
TOPOLOGY PROCEEDINGS



Volume 4, 1979

Pages 407–435

<http://topology.auburn.edu/tp/>

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Topology Proceedings

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ISSN: 0146-4124

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ISOMORPHISMS OF SOME COMPLETIONS OF $C(X)$

Anthony W. Hager

$C(X)$ is the ring or vector lattice of real valued continuous functions on the Tychonoff space X , and $C^*(X)$ is the substructure of bounded functions. Most canonical completions of these can be realized as or in certain direct limits $C[\mathcal{S}(X)] \equiv \varinjlim \{C(S) \mid S \in \mathcal{S}(X)\}$, $\mathcal{S}(X)$ being a filter base of dense sets in X .

For example, when X is a Baire space, the Dedekind-MacNeille completion of $C^*(X)$ is $C^*[\mathcal{G}_\delta(X)]$, $\mathcal{G}_\delta(X)$ being the dense G_δ 's $[W_1]$, $[FGL]$. This readily implies that a homeomorphism $\mathcal{G}_\delta(Y) \ni S \xrightarrow{\tau} T \in \mathcal{G}_\delta(X)$ induces an isomorphism $C^*[\mathcal{G}_\delta(Y)] \xrightarrow{\phi} C^*[\mathcal{G}_\delta(X)]$ of completions, by $\phi(f) = f \circ \tau$ (roughly speaking). We prove here a converse:

(I) An isomorphism of the Dedekind-MacNeille completions of $C^*(Y)$ and $C^*(X)$ is induced by a homeomorphism of dense G_δ 's, provided X and Y are completely metrizable (or a bit more generally).

Likewise, with $\mathcal{G}(X)$ = the dense open sets, $C[\mathcal{G}(X)]$ is

I am indebted to: Nathan Fine for pointing out the problem which Theorem (II) addresses, and for many conversations about this problem (and other matters); W. W. Comfort for some suggestions about some proofs in §3; Melvin Henriksen for listening attentively and critically to the proof of Theorem (II), and for some useful suggestions; Scott Williams for various helpful and educational conversations about Theorems (I) and 3.1, documented further in the text; the referee, for his helpful and civilized assistance in improving the paper.

both the maximal ring of quotients of the ring $C(X)$ [FGL] and the lateral completion of the vector lattice $C(X)$ [VG]. Again, a homeomorphism $\mathcal{G}(Y) \ni S \mapsto T \in \mathcal{G}(X)$ induces an isomorphism $C[\mathcal{G}(Y)] \rightarrow C[\mathcal{G}(X)]$. We prove a converse:

(II) An isomorphism of the rings of quotients (lateral completions) of $C(Y)$ and $C(X)$ is induced by a homeomorphism of dense open sets, provided X and Y are separable metrizable (or rather more generally).

§'s 2 and 3 are devoted to proving (I) and §'s 4 and 5 to (II). These are almost completely independent; in spite of the similarity in form between (I) and (II), the proofs are very different.

(I) is equivalent to a theorem on co-absolutes obtained independently (of this work and each other) by C. Gates, D. Maharam and A. H. Stone, and to some extent S. Williams. See 3.2(e).

(II) I proved and announced some time ago [H₁]; it is the solution to a problem of Fine, Gillman, and Lambek.

1. Preliminaries

This section describes some of the features of the structures $C[\mathcal{J}(X)]$ needed in the sequel. We particularly draw on [FGL] and [S], though we try to be reasonably self-contained.

1.1 $C[\mathcal{J}]$. Let X be a space (always Tychonoff), and let $\mathcal{J}(X)$ be a filter base of dense subsets of X . That is, if $S_1, S_2 \in \mathcal{J}(X)$, then there is $S_3 \in \mathcal{J}(X)$ with $S_3 \subseteq S_1 \cap S_2$. Let

$$C[\mathcal{J}(X)] \equiv \cup\{C(S) \mid S \in \mathcal{J}(X)\} \text{ modulo } \sim$$

where: for $f \in C(S_1)$ and $g \in C(S_2)$, $f \sim g$ means $f(x) = g(x)$ for each $x \in S_1 \cap S_2$. (This is easily seen to be an equivalence relation.)

Usually, we can suppress mention of equivalence classes, and, e.g., write $\text{dom } f$ for $f \in C[\mathcal{J}(X)]$ to mean some $S \in \mathcal{J}(X)$ on which some representative of f is defined. We also write $C(X)$, $C(S) \subseteq C[\mathcal{J}(X)]$, with no confusion.

$\mathcal{J}(X)$ is a directed set under set inclusion. For $S \subseteq T$, define $\rho_S^T: C(T) \rightarrow C(S)$ as $\rho_S^T(f) = f|_S$ (the restriction). This function clearly preserves pointwise operations and order, and so is a homomorphism between rings, vector lattices, lattice-ordered algebras, etc. Because S is dense, ρ_S^T is 1 - 1.

Thus, we have a direct system $\{C(S); \rho_S^T\}$ of rings, vector lattices, etc. It follows without difficulty that $C[\mathcal{J}(X)]$ is a realization of the direct limit $\varinjlim \{C(S) \mid S \in \mathcal{J}(X)\}$ (in rings, vector lattices, etc.).

Two important substructures (rings, vector lattices, etc.) of $C[\mathcal{J}(X)]$ are

$$\begin{aligned} C^*[\mathcal{J}(X)] &= \{f \in C[\mathcal{J}(X)] \mid f \text{ is bounded}\} \\ &= \cup\{C^*(S) \mid S \in \mathcal{J}(X)\} \text{ modulo } \sim \\ &= \varinjlim \{C^*(S) \mid S \in \mathcal{J}(X)\}, \text{ and} \\ C_X[\mathcal{J}(X)] &= \{f \in C[\mathcal{J}(X)] \mid |f| \leq g \text{ for some } g \in C(X)\}. \end{aligned}$$

1.2 Filter Bases and Completions. Some of the naturally occurring filter bases $\mathcal{J}(X)$ are, first $\mathcal{G}(X)$ and $\mathcal{G}_\delta(X)$ mentioned in the introduction; also, $\mathcal{C}(X)$ = all dense cozero sets, and $\mathcal{C}_\delta(X)$ = all dense countable intersections of dense

cozero sets. Actually, $\mathcal{G}_\delta(X)$ and $C_\delta(X)$ are not always filter bases; assuming X to satisfy the Baire Category Theorem (" X is a Baire space") ensures that they are. For most purposes though, the filter bases $\mathcal{G}_\delta(\beta X)$ and $C_\delta(\beta X)$ will do. The "natural occurrences" of all these are listed below:

1.2.1 $C[\mathcal{G}(X)]$ is the maximal ring of quotients of the ring $C(X)$ [FGL], and the lateral completion of the vector lattice $C(X)$ [VG].

$C^*[\mathcal{G}_\delta(\beta X)]$ is the Dedekind-MacNeille completion of the lattice, vector lattice, or ℓ -algebra $C(\beta X) = C^*(X)$ [W₁], [FGL], [S], and $= C^*[\mathcal{G}_\delta(X)]$ for X Baire.

$C_X[\mathcal{G}_\delta(\beta X)]$ likewise for $C(X)$.

$C[\mathcal{C}(X)]$ is the classical ring of quotients of the ring $C(X)$ [FGL].

$C^*[\mathcal{C}_\delta(\beta X)]$ is the Cantor (order-Cauchy) completion of the ℓ -group, vector lattice, or ℓ -algebra $C(\beta X) = C^*(X)$ [DHH], and $= C^*[\mathcal{C}_\delta(X)]$ for X Baire.

$C_X[\mathcal{C}_\delta(\beta X)]$ likewise for $C(X)$.

The above facts are motivational, and we don't need to explain them further. Also, we won't say anything in the sequel about $\mathcal{C}(X)$ or $\mathcal{C}_\delta(X)$, essentially because our hypotheses for (I) and (II) largely obliterate any differences between \mathcal{G}_δ and \mathcal{C}_δ , and \mathcal{G} and \mathcal{C} .

1.3 *Maximal Ideal Spaces.* Again with a general $\mathcal{S}(X)$, for $S \subseteq T$ we have a continuous map $\pi_S^T: \beta S \rightarrow \beta T$ of the

Stone-Čech compactifications, extending the inclusion $S \hookrightarrow T$. There results an inverse system $\{\beta S; \pi_S^T\}$ of compact spaces, with a nonempty inverse limit.

1.3.1 $\beta \mathcal{S}(X) \equiv \lim\{\beta S \mid S \in \mathcal{S}(X)\}$.

Let $\pi_S: \beta \mathcal{S}(X) \rightarrow \beta S$ be the natural projection.

Let $\bar{\mathbb{R}}$ be the two point compactification of the reals. For $f \in C(S)$, there is an extension $\beta f: \beta S \rightarrow \bar{\mathbb{R}}$, and then $\hat{f} \equiv \beta f \circ \pi_S$ defines $\hat{f} \in C(\beta \mathcal{S}(X), \bar{\mathbb{R}})$. Without going into the details, this process isomorphically represents $C[\mathcal{S}(X)]$ as a lattice-ordered algebra of $\bar{\mathbb{R}}$ -valued functions on $\beta \mathcal{S}(X)$: it is easily seen from the construction of the inverse limit that the functions \hat{f} separate the points. Consequently

- 1.3.2 $\beta \mathcal{S}(X)$ "is" (a) The space of maximal ℓ -ideals in the ℓ -algebra $C[\mathcal{S}(X)]$ (or in $C^*[\mathcal{S}(X)]$ or $C_X[\mathcal{S}(X)]$), and
(b) The space of vector lattice ideals maximal with respect to not containing the constant function 1, in the vector lattice $C[\mathcal{S}(X)]$ (or $C^*[\mathcal{S}(X)]$, or $C_X[\mathcal{S}(X)]$).

Here, the ideal spaces are given the hull-kernel topology. The above either is in [FGL] or follows from considerations in §'s 2, 3, 4 or [HR₁]. 1.3.2 implies that the representation $f \mapsto \hat{f}$ is that of Henriksen-Johnson for ℓ -algebras [HJ], and that of Yosida for vector lattices. (See [HR₁] and [LZ].)

1.3.3 In $C[\mathcal{S}(X)]$, we introduce the metric of uniform

convergence, $\rho(f, g) = \sup\{|f(x) - g(x) \wedge 1| \mid x \in \text{dom } f \cap \text{dom } g\}$. For $f - g$ bounded, we shall write $\|f - g\|$ for $\rho(f, g)$.

If $(f_n) \subseteq C[\mathcal{J}(X)]$ and if $\bigcap_n \text{dom } f_n$ is dense, then $\rho(f_n, f_0) \rightarrow 0$ means that $f_n \rightarrow f_0$ uniformly on $\bigcap_n \text{dom } f_n$. In case $\mathcal{J}(X)$ is closed under countable intersections, standard methods yield that $C[\mathcal{J}(X)]$ is complete in ρ [FGL], or, as we shall say, "uniformly closed"; likewise for $C^*[\mathcal{J}(X)]$.

Combining the representation on $\beta\mathcal{J}(X)$ with the Stone-Weierstrass Theorem, we obtain this: If $\mathcal{J}(X)$ is closed under countable intersection, then $C^*[\mathcal{J}(X)] \ni f \mapsto \hat{f} \in C(\beta\mathcal{J}(X))$ is an isomorphism (onto).

1.4 Homomorphisms. We consider homomorphisms between structures of the form $C[\mathcal{J}]$. Now and in the sequel, *every homomorphism is supposed to preserve the constant function 1*. In the following two propositions, we have topological spaces Y and X , and suppose given filter bases $\mathcal{J}(Y)$ and $\mathcal{J}(X)$ of dense sets in Y and X , respectively. By 1.3.2(b) and 2.10 of [HR₁], we have:

1.4.1 If $C[\mathcal{J}(Y)] \xrightarrow{\phi} C[\mathcal{J}(X)]$ is a vector lattice homomorphism, then there is a unique continuous map $\beta\mathcal{J}(Y) \xrightarrow{\tau} \beta\mathcal{J}(X)$ for which $\phi(f)^\wedge = \hat{f} \circ \tau$ for each $f \in C[\mathcal{J}(Y)]$. If ϕ is an isomorphism, then τ is a homeomorphism. (Here " $^\wedge$ " refers to 1.3.)

1.4.2 Let $C[\mathcal{J}(Y)] \xrightarrow{\phi} C[\mathcal{J}(X)]$ be a function with $\phi(1) = 1$. Then ϕ is a vector lattice homomorphism iff ϕ is a ring homomorphism.

Proof. (Sketch). \Rightarrow is 4.3 of [HR₁] (and also follows from 6.7 of [HR₁]). The first proof, using 1.4.1, is just

the observation that the operation $f \mapsto f \circ \tau$ preserves existing multiplications.

Let ϕ be a ring homomorphism. In the structures $C[S]$, $f \geq g$ iff $f - g$ is a square. Since ϕ preserves squares, we have $f \geq g \implies \phi(f) \geq \phi(g)$. Then, $\phi(|f|)^2 = \phi(f)^2$, and $\phi(|f|) \geq 0$; by taking square roots, $\phi(|f|) = |\phi(f)|$. Then easily $\phi(f \vee g) = \phi(f) \vee \phi(g)$ and likewise for \wedge , so ϕ is a lattice homomorphism. It remains to show $\phi(rf) = r\phi(f)$ for $r \in R$, and this will follow from $\phi(r) = r$: This is so for integers since $\phi(1) = 1$; $\phi(\frac{1}{n}) = \frac{1}{n}$, for integers n , follows easily; hence $\phi(r) = r$ for rationals r . For $r \in R$, let $r_n \downarrow r$ with r_n rational. Then $0 \leq \phi(r_n - r) = r_n - \phi(r)$, and taking the limit, $0 \leq r - \phi(r)$. Likewise $0 \geq r - \phi(r)$.

Because of 1.4.2, we shall not qualify the word "homomorphism."

2. Dedekind Completion and Co-Absolutes

This very sketchy section is preliminary to proving (I) of the introduction.

2.1 The *absolute* (projective cover, projective resolution) of a space X is an extremally disconnected space aX and a perfect irreducible map $aX \xrightarrow{\pi} X$. ("Perfect" means continuous, closed, with point-inverses compact; equivalently, the extension over Stone-Ćech compactifications preserves remainder. "Irreducible" means continuous, onto, and mapping no proper closed set onto.)

Each space has an essentially unique absolute. ($[G_2]$ for compact spaces; see the bibliography of $[H_2]$.)

Spaces with homeomorphic absolutes are said to be co-absolute.

2.2 The absolute of βX is $\beta \mathcal{G}_\delta(\beta X)$. This is homeomorphic to $\beta \mathcal{G}(\beta X)$, to $\beta \mathcal{G}(X)$, and for X Baire, to $\beta \mathcal{G}_\delta(X)$. ($[W_1]$, $[FGL]$, $[S]$.)

Thus, $C(a\beta X)$ is the Dedekind-MacNeille completion of $C^*(X)$ (using 1.3.3 and the foregoing).

2.3 A homomorphism of the completion of $C^*(Y)$ to that of $C^*(X)$ is a homomorphism $C(a\beta Y) \xrightarrow{\psi} C(a\beta X)$, and these correspond bi-uniquely (and bi-functorially) to continuous maps $a\beta Y \xleftarrow{\tau} a\beta X$, with isomorphisms corresponding to homeomorphisms, as $\psi(f) = f \circ \tau$.

Proof. By the usual duality for $C(X)$, X compact ($[GJ]$, ch. 10); or by its generalization 1.4.1.

2.4 Suppose Y and X are Baire, that $C^*[\mathcal{G}_\delta(Y)] \xrightarrow{\phi} C^*[\mathcal{G}_\delta(X)]$ is an isomorphism, that $C[a\beta Y] \xrightarrow{\psi} C[a\beta X]$ is the isomorphism satisfying $\psi(\hat{f}) = \phi(f)^\wedge$ (" \wedge " referring to 1.3), and that $a\beta Y \xleftarrow{\tau} a\beta X$ is the homeomorphism corresponding to ψ as in 2.3. Suppose there are $S \in \mathcal{G}_\delta(Y)$, $T \in \mathcal{G}_\delta(X)$ and a homeomorphism $S \xleftarrow{\sigma} T$ for which the diagram commutes:

$$\begin{array}{ccc} a\beta Y & \xleftarrow{\tau} & a\beta X \\ \pi_S \downarrow & & \downarrow \pi_T \\ \beta S & \xleftarrow{\beta\sigma} & \beta T \end{array}$$

Here $a\beta Y = \beta \mathcal{G}_\delta(Y) \xrightarrow{\pi_S} \beta S$ is the natural projection (1.1), π_T likewise, and $\beta\sigma$ is the Stone-Ćech extension.

Then σ induces ϕ as follows: If $f \in C^*[\mathcal{G}_\delta(Y)]$, then

$\text{dom } f \in \mathcal{G}_\delta(Y)$, $\tau^+ (\text{dom } f) \in \mathcal{G}_\delta(X)$, and $f \circ \tau \in C^*(\tau^+ (\text{dom } f))$ is a representative of $\phi(f)$. We write just " $\phi(f) = f \circ \tau$ for each f ."

To prove this, one merely chases diagrams. The point is that proving (I) is now explicitly reduced to 3.1 below.

2.3 says that (for X and Y Baire) an isomorphism $C^*[\mathcal{G}_\delta(Y)] \xrightarrow{\phi} C^*[\mathcal{G}_\delta(X)]$ implies a homeomorphism $a\beta Y \xrightarrow{\tau} a\beta X$ (inducing ϕ through combination of 1.4.1 and " \wedge " of 1.3). The following generalization seems of interest:

2.5 *Let $\mathcal{J}(Y)$ and $\mathcal{J}(X)$ be filter bases of dense sets in Y and X , respectively. Let A be a large sub-vector-lattice of $C[\mathcal{J}(Y)]$ with $1 \in A$, and suppose that $A \xrightarrow{\phi} C[\mathcal{J}(X)]$ is an embedding, with $\phi(A)$ large in $C[\mathcal{J}(X)]$. Then there is a homeomorphism $a\beta Y \xrightarrow{\tau} a\beta X$ (inducing ϕ as above).*

Proof. (Sketch). The proof consists in the following diagram, which we shall briefly explain.

$$\begin{array}{ccc}
 a\beta Y & \xleftarrow{\tau} & a\beta X \\
 \pi_1 \downarrow & & \downarrow \pi_2 \\
 \beta \mathcal{J}(Y) & & \beta \mathcal{J}(X) \\
 \sigma_1 \downarrow & & \downarrow \sigma_2 \\
 X_A & \xleftarrow{h} & X_{\phi(A)}
 \end{array}$$

Here, X_A and $X_{\phi(A)}$ are spaces of ideals referred to in 1.3.2(b). And σ_1 is the dual (in the sense of 2.10 of [HR₁]) of the embedding $A \hookrightarrow C[\mathcal{J}(Y)]$, and is irreducible because A is large (4.1 of [HR₂]; see also p. 17 of [W₁]). Likewise, σ_2 exists and is irreducible. h is the homeomorphism dual to $A \xrightarrow{\phi} \phi(A)$. The absolute $a\beta Y$ maps irreducibly onto any

space like $\beta\mathcal{S}(Y)$, hence π_1 ; likewise, π_2 . This shows that $a\beta Y$ is the absolute of X_A , and $a\beta X$ of $X_\phi(A)$. Thus τ exists as the "lift" of h (such lifts follow from projectivity of $a\beta X$ [G_1]). The position of τ in the diagram shows τ is irreducible, and since $a\beta Y$ is extremally disconnected, it follows that τ is a homeomorphism [G_2].

2.6 Getting ahead of ourselves, we raise the question of how to identify a homeomorphism $a\beta Y \xrightarrow{\tau} a\beta X$ which comes from an isomorphism $C[\mathcal{G}(Y)] \rightarrow C[\mathcal{G}(X)]$. When we have Theorem (II), this asks what τ looks like when there is a commutative diagram

$$\begin{array}{ccc} a\beta Y & \xleftarrow{\tau} & a\beta X \\ \pi_S \downarrow & & \downarrow \pi_T \\ \beta S & \xleftarrow{\beta\sigma} & \beta T \end{array}$$

for a homeomorphism $\mathcal{G}(Y) \ni S \xleftarrow{\sigma} T \in \mathcal{G}(X)$.

3. Co-Absolute Implies Homeomorphic Dense G_δ 's

By the preceding section, in order to prove Theorem (I) of the introduction it suffices to prove

3.1 *Theorem.* Let X and Y be spaces which are Čech-complete with G_δ -diagonal. If $a\beta Y \xleftarrow{\tau} a\beta X$ is a homomorphism of the absolutes, then there are dense G_δ 's $S \subseteq Y$ and $T \subseteq X$, and a homeomorphism $S \xleftarrow{\sigma} T$, such that the following diagram commutes:

$$\begin{array}{ccc} a\beta Y & \xleftarrow{\tau} & a\beta X \\ \pi_S \downarrow & & \downarrow \pi_T \\ \beta S & \xleftarrow{\beta\sigma} & \beta T \end{array}$$

3.2 *Remarks.* (a) X is called Čech-complete if X is a G_δ in βX , equivalently, in any/every compactification. Such a space is Baire. Any completely metrizable space has this property, and any metrizable Čech-complete space admits a complete metric. (Čech's Theorem; see $[W_2]$.) This property is inversely preserved under perfect maps. X has G_δ -diagonal means that the diagonal in $X \times X$ is a G_δ -set. Metrizable spaces have this property, and any space with G_δ -diagonal which is Čech-complete and paracompact is completely metrizable [C].

(b) In $[W_4]$, Remark (2), Scott Williams shows that any Čech-complete space with G_δ -diagonal contains a dense G_δ -set which is completely metrizable. This shows that 3.1 is essentially no stronger than the same assertion for completely metrizable spaces; my proof of 3.1, below, becomes no simpler, however.

(c) Williams points out the following companion to 3.1 and results in $[W_4]$: for spaces X and Y which are first countable regular with countable π -base, a homeomorphism $a\beta Y \leftarrow a\beta X$ is equivalent to homeomorphic dense subsets of X and Y . This follows from the fact that such a space contains a dense set homeomorphic to a space $Q \cup F$, Q being the rationals and F some subspace of the integers (itself following from: a countable metrizable space without isolated points contains a copy of Q). In this connection, see Remark (4) of $[W_4]$.

(d) In the diagram in 3.1, the map π_S is both (1) The projection from $\beta\mathcal{G}_\delta(Y)$ onto βS , and (2) The irreducible map

of $a\beta S$ onto βS (since $a\beta Y = a\beta S$); likewise for π_T . Thus, of course, σ determines τ in each of two ways: (1) $\tau = \lim_{\leftarrow} \{\beta\sigma_{T'} : T' \in \mathcal{G}_\delta(X), T' \subseteq T\}$, where $\sigma_{T'} = \sigma|_{T'}$, and (2) from projectivity of $a\beta X$: Let $f = (\beta\sigma) \circ \pi_T$; then there is τ with $\pi_S \circ \tau = f$.

(e) Theorem 3.1 has, from the author's perspective, the following history: I proved it under the (stronger) hypothesis of a homeomorphism $aY + aX$ in the Fall of 1977, and submitted [H₃] for publication in Fall 1978. After this, I noticed that D. Maharam had proved the theorem for $Y = X =$ the Cantor set $[M_1]$. Melvin Henriksen told Scott Williams the theorem in Biloxi, January, 1979, and Williams noted that 3.1 followed from his results in [W₃], and then that my proof yielded the full 3.1. In Athens in March, 1979, A. H. Stone told me that he and Maharam had proved 3.1 [MS₁] and more [MS₂]; then Williams discovered that 3.1 appears in the 1976 thesis of Catherine Gates [G₁].

The proof of 3.1 which we now present differs substantially from the other proofs.

Proof of 3.1. We begin by constructing the "least common cover" of the co-absolute spaces βY and βX . Proceeding more generally:

3.3 *Lemma.* Let $\{X_\alpha | \alpha \in A\}$ be a set of co-absolute spaces: For each α , let $E \xrightarrow{\pi_\alpha} X_\alpha$ be the projective cover. Let $E \xrightarrow{\pi} Z$ be the evaluation map $\pi(e) = (\pi_\alpha(e)) \in \prod \{X_\alpha | \alpha \in A\}$ onto the range $Z = \pi[E]$, and let $Z \xrightarrow{q_\alpha} X_\alpha$ ($\alpha \in A$) be the α -th projection, restricted to Z . So $q_\alpha \circ \pi = \pi_\alpha$ for each α , and:

- (a) π and each q_α are perfect and irreducible,
- (b) Z is a closed subspace of $\prod\{X_\alpha \mid \alpha \in A\}$,
- (c) Let W be a space and for each α , $W \xrightarrow{f_\alpha} X_\alpha$ a perfect irreducible map. Then there is unique $W \xrightarrow{f} Z$ with $q_\alpha \circ f = f_\alpha$ for each α .

(Actually, we shall need only (a) in the proof of 3.1, but (b) and (c) seem of interest.)

(a) is immediate from the following, which is known and easily proved.

3.4 Lemma. Suppose given maps $B \xrightarrow{f} C \xrightarrow{g} D$.

- (a) If $g \circ f$ is irreducible (respectively, perfect), then f is irreducible (resp., perfect).
- (b) If $g \circ f$ is irreducible and f is onto (resp., $g \circ f$ is perfect and f is dense) then g is irreducible (resp., perfect).

To prove 3.3(b): $E \xrightarrow{\pi} \prod\{X_\alpha \mid \alpha \in A\}$ (to the full product) is also perfect, by 3.4(a), and as is well known, any perfect map has closed range.

We omit the proof of 3.3(c).

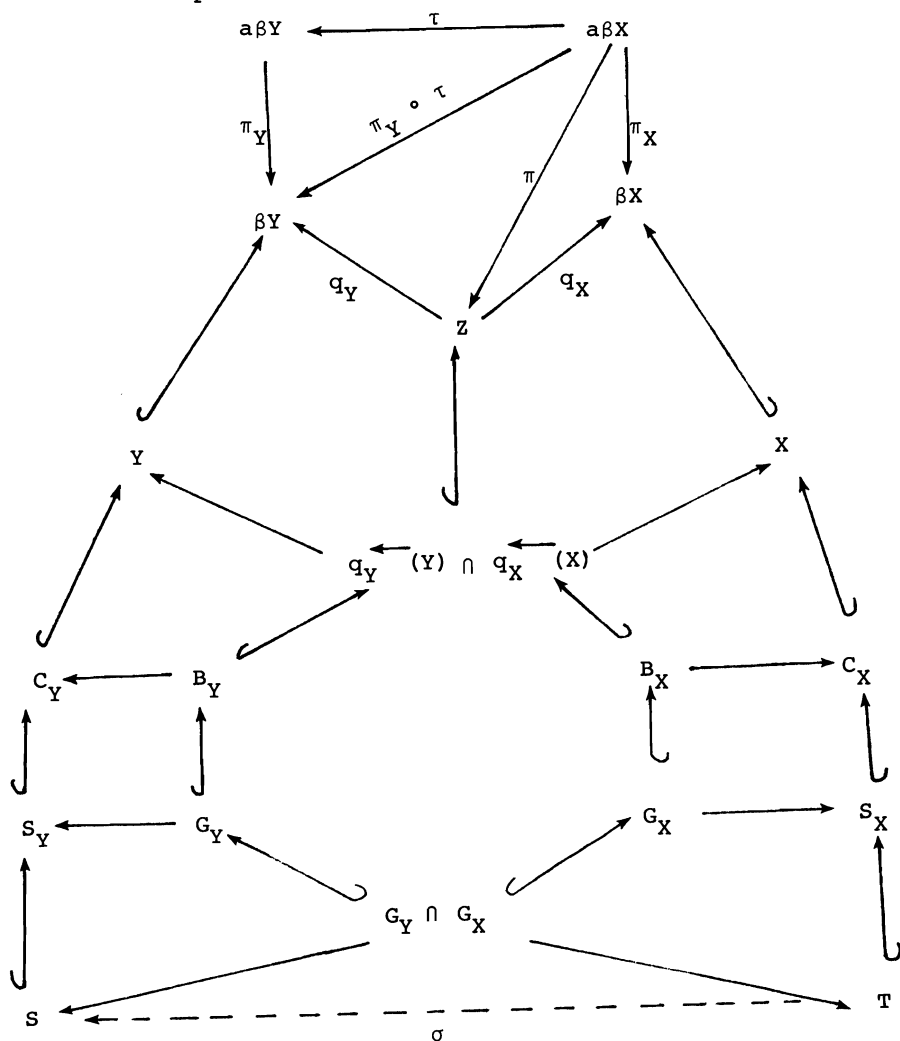
The following is crucial.

3.5 Theorem. (Mishkin, Theorem 4 of $[M_2]$). Suppose $B \xrightarrow{f} C$ is closed and irreducible, that B has G_δ -diagonal and that C is a Baire space. Then there is a dense G_δ -set G in B with $f|G$ a homeomorphism onto a dense G_δ -set in C .

(Mishkin's result is stated a bit differently than this, but his proof yields the above.)

The proof of 3.1 consists in putting the foregoing

together according to the commuting diagram below, which we shall explain.



Here σ shall be a homeomorphism of the dense G_δ 's T and S ; thus we obtain 3.1 (as explained in 3.2(b)).

To begin, 3.3 is applied to the co-absolute spaces βY and βX , with $a\beta X \xrightarrow{\pi_X} \beta X$ and $a\beta X \xrightarrow{\pi_Y \circ \tau} \beta Y$ taken as the projective covers. This yields Z , π , and q_Y, q_X . Note that

$Z \subseteq \beta Y \times \beta X$.

We now shall construct dense G_δ 's B_Y and C_Y as shown, so that $C_Y \xleftarrow{q_Y} B_Y$ is perfect and irreducible. Then 3.5 will be applied.

Since Y is a G_δ in βY , $q_Y^{-1}(Y)$ is a G_δ in Z , and is dense because q_Y , being irreducible, inversely preserves density. Likewise, $q_X^{-1}(X)$ is a dense G_δ in Z . Hence, so is $q_Y^{-1}(Y) \cap q_X^{-1}(X) \equiv A$. Thus, A is Čech-complete (being a G_δ in a compact space $[W_2]$), hence a Baire space. Note that $A \subseteq Y \times X$.

Observe that $q_Y^{-1}(Y) \xrightarrow{q_Y} Y$ is perfect (because $q_Y(Z - q_Y^{-1}(Y)) \subseteq \beta Y - Y$ and irreducible (because $Z \xrightarrow{q_Y} \beta Y$ is irreducible; see $[W_1]$, p. 17). And A is a dense G_δ in $q_Y^{-1}(Y)$. We need the following.

3.6 Lemma. *Let $D \xrightarrow{q} E$ be perfect and irreducible, with E a Baire space. Let A be a dense G_δ in E , let $C \equiv E - q(D - A)$, and $B \equiv q^{-1}(C)$. Then C and B are dense G_δ 's, $B \subseteq A$, and $B \xrightarrow{q} C$ is perfect and irreducible.*

Proof. A perfect map is closed. Thus, since $D - A$ is an F_σ , so is $q(D - A)$, so that C is a G_δ ; therefore, B is a G_δ .

C is dense: We have $D - A = \bigcup_n F_n$, with each F_n closed and nowhere dense. Because q is irreducible, each $q(F_n)$ is nowhere dense. (If $\overline{q(F_n)} \supseteq$ open $0 \neq \emptyset$, then $\overline{F_n}$ would contain $q^{-1}(0)$: if not, there would be open nonvoid $V \subseteq q^{-1}(0)$ with $V \cap F_n = \emptyset$. Then $q(D - V) = q(D)$, contradicting irreducibility.) Thus, $q(D - A) = \bigcup_n q(F_n)$ is meager, and the complement C is residual, hence dense (since E is Baire).

Irreducible maps inversely preserve density; thus, B is dense.

Finally, $B \xrightarrow{q} C$ is perfect, since $q(D - B) \subseteq E - C$ and $D \xrightarrow{q} E$ is perfect, and is irreducible since $D \xrightarrow{q} E$ is irreducible (and $[W_1]$, p. 17).

Now, applying 3.6 to $q_Y^{-1}(Y) \cap q_X^{-1}(X)$ produces $B_Y \xrightarrow{q_Y} C_Y$ perfect and irreducible, with B_Y and C_Y dense G_δ 's. Note that $B_Y \subseteq Y \times X$, hence B_Y has G_δ diagonal (since Y and X do, and the property is finitely productive and hereditary); also, C_Y is Baire.

Mishkin's Theorem 3.5 now yields dense G_δ 's G_Y and S_Y , with $G_Y \xrightarrow{q_Y} S_Y$ a homeomorphism. The same procedure applied to $q_X^{-1}(X) \xrightarrow{q_Y} X$ yields dense G_δ 's G_X and S_X , with $G_X \xrightarrow{q_X} S_X$ a homeomorphism.

Now, G_Y and G_X are dense G_δ 's in the Baire space $q_Y^{-1}(Y) \cap q_X^{-1}(X)$, so that $G_Y \cap G_X$ is a dense G_δ . Then $S \equiv q_Y(G_Y \cap G_X)$ and $T \equiv q_X(G_Y \cap G_X)$ are dense G_δ 's, and are homeomorphic *via* $\sigma \equiv q_Y \circ (q_X^{-1} \upharpoonright q_X(G_Y \cap G_X))$.

The proof is complete.

4. Direct Limit Homomorphisms

§'s 4 and 5 are devoted to Theorem (II) of the introduction. In §4, we show how a homomorphism $C[\mathcal{J}(Y)] \rightarrow C[\mathcal{J}(X)]$ which "respects the direct limits" is induced by a continuous function $S \leftarrow T$ (for some $S \in \mathcal{J}(Y)$ and $T \in \mathcal{J}(X)$). In §5, we show that for $\mathcal{J} = \mathcal{G}$, and under some hypotheses on X , each homomorphism indeed *does* respect the direct limits. From these Theorem (II) follows.

First, we interject some remarks about the situation

surrounding Theorems (II) and (I). (1) The "algebraic" hypothesis of (II) is stronger than that of (I), for: an isomorphism $C[\mathcal{J}(Y)] \xrightarrow{\phi} C[\mathcal{J}(X)]$ implies by 2.5 a homeomorphism $a\beta Y \xleftarrow{\tau} a\beta X$ (inducing ϕ in a certain way), and then there is an isomorphism of the completion of $C^*(Y)$ onto the completion of $C^*(X)$, by 2.3. (2) The conclusion of (II) is stronger than that of (I). (3) As pointed out in $[H_1]$: there is a natural isomorphism $C[\mathcal{J}(Q)] \rightarrow C[\mathcal{J}(\beta Q)]$ (Q = the rationals), but Q and βQ do not have homeomorphic dense open sets. Thus, some topological hypotheses are needed in (II). (4) The remarks of 3.1(c) help place Theorem (II) in perspective.

Throughout this section, $\mathcal{J}(Y)$ and $\mathcal{J}(X)$ are fixed filter bases of dense sets in Y and X , respectively.

4.1 *Functions Induce Homomorphisms.* (a) Consider a continuous function $Y \xleftarrow{\tau} \text{dom } \tau \subseteq X$ which satisfies the condition

$$(\mathcal{J} + \mathcal{J}) \text{ if } S \in \mathcal{J}(Y), \text{ then } \tau^+(S) \text{ contains some } T \in \mathcal{J}(X).$$

Then, for $f \in C[\mathcal{J}(Y)]$, choose S with $f \in C(S)$, then choose $T \in \mathcal{J}(X)$ with $\tau^+(S) \supseteq T$, and define $\phi(f) \in C(\mathcal{J}(X))$ to be the equivalence class of the function $f \circ (\tau|_T)$. It is easily seen that this is well defined, and that there results a homomorphism $C[\mathcal{J}(Y)] \xrightarrow{\phi} C[\mathcal{J}(X)]$. We call ϕ the homomorphism induced by τ , and express this by writing " $\phi(f) = f \circ \tau$ for each f ." Evidently, such ϕ has the property

$$(d1) \text{ if } S \in \mathcal{J}(Y), \text{ then there is } T \in \mathcal{J}(X) \text{ with } \phi(C(S)) \subseteq C(T).$$

Here, "dl" is short for "direct limit": It is easy to see that a homomorphism satisfying (dl) is the direct limit (in rings, vector lattices, etc.) of "partial" homomorphisms $C(S) \xrightarrow{\phi_S} C(T)$.

If ϕ is an isomorphism with both ϕ and ϕ^{-1} satisfying (dl), we call ϕ a bidl isomorphism.

(b) Note that for the cases $\mathcal{J} = \mathcal{I} = \mathcal{G}$, or \mathcal{G}_δ , or \mathcal{C} , or \mathcal{C}_δ (assuming Baire spaces where appropriate, a la 1.2), the condition $(\mathcal{J} + \mathcal{I})$ will be satisfied by any continuous function $Y \xleftarrow{\tau} T \in \mathcal{I}(X)$ which inversely preserves dense sets. (Here, of course, $(\mathcal{J} + \mathcal{I})$ reduces to " $\tau^+(\mathcal{J}) \subseteq \mathcal{I}$." Relatively familiar classes of maps which do this are the irreducible maps, Henriksen-Jerison maps, and the skeletal maps (all of which occur naturally in the theory of the absolute).

(c) We are aiming towards proving that certain homomorphisms are induced by functions in the manner of (a). We require the following fragment of the duality theory for $C(X)$'s:

Each homomorphism $C(Y) \xrightarrow{\phi} C(X)$ is induced by a unique continuous map $\nu Y \xleftarrow{\tau} X$, as $\phi(f) = (\nu f) \circ \tau$ for $f \in C(Y)$.

Here, νY is the Hewitt realcompactification of Y , and νf is the extension of f in $C(\nu Y)$. See Chapter 10 of [GJ] for details.

We are going to apply this fact to a homomorphism $C[\mathcal{J}(Y)] \xrightarrow{\phi} C[\mathcal{I}(X)]$ satisfying (dl). This will produce, for each $S \in \mathcal{J}(Y)$ a continuous map $\nu S \xleftarrow{\tau_S} T_S$, for some $T_S \in \mathcal{I}(X)$. For simplicity, we shall suppress the ν 's by assuming that $\mathcal{J}(Y)$ consists entirely of realcompact sets. Since our

primary interest is in $\mathcal{G}(Y)$, we recall the following from 8.17 of [GJ].

These are equivalent: Y is hereditarily realcompact; each $Y - \{p\}$ is realcompact; each member of $\mathcal{G}(Y)$ is realcompact.

We now have a converse of (a).

4.2 Proposition. *Let $C[\mathcal{J}(Y)] \xrightarrow{\phi} C[\mathcal{J}(X)]$ satisfy (dl), suppose that $\mathcal{J}(Y)$ consists of realcompact sets and is closed under finite union. Then there is $T \in \mathcal{J}(X)$ and continuous $Y \xleftarrow{\tau} T$ (which is essentially unique and satisfies $(\mathcal{J} + \mathcal{J})$) such that $\phi(f) = f \circ \tau$ for each f .*

Proof. Given S , we have $\phi(C(S)) \subseteq C(T_S)$. So $\phi_S \equiv \phi|C(S)$ is a homomorphism $C(S) \rightarrow C(T_S)$, and is induced by a unique $S \xleftarrow{\tau_S} T_S$.

Given S_1 and S_2 , we claim that $\tau_{S_1}|_{T_{S_1} \cap T_{S_2}} = \tau_{S_2}|_{T_{S_1} \cap T_{S_2}}$. Supposing not, there is $p \in T_{S_1} \cap T_{S_2}$ with $\tau_{S_1}(p) \neq \tau_{S_2}(p)$. Choose $f \in C(S_1 \cup S_2)$ with $f(\tau_{S_1}(p)) \neq f(\tau_{S_2}(p))$, by complete regularity. Since $f \in C(S_i)$, $\phi_{S_i}(f) = f \circ \tau_{S_i}$. Since $\phi_{S_i} = \phi|C(S_i)$, we have $\phi_{S_1}(f) = \phi_{S_2}(f)$ as members of $C[\mathcal{J}(X)]$, and this implies pointwise equality of representatives on any dense set on which the representatives are defined, e.g., $T_{S_1} \cap T_{S_2}$; this contradicts $f(\tau_{S_1}(p)) \neq f(\tau_{S_2}(p))$.

Specifically now, fix any S_0 (e.g., Y , if $Y \in \mathcal{J}(Y)$), take $T = T_{S_0}$ and let $\tau = \tau_{S_0}$. Then $Y \xleftarrow{\tau} T$ induces ϕ .

$(\mathcal{J} + \mathcal{J})$ holds because given S , there is T_S with

$S \xleftarrow{\tau_S} T_S$ inducing ϕ_S , while $\tau_S|_T \cap T_S = \tau|_T \cap T_S$, so that $\tau^+(S) \supseteq T_S$.

4.3 Proposition. *Let $C[S(Y)] \xrightarrow{\phi} C[J(X)]$ be a bidl isomorphism, and suppose that $S(Y)$ and $J(X)$ are each closed under finite union and consist of realcompact sets. Then ϕ is induced by a homeomorphism $Y_1 \xleftarrow{\tau} X_1$, where $Y \supseteq Y_1$ contains members of $S(Y)$ and $X \supseteq X_1$ contains members of $J(X)$.*

Prior to the proof, we remark: the conditions on Y' and X' above come from the conditions $(S \leftarrow J)$ on τ and $(S \rightarrow J)$ on τ^+ in the conclusion of 4.2, and for our examples (\mathcal{G} , etc.). These reduce, per 4.1(b), to $\tau^+(S) \subseteq J$ and $S \supseteq (\tau^+)^+(J)$. Further, looking at the proofs shows that, for the examples, $Y_1 \in S(Y)$ and $X_1 \in J(X)$. Explicitly, then, for the case of $S = J = \mathcal{G}$, we have

4.4 Corollary. *If Y and X are hereditarily realcompact, and if $C[\mathcal{G}(Y)] \xrightarrow{\phi} C[\mathcal{G}(X)]$ is a bidl isomorphism, then there are $S \in \mathcal{G}(Y)$, $T \in \mathcal{G}(X)$, and a homeomorphism $S \xleftarrow{\tau} T$ with $\phi(f) = f \circ \tau$ for each f .*

Proof of 4.3. By 4.2, ϕ is induced by $Y \xleftarrow{\tau} T_0$ satisfying $(S \leftarrow J)$, and ϕ^{-1} is induced by $S_0 \xrightarrow{\sigma} X$ satisfying $(S \rightarrow J)$.

Set $X_1 \equiv \tau^+(S_0) \cap T_0$; this contains a $T \cap T_0$, and $\sigma \circ \tau$ is defined on X_1 . Let $\tau_1 \equiv \tau|_{X_1}$.

We claim that $\sigma \circ \tau_1$ is the identity function on X_1 : If not, there is $p \in X_1$ with $\sigma(\tau(p)) \neq p$. Choose $f \in C(X)$ with

$$f = \begin{cases} 0 & \text{on a neighborhood } G \text{ of } p, \\ 1 & \text{on a neighborhood } H \text{ of } \sigma(\tau(p)). \end{cases}$$

Then, $f = \phi(\phi^{-1}(f))$ as elements of $C[S(X)]$, which translates

to: $f = (f \circ \sigma) \circ \tau$ pointwise on some $T \in \mathcal{J}(X)$. Let G' be a neighborhood of p with $\sigma(\tau(G')) \subseteq H$. Then, f and $f \circ \sigma \circ \tau$ are defined and equal on $N \equiv G \cap G' \cap T \neq \emptyset$. But $f|N = 0$ while $f \circ \sigma \circ \tau|N = 1$. Contradiction.

So, $X_1 \xrightarrow{\sigma \circ \tau} X_1$ is the identity, and thus $\sigma(S_0) \supseteq X_1$.

Set $Y_1 \equiv \sigma^{-1}(X_1)$ and $\sigma_1 \equiv \sigma|Y_1$. As before, Y_1 contains an S and $\tau_1 \circ \sigma_1$ is defined on Y_1 . As before, we show that $Y_1 \xrightarrow{\tau \circ \sigma} Y_1$ is the identity. Thus, $\tau_1(X_1) \supseteq Y_1$.

Now $\tau_1^{-1}(Y_1) = \tau_1^{-1}(\sigma^{-1}(X_1)) = (\sigma \circ \tau_1)^{-1}(X_1) = X_1$. We thus have $X_1 \xrightarrow{\tau_1} Y_1 \xrightarrow{\sigma_1} X_1$ with $\sigma_1 \circ \tau_1$ and $\tau_1 \circ \sigma_1$ both identities. Consequently, τ_1 and σ_1 are mutually inverse, hence homeomorphisms.

Specifically, the τ in the statement of 4.3 is the above τ_1 .

5. \mathcal{dl} Targets for \mathcal{G}

We now complete the proof of Theorem (II). Some of the propositions here are true for arbitrary filter bases of dense sets; the symbols $\mathcal{J}(Y)$ and $\mathcal{J}(X)$ as usual have this meaning.

5.1 *Notation.* The following hypotheses on a space X will be needed.

- (α) If $D \subseteq X$ and $\text{int } \bar{D} \neq \emptyset$, then there is countable $D_0 \subseteq D$ with $\text{int } \bar{D}_0 \neq \emptyset$.
- (β) If $p \notin T \in \mathcal{G}(X)$, then there is a sequence $(p_n) \subseteq T$ with $p_n \rightarrow p$.

For example, any hereditarily separable space, or a space with a π -base of hereditarily separable sets, satisfies (α). And any Frechét space (in particular, a

first-countable space) satisfies (β) .

It will be clear too, that for our purposes (including Theorem (II)) it suffices to have (β) on some $T_0 \in \mathcal{G}(X)$. This permits excluding obnoxious points lying in a closed nowhere dense set.

It will be convenient to consider homomorphisms $C[\mathcal{S}(Y)] \xrightarrow{\phi} C[\mathcal{I}(X)]$ or $C^*[\mathcal{S}(Y)] \xrightarrow{\phi} C^*[\mathcal{I}(X)]$ which satisfy (dl^*) for each $S \in \mathcal{S}(Y)$, there is $T \in \mathcal{I}(X)$ with $\phi(C^*(S)) \subseteq C^*(T)$.

For now, (dl^*) will be only an item in the proof of Theorem (II), but we intend to examine these things further in later work.

The following is the main theorem of this section

5.2 Theorem. (a) If X satisfies (α) , then any homomorphism $C^*[\mathcal{S}(Y)] \rightarrow C^*[\mathcal{G}(X)]$ is dl^* .

(b) If X satisfies (β) , then any homomorphism $C[\mathcal{S}(Y)] \rightarrow C[\mathcal{G}(X)]$ which is dl^* is also dl .

(c) If X satisfies (α) and (β) , then any homomorphism $C[\mathcal{S}(Y)] \rightarrow C[\mathcal{G}(X)]$ is dl .

Combining 5.2(c) with 4.4, we obtain the following (which implies Theorem (II)):

5.3 Corollary. Suppose Y and X are each hereditarily realcompact and satisfy (α) and (β) . If $C[\mathcal{G}(Y)] \xrightarrow{\phi} C[\mathcal{G}(X)]$ is an isomorphism, then there are $S \in \mathcal{G}(Y)$, $T \in \mathcal{G}(X)$, and a homeomorphism $S \xleftarrow{\tau} T$ with $\phi(f) = f \circ \tau$ for each f .

The rest of this section consists of the proof of 5.2. We first require some terminology and elementary properties

of homomorphisms.

5.4 Let $A \subseteq C[\mathcal{J}(X)]$. We say that:

A is closed under composition if $a \in A$ and $g \in C(R)$ imply that $g \circ a \in A$.

A is closed under bounded inversion if $a \in A$ and $a \geq 1$ imply that $1/a \in A$.

A is uniformly closed if A is complete with respect to the uniform metric on $C[\mathcal{J}(Y)]$. (See 1.3.3.)

Notice that, if $S \in \mathcal{J}(X)$, then $A = C(S)$ has these properties.

(The definitions can be made for various more general ℓ -algebraic structures, e.g., [HJ], [LZ].)

5.5 *Lemma.* Let $A \subseteq C^*[\mathcal{J}(X)]$ be a uniformly closed vector lattice (or ring) containing the constants. Then A is closed under composition.

Proof. Let $a \in A$. And consider the compact set $K = \overline{\text{range } a}$. (This is easily seen to be independent of representative for a .) Let $g \in C(R)$. By the Stone-Weierstrass Theorem (or less) there is a sequence (p_n) of piecewise linear functions in $C(R)$ with $p_n \rightarrow g$ uniformly on K . Since A is a vector lattice containing constants, each $p_n \circ a \in A$. The sequence $(p_n \circ a)$ is Cauchy in A , evidently converging to $g \circ a$. Since A is uniformly closed, $g \circ a \in A$.

When A is a ring, we use polynomials (p_n) as above.

5.6 *Lemma.* Let $C[\mathcal{J}(Y)] \xrightarrow{\phi} C[\mathcal{J}(X)]$ be a homomorphism, and let $C \subseteq C[\mathcal{J}(Y)]$ be a vector lattice containing constants. Then

- (a) $\phi(C)$ is a vector lattice containing constants.
 (b) If C is uniformly closed, then so is $\phi(C)$; then $\phi(C)^*$ is closed under composition.
 (c) If C is closed under bounded inversion, then so is $\phi(C)$.

Proof. (Sketch). (a) because ϕ is a vector lattice homomorphism with $\phi(1) = 1$.

(b) The second part follows from the first and 5.5. To see that $\phi(C)$ is uniformly closed: Let $(\phi(f_n))$ be Cauchy in $\phi(C)$. By extracting a subsequence, we suppose that $||\phi(f_{n+1}) - \phi(f_n)|| \leq 2^{-n}$ for each n . Then we define a Cauchy sequence (g_n) in C with $\phi(g_n) = \phi(f_n)$ by: $g_1 = f_1$, $h_{n+1} = ((f_{n+1} - g_n) \wedge 2^{-n}) \vee (-2^{-n})$ and $g_{n+1} = g_n + h_{n+1}$. Then $g_n \rightarrow$ some $g \in C$, whence $\phi(g_n) \rightarrow \phi(g) \in \phi(C)$ (it being easily seen that a homomorphism is continuous for the uniform metrics).

(c) If $\phi(f) \geq 1$, then $\phi(f \vee 1) = \phi(f)$, so $1/(f \vee 1) \in C$, whence $1/\phi(f) = \phi(1/(f \vee 1)) \in \phi(C)$.

In 5.6, we can take $C = C^*(S)$ for $S \in \mathcal{J}(Y)$. Consequently, the following will prove 5.2(a).

5.7 Proposition. *Let X satisfy (α) and let $A \subseteq C^*[\mathcal{J}(X)]$ be a uniformly closed vector lattice containing constants. Then there is $T \in \mathcal{J}(X)$ with $A \subseteq C^*(T)$.*

The proof uses the oscillation function: For $f \in C[\mathcal{J}(X)]$, and $p \in X$, $\omega(f)(p) = \inf\{\sup \overline{f(U)} - \inf \overline{f(U)} \mid U \text{ a neighborhood of } p\}$. This is clearly independent of representative for f . As usual, f is continuously definable at p iff $\omega(f)(p) = 0$. Note that $\omega(f)(p) = +\infty$ can occur, but

for $f \in C^*[J(X)]$, $\omega(f)(p) < +\infty$.

Proof of 5.7. Let $D = \{p \in X \mid \omega(f)(p) > 0 \text{ for some } f \in A\}$. If $\text{int } \bar{D} = \emptyset$, then $T = X - \bar{D} \in \mathcal{G}(X)$ and $A \subseteq C^*(T)$.

Otherwise, $\text{int } \bar{D} \neq \emptyset$, and by (a) there is countable $D_0 \subseteq D$ with $\text{int } \bar{D}_0 \neq \emptyset$. Write $D_0 = \{p_1, p_2, \dots\}$. For each n , there is $f_n \in A$ with $\omega(f_n)(p_n) > 0$, and we can arrange it that $0 \leq f_n \leq 1$ and $\omega(f_n)(p_n) = 1$.

We shall choose numbers (a_n) so that the partial sums of $\sum_n a_n f_n$ form a Cauchy sequence in A , but the limit f (in A , since A is uniformly closed) has $\omega(f)(p_n) > 0$ for each n . Since any $T \in \mathcal{G}(X)$ contains some p_n , this means that $f \notin C[\mathcal{G}(X)]$, a contradiction.

In choosing (a_n) , we have to be sure that, for example, $\omega(f_1)(p_2)$ doesn't cancel $\omega_{f_2}(p_2)$. For this, note: if $\omega(g)(p) > 0$ and $\omega(f+g)(p) = 0$, then for any $a \neq 1$, $\omega(f+ag) \neq 0$.

Now define (a_n) and oscillations (ε_n) , as follows.

$$a_1 = \frac{1}{2} \text{ and } \varepsilon_1 = \omega(a_1 f_1)(p) (> 0).$$

And inductively,

$$a_{n+1} = \begin{cases} \frac{\varepsilon_n a_n}{2^2} & \text{if } \omega(a_1 f_1 + \dots + a_n f_n + \frac{\varepsilon_n a_n}{2^2} f_{n+1})(p_{n+1}) > 0 \\ \frac{\varepsilon_n a_n}{2^3} & \text{otherwise} \end{cases}$$

and $\varepsilon_{n+1} = \omega(a_1 f_1 + \dots + a_{n+1} f_{n+1})(p_{n+1}) (> 0)$.

It is easy to check that (a_n) does what is required.

The proof of 5.7 is complete.

We turn to the proof of 5.2(b).

5.8 Lemma. *Let $A \subseteq C[J(X)]$ be a ring and vector lattice containing constants, closed under bounded inversion, with A^* closed under composition. Suppose $f \in A$, $p \in X$, and that f has no real continuous extension to p . Then, if there is a sequence $(p_n) \subseteq \text{dom } f$ with $p_n \rightarrow p$, then there is $g \in A^*$ with $\omega(g)(p) > 0$.*

Proof. If $\liminf_{x \rightarrow p} f(x) \neq \limsup_{x \rightarrow p} f(x)$, then for some real M , a function $g = (f \wedge M) \vee (-M)$ will work. So we suppose $\liminf = \limsup$. Take $(p_n) \subseteq \text{dom } f$ with $p_n \rightarrow p$, and replace f by $|f| \wedge 1$ if necessary. We then have $f(p_n) \rightarrow +\infty$ and $f \geq 1$. By passing to a subsequence, we suppose $f(p_n) \uparrow +\infty$.

Now $h \equiv 1/f \in A^*$, and $h(p_n) \downarrow 0$. Construct $i \in C(R)$, by linear interpolation say, with

$$i(h(p_{2n})) = \sqrt{n}(p_{2n}), \quad i(h(p_{2n+1})) = h(p_{2n+1}).$$

Then $i \circ h \in A^*$.

Finally, let $g = ((i \circ h) \circ f) \wedge 2$. Then $g \in A^*$, $g(p_{2n}) = 2$ and $g(p_{2n+1}) = 1$. So $\omega(g)(p) \geq 1$.

Remark. There are other hypotheses on A which entail the conclusion, e.g., A is a vector lattice containing constants, closed under composition. Here, we reduce to $f(p_n) \uparrow +\infty$ as above, then choose $h \in C(R)$ (by linear interpolation) so that $\sin(h(f(p_n))) = +1$ for n odd and $= -1$ for n even. Then $\sin \circ h \circ f = g$ works. There are two reasons that we prefer the hypotheses in 5.8. (1) They are weaker: If A is closed under composition, then A is a ring closed under bounded inversion. (2) The application of 5.8 is to $A = \phi(C(S))$. This structure is closed under composition, but

proving this seems to need the duality for vector lattices in $[HR_1]$ (as in 1.4.1), and we prefer to keep this section devoid of the maximal ideal spaces.

The following implies 5.2(b).

5.9 *Proposition.* *If X satisfies*

$(\beta(J))$ whenever T and $T' \in J(X)$ and $p \in T - T'$,

then there is $(p_n) \subseteq T'$ with $p_n \rightarrow p$,

then any homomorphism $C[S(Y)] \xrightarrow{\phi} C[J(X)]$ which is dl^ is also dl .*

Proof. Let $S \in S(Y)$. Then there is $T \in J(X)$ with $\phi(C^*(S) \subseteq C^*(T)$. If there is $f \in \phi(C(S)) - C(T)$, then there is $p \in T$ such that f has no real extension to p . Take $T' = \text{dom } f$. By $(\beta(J))$, 5.8 applies to produce $g \in \phi(C(S))^* = \phi(C(S)^*)$ with $\omega(g)(p) > 0$. This is a contradiction.

This completes the proof of 5.2, hence of (II).

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