_2, \cdots, v_n))) = v_1$; note that λ is a continuous map such that $f\lambda = \gamma$. Define $g\colon \tilde{X} \to V$ by $g(x) = \lambda\, \emptyset' h(x)$. Note that $fg = f\lambda\, \emptyset' h = \gamma\, \emptyset' h = \beta\rho\, \emptyset' h = \beta\, \emptyset\, \rho' h = p$; hence g is "fiber-preserving." In order to show that g is one-to-one, suppose that $g(x) = g(y)$; thus $p(x) = p(y)$. If

$$h(x) = (p(x), \mu(x', (v_1, \cdots, v_n)))$$

and

$$h(y) = (p(y), \mu(y', (w_1, \cdots, w_n))),$$

then, from the definition of pullback, $\emptyset(p(x)) =$

$\rho\mu(x',(v_1,\cdots,v_n))$ and

$$p(x) = \beta\emptyset p(x) = \beta\rho\mu(x',(v_1,\cdots,v_n)) = \alpha(x',(v_1,\cdots,v_n)) = x'$$

Similarly, $p(y) = y'$. Since $\emptyset p = \emptyset\rho' h = \rho\emptyset' h$ and $\tau = \rho\mu$, $\{v_1,\cdots,v_n\} = \{w_1,\cdots,w_n\}$. Also, $g(x) = \lambda\,\emptyset' h(x) = \lambda(\mu(x',(v_1,\cdots,v_n))) = v_1$ and $g(y) = w_1$, similarly. Hence, $h(x) = h(y)$ and since h is one-to-one, $x = y$. g is the desired embedding.

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\pi_1 F_n(\mathbf{R}^k) & \longrightarrow & \pi_1 C_n(\mathbf{R}^k) & \xrightarrow{\partial_1'} & \Sigma_n & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \| & & \\
\pi_1 F_n(V) & \xrightarrow{\tau_*} & \pi_1 C_n(V) & \xrightarrow{\partial_1} & \Sigma_n & \longrightarrow & 1 \\
\downarrow \alpha_* & & \downarrow \beta_* & & & & \\
\pi_1 X & = & \pi_1 X & & & & \\
\downarrow & & \downarrow & & & & \\
1 & & 1 & & & &
\end{array}$$

where the rows and columns are from the exact sequences of bundles and covering spaces.

Lemma 4. *If* $k \geq 3$, *then* $\beta_* \times \partial_1 \colon \pi_1 C_n(V) \to \pi_1 X \times \Sigma_n$ *is an isomorphism whose inverse is given by*

$$(\beta_* \times \partial_1)^{-1}(z_1,z_2) = \tau_*\,\alpha_*^{-1}(z_1) \cdot j_*(\partial_1')^{-1}(z_2).$$

Proof. If $k \geq 3$, then $\pi_1 F_n(\mathbf{R}^k) = 1$ [F-N] and, hence, α_* is an isomorphism. The rest of the proof is a straightforward diagram chasing argument.

In the proof of the latter lemma, one shows the following.

Lemma 5. *If* $k \geq 3$, *then* j_* *is one-to-one.*

3. Characteristic Maps for n-Fold Covering Maps

Let $q: \tilde{Y} \to Y$ be an n-fold covering map, $y_0 \in Y$ and let $q^{-1}(y_0) = \{y_1, \cdots, y_n\}$. Let $k: [0,1] \to Y$ represent an element of $\pi_1(Y, y_0)$ and let $k_i: [0,1] \to \tilde{Y}$ be a lifting of k so that $k_i(0) = y_i$. Let $\sigma \in \Sigma_n$ be such that $k_i(1) = y_{\sigma(i)}$. Let $\chi(q): \pi_1(Y, y_0) \to \Sigma_n$ be the homomorphism defined by $\chi(q)[k] = \sigma$; $\chi(q)$ is called a *characteristic map* for q, [S] [H]. $\chi(q)$ depends upon the ordering of $q^{-1}(y_0)$ and is well-defined up to conjugacy class; i.e. if $\chi'(q)$ is defined by using a different ordering, then there exists $\sigma_0 \in \Sigma_n$ such that $\chi(q)(\alpha) = \sigma_0[\chi'(q)(\alpha)]^{-1}\sigma_0$ for all $\alpha \in \pi_1(Y, y_0)$. Characteristic maps determine the n-fold coverings of Y: two n-fold coverings $q_i: \tilde{Y}_i \to Y$ are equivalent if and only if their characteristic maps are conjugate.

Proposition 6. *The boundary homomorphism* $\partial_1: \pi_1 C_n(V) \to \Sigma_n$ *from the long exact homotopy sequence of the fiber space* $\tau: F_n(V) \to C_n(V)$ *is a characteristic map for the covering* $\rho: E_n(V) \to C_n(V)$.

Proof. Let v_0 be the base point of $C_n(V)$ and let us identify $\tau^{-1}(v_0) = \Sigma_n$. Consider the cosets of Σ'_{n-1} in Σ_n, $\{w_1 \Sigma'_{n-1}, w_2 \Sigma'_{n-1}, \cdots, w_n \Sigma'_{n-1}\}$, where w_1 = identity and w_i is a permutation such that $w_i(1) = i$. Let $k: [0,1] \to C_n(V)$ represent an element of $\pi_1(C_n(V), v_0)$. Choose liftings $\tilde{k}_i: [0,1] \to F_n(V)$ such that $\tilde{k}_i(0) = w_i$.

Recall that the long exact homotopy sequence of the fiber space $\tau: F_n(V) \to C_n(V)$ is obtained from the long exact homotopy sequence of the pair $(F_n(V), \Sigma_n = \tau^{-1}(v_0))$ using the isomorphism $\tau_*: \pi_*(F_n(V), \Sigma_n; w_1) \to \pi_*(C_n(V), v_0)$. In order to determine $\partial_1([k])$, we consider the class of $[\tilde{k}_1] \in \pi_1(F_n(V),\Sigma_n;w_1)$; then $\partial_1([k]) = \tilde{k}_1(1) \equiv \sigma$.

Since τ is a principal Σ_n-bundle, $\tilde{k}_i(1) = \sigma \circ w_i$. Consider $k_i = \mu \circ \tilde{k}_i; k_i(0) = \mu(w_i) \equiv v_i$. Note that $\rho^{-1}(v_0) = \{v_1,v_2,\cdots,v_n\}$ and that $k_i(1) = \mu\tilde{k}_i(1) = \mu(\sigma \circ w_i) = \mu(w_{\sigma(i)}) = v_{\sigma(i)}$ since $\sigma \circ w_i(1) = \sigma(i)$. Hence, ∂_1 is a characteristic map for ρ.

The proof of the following is straightforward.

Proposition 7. *Let* $\rho\colon \tilde{Y} \to Y$ *be an* n*-fold covering map and let* $\alpha\colon Z \to Y$ *be a continuous map. Then* $\chi(\tilde{\rho}) = \chi(\rho) \circ \alpha_*$ *is a characteristic map for the pullback* $\tilde{\rho}\colon \tilde{Z} \to Z$.

Lemma 8. *If* $k \geq 3$, *there exists a homomorphism* $\lambda\colon \pi_1 X \to \pi_1 C_n(V)$ *such that* $\beta_*\lambda = $ *identity and* $\chi(p) = \partial_1\lambda$.

Proof. Define

$$\lambda(z) = (\beta_* \times \partial_1)^{-1}(z,\chi(p)(z))$$
$$= \tau_*\alpha_*^{-1}(z) \cdot j_*(\partial_1')^{-1}(\chi(p)(z)) \text{ [cf. Lemma 4].}$$

Clearly $\beta_*\lambda(z) = z$ and $\chi(p)(z) = \partial_1\lambda(z)$.

Lemma 8'. *If* X *has the homotopy type of a locally finite* 1*-dimensional simplicial complex, then there exists a homomorphism* $\lambda\colon \pi_1 X \to \pi_1 C_n(V)$ *such that* $\beta_*\lambda = $ *identity and* $\chi(p) = \partial_1\lambda$.

Proof. $\pi_1 X$ is free on generators $\{x_i\}$. Since $\partial_1'\colon \pi_1 C_n(\mathbf{R}^k) \to \Sigma_n$ is onto, there exists for each i, $y_i \in \pi_1 C_n(\mathbf{R}^k)$ such that $\chi(p)(x_i) = \partial_1'(y_i)$. Since $\alpha_*\colon \pi_1 F_n(V) \to \pi_1 X$ is onto, there exists for each i, $z_i \in \pi_1 F_n(V)$ such that $\alpha_*(z_i) = x_i$.

Define $\lambda: \pi_1 X \to \pi_1 C_n(V)$ by

$$\lambda(x_i) = \tau_*(z_i) \cdot j_*(y_i).$$

λ is the desired homomorphism.

4. Proof of Theorem 1

Proposition 9. *Let* X *be a locally finite simplicial complex and suppose that* $k \geq 3$*; then there exists a section* $\emptyset: X^{k-1} \to C_n(V)$ *of* β *such that the pullback of* ρ *by* $\emptyset$ *is equivalent to* $p|p^{-1}(X^{k-1})$ *where* X^{k-1} *is the* (k-1)*-skeleton of* X.

Proof. For each vertex $v \in X$, choose a point $\emptyset(v) \in \beta^{-1}(v)$. Let T be a maximal tree in the 1-skeleton of X. Since the fiber of β is path-connected, we can extend $\emptyset$ to a cross-section $\emptyset$ over T as in [S; p. 148]. Recall the calculation of $\pi_1(X,v_0)$ using edge-paths [H-W; p. 241]: order the vertices of X, $v_0, v_1, v_2, \cdots,$. For each i, let e_i be an edge-path in T from v_0 to v_i. We may assume that each e_i is an arc.

Let $[v_i v_j]$ be a 1-simplex in $X \setminus T$, $i < j$; $[v_i v_j]$ determines an element ξ_{ij} of $\pi_1(X,v_0)$ which is given by the edge-loop $e_i * v_i v_j * \bar{e}_j$. Let $F: \beta^{-1}(e_i \cup [v_i v_j] \to (e_i \cup [v_i v_j] \times C_n(\mathbb{R}^k)$ be a homeomorphism such that $\rho_1 F = \beta$ where ρ_t denotes the projection of $(e_i \quad [v_i v_j]) \times C_n(\ ^k)$ onto the t^{th} factor.

Let $K: [0,1] \to [v_i v_j]$ be a homeomorphism such that $K(0) = v_i, K(1) = v_j$ and let $K': [0,1] \to C_n(\mathbb{R}^k)$ be a path such that $K'(0) = \rho_2 F\emptyset(v_i)$ and $K'(1) = \rho_2 F\emptyset(v_j)$. K' determines a path $\check{K}$ in $\beta^{-1}(e_i \quad [v_i v_j])$ defined by $\check{K}(t) = F^{-1}(K(t), K'(t))$ and, hence, a loop $\check{\xi}_{ij} = \emptyset(e_i) * K * \emptyset\ (\bar{e}_j)$ in $C_n(V)$. Note that $\beta_*(\check{\xi}_{ij}) = \xi_{ij}$. Let $\lambda: \pi_1(X,v_0) \to \pi_1(C_n(V), \emptyset(v_0))$ be the homomorphism obtained in Lemma 8. Consider the element $\check{\xi}_{ij}^{-1}\lambda(\xi_{ij}) \in \pi_1(C_n(V),\emptyset(v_0))$; since $\beta_*(\check{\xi}_{ij}^{-1}\lambda(\xi_{ij})) = 1$, $\check{\xi}_{ij}^{-1}\lambda(\xi_{ij})$ lies in the image of $\pi_1(C_n(\mathbf{R}^k))$ in $\pi_1(C_n(V))$. Hence, if we choose the path K' carefully, we can obtain $\check{\xi}_{ij} = \lambda(\xi_{ij})$. Extend $\emptyset$ to $[v_iv_j]$ by $\emptyset(x) = \check{K}K^{-1}(x)$.

If we perform this construction for each 1-simplex in $X\backslash T$, we obtain a section $\emptyset$ defined on the 1-skeleton of X such that $\emptyset_*$ takes the generators ξ_{ij} of $\pi_1(X,v_0)$ to $\lambda(\xi_{ij}) \in \pi_1(C_n(V),\emptyset(v_0))$.

Let $[v_iv_jv_k]$, $i < j < k$, be an ordered 2-simplex in X. Note that for one of the paths $e_s \in \{e_i,e_j,e_k\}$, $e_s \cap [v_iv_jv_k] = \{v_s\}$. Suppose $s = i$. Let $F: \beta^{-1}(e_i \cup [v_iv_jv_k]) \to (e_i \cup [v_iv_jv_k]) \times C_n(\mathbf{R}^k)$ be a homeomorphism such that $\rho_1 F = \beta$. Note that $\Gamma = e_i \cup \text{bdry } [v_iv_jv_k]$ represents the loop $\xi_{ij}\xi_{jk}\xi_{ik}$ which is trivial in X. Consider $\emptyset_*(\Gamma) = \emptyset_*(\xi_{ij})\, \emptyset_*(\xi_{jk})\, \emptyset_*(\xi_{ik})^{-1} = \lambda(\xi_{ij})\, \lambda(\xi_{ij})\, \lambda(\xi_{ik})^{-1} = \lambda(\Gamma) = 1$. By Lemma 5, $\rho_1 F\, \emptyset\, (\Gamma)$ is homotopically trivial in $C_n(\mathbf{R}^k)$. By standard techniques [S; p. 149] we can extend $\emptyset$ to a section over $[v_iv_jv_k]$ and, hence, over the 2-skeleton X^2 of X. Note that $\emptyset_*: \pi_1(X^2) \to \pi_1(C_n(V))$ is the homomorphism λj_* where $j: X^2 \to X$ is inclusion.

If $k \geq 4$, $\pi_i(C_n(\mathbf{R}^k)) = 0$ for $2 \leq i \leq k-2$ [F-N]; hence, by classical techniques [S] $\emptyset$ extends to a section of X^{k-1} into $C_n(V)$.

Let $\tilde{\rho}: \xi \to X^{k-1}$ be the pullback of ρ by $\emptyset$. A characteristic map for $\tilde{\rho}$, $\chi(\tilde{\rho}) = \chi(\rho) \circ \emptyset_*$ by Proposition 7; but $\chi(\rho) = \partial_1$ by Proposition 6. Hence $\chi(\tilde{\rho}) = \partial_1 \circ \emptyset_* = \partial_1 \circ \lambda \circ j_* = \chi(p) \circ j_*$ by Lemma 8. Again, by Proposition 7,

$\chi(p|p^{-1}(X^{k-1})) = \chi(p) \circ j_*$. Hence, $\tilde{\rho}$ and $p|p^{-1}(X^{k-1})$ are equivalent.

If $\ell \geq 2$, then Propositions 3 and 9 yield Theorem 1. If $\ell = 1$, the following Proposition, whose proof is similar to the proof of Proposition 9, and Proposition 3 yield the remaining case of Theorem 1.

Proposition 9'. *Let* X *be a locally finite* 1*-dimensional simplicial complex; then there exists a section* $\emptyset: X \to C_n(V)$ *of* β *such that the pullback of* ρ *by* $\emptyset$ *is equivalent to* p *when* V *is an* $\mathbf{R}^2$*-bundle over* X.

References

[B] H. J. Baues, *Obstruction theory*, Lecture Notes in Mathematics 628 (1977), Springer-Verlag, New York.

[D-H] P. F. Duvall and L. S. Husch, *Embedding finite covers into vector bundles with an application to embedding continua up to shape* (in preparation).

[F-N] E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand. 10 (1962), 111-118.

[H] V. L. Hansen, *Embedding finite covering spaces into trivial bundles*, Math. Ann. 236 (1978), 239-243.

[H-W] P. J. Hilton and S. Wylie, *Homology theory*, Cambridge University Press, Cambridge (1962).

[M-Z] D. Montgomery and L. Zippin, *Topological transformation groups*, John Wiley and Sons, Inc., New York (1955).

[S] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton (1951).

Oklahoma State University

Stillwater, Oklahoma 74074

and

University of Tennessee 37916

Knoxville, Tennessee