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A DIFFERENTIABLE, PERFECTLY NORMAL, NONMETRIZABLE MANIFOLD

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In answer to a question originally raised by Alexandroff in [A], Rudin and Zenor, using the continuum hypothesis, displayed an example of a perfectly normal, hereditarily separable, non-metrizable topological manifold [R,Z]. In this paper, we show that the Rudin-Zenor manifold can be constructed so that it is analytic. A key step in our construction is a modification of a theorem of Brown [B] which is interesting in its own light; namely, we show that if a differentiable manifold M has an atlas $\{(V_i,\phi_i) | i \in \omega_0\}$ such that $V_{i+1} \supset V_i$ and $\phi_i(V_i) = \mathbf{R}^n$ for all $i \in \omega_0$, then M is diffeomorphic to $\mathbf{R}^n$.

The construction of the manifold follows very closely that of [R,Z] and we recommend that the reader be familiar with that paper before proceeding.

Let X be a set, and let n be a fixed positive integer.

A *chart* is a pair (U,ϕ) where $\phi: U \to \mathbf{R}^n$ is an injective function of a subset U of X onto an open subset ϕU of $\mathbf{R}^n$.

Two charts (U,ϕ), (V,ψ) are *compatible*, if $\phi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbf{R}^n$ and $\psi\phi^{-1}|\phi(U \cap V): \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

An *atlas* on the set X is a collection $\{(U_j,\phi_j) | j \in J\}$

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of charts such that $X = \cup\{U_j | j \in J\}$ and any two charts are compatible.

A *differential structure* $\mathcal{D}$ on a set X is a maximal atlas. It is clear that any atlas is contained in a unique differential structure which is said to generate.

If $\mathcal{A}$ is an atlas on the set X, it is also clear that there is a unique topology on X with the property that $\phi: U \to \phi U$ is a homeomorphism of the open set U onto ϕU for every chart (U,ϕ).

A smooth manifold is a set X together with a differential structure $\mathcal{D}$ or X; notation: $(X,\mathcal{D})$. When there is no danger of confusion, one simply refers to the smooth manifold X.

Let $D(r) = \{u \in \mathbf{R}^n | \ |u| \leq r\}$, and let M be a smooth n-manifold. A subset D of M is said to be an n-*disk*, provided there is a chart (U,ϕ) of M such that $\phi D = D(r)$ for some psoitive number r. (This definition allows us to avoid some technicalities regarding differentiability on sets which are not open.)

If D is an n-disk in M, then a map $f: M \to M$ is said to be a *radial diffeomorphism* in D, if there exist a chart (U,ϕ) of M, a positive number ε, and a diffeomorphism $\lambda: \mathbf{R} \to \mathbf{R}$ such that $\phi D = D(1)$, $\lambda(t) = t$ for all $t < \varepsilon$ and all $t > 1 - \varepsilon$, $f(x) = x$ for all $x \in M - D$, and $f(x) = \phi^{-1}\Lambda\phi(x)$ for $x \in D$, where $\Lambda: \mathbf{R}^n \to \mathbf{R}^n$ if defined by $\Lambda(u) = \lambda(|u|)u/|u|$ if $u \neq 0$ and $\Lambda(0) = 0$. Because f is the identity on $M - D$ and a diffeomorphism of Int D, $f: M \to M$ is in fact a diffeomorphism.

Lemma 1. *If* D_1, D_2, D_3, D_4 *are* n-*disks in a smooth* *manifold* M *such that* $D_i \subset \text{Int } D_{i+1}$ *for* $i = 1,2,3$, *then there is a diffeomorphism* $f: M \to M$ *such that* $f(x) = x$ *for* $x \in D_1 \cup (M - D_4)$ *and* $\text{Int } fD_2 \supset D_3$.

Proof. There is a radial diffeomorphism $g: M \to M$ in D_4 which is the identity on a nonempty open subset B of $\text{Int } D_1$ and which maps D_3 into D_1, and there is a radial diffeomorphism $h: M \to M$ in D_2 which maps D_1 into V. Put $f = h^{-1}g^{-1}h$. If $x \in D_3$, then $h(x) \in D_3$ and $gh(x) \in D_1$ and consequently $h^{-1}gh(x) \in \text{Int } D_2$; hence $f^{-1}D_3 \in \text{Int } D_2$, and therefore $D_3 \subset g(\text{Int } D_2) = \text{Int } fD_2$.

Theorem 1. *If a differentiable manifold* M *has an atlas* $\{(U_i,\phi_i) | i \in \omega_0\}$ *such that* $U_i \subset U_{i+1}$ *and* $\phi_i U_i = \mathbf{R}^n$ *for all* $i \in \omega_0$, *then* M *is diffeomorphic to* $\mathbf{R}^n$.

Proof. Let $h_i = \phi_i^{-1}: R^n \to U_i \subset M$. From the hypothesis that $U_i \subset U_{i+1}$ for $i \in \omega_0$ it follows that there is a strictly increasing sequence of positive integers r_i, $i \in \omega_0$ such that $\cup\{h_i D(r_i) | i \in \omega_0\} = M$ and $h_i D(r_i) \subset \text{Int } h_{i+1} D(r_{i+1})$ for $i \in \omega_0$. Put $Q_i = h_i D(r_i)$.

We assert that there exist a sequence of diffeomorphisms $f_i: M \to M$, $i \in \omega_0$ and a strictly increasing sequence of positive numbers s_i, $i \in \omega_0$ with limit r_1 such that $A(i)$: f_i is the identity on $M - Q_{i+1}$ and on $f_{i-1} \cdots f_1 f_0 h_1 D(s_{i-1})$ and such that $B(i)$: $f_i \cdots f_1 f_0 h_1 D(s_i) \supset Q_i$. To verify this assertion assume inductively that f_i and s_i for $i = 0,1,\cdots,k$ satisfy $A(i)$ and $B(i)$ for $i = 0,1,\cdots,k$. Since $f_k \cdots f_1 f_0 Q_1 \subset \text{Int } Q_{k+1}$, there is $s_{k+1} > s_k$ such that $0 < r_i - s_{k+1} < 1/(k+1)$, and the lemma applies to $D_1 = f_k \cdots f_1 f_0 h_1 D(s_k)$, $D_2 = f_k \cdots f_1 f_0 h_1 D(s_{k+1})$, $D_3 = Q_{k+1}$, and $D_4 = Q_{k+2}$ to provide a diffeomorphism f_{k+1}: $M \to M$ such that A(k+1) and B(k+1) hold.

To complete the proof of the Theorem, define F: Int $Q_1 \to M$ by $F(x) = \lim_{k\to\infty} F_k(x)$ where $F_k = f_k \cdots f_1 f_0$: $M \to M$. Since $F(x) = F_k(x)$ for $x \in h_1 D(s_k)$, F is well-defined and clearly a homeomorphism onto M. Since F is a diffeomorphism on each of the open sets Int $h_1 D(s_k)$, $k \in \omega_0$, it is a diffeomorphism of Int Q_1 (which is diffeomorphic to $\mathbf{R}^n$) onto M.

Lemma 2. *Any closed smooth embedding* $\mathbf{R} \to \mathbf{R}^2$ *extends to a diffeomorphism of* $\mathbf{R}^2$ *onto itself.*

Proof. Any closed embedding of $\mathbf{R}$ into $\mathbf{R}^2$ extends to a closed embedding f: $\mathbf{R} \times [-2,2] \to \mathbf{R}^2$ by means of the Collaring Theorem.

Take a rectilinear triangulation T of $\mathbf{R}^2 \setminus f(\mathbf{R} \times \{0\})$. The 1-simplices of T which are not contained in $f(\mathbf{R} \times [-1,1])$ comprise a sequence $\{A(j) \mid j \in \omega\}$ with the property that for any compact set K in $\mathbf{R}^2$ there is an index j(K) such that $A(j) \cap K = \emptyset$ for all $j \geq j(K)$.

For each positive real number r define the band $B(r) = \mathbf{R} \times [-2 + 1/r, 2 - 1/r]$. We claim there is a sequence of closed embeddings F_m: $\mathbf{R} \times [-2,2] \to \mathbf{R}^2$ $(n \in \omega)$ such that $F_0 = F$ and for all $n \in \omega$:

(1) $F_{n+1}(x) = F_n(x)$ for the points x of B(n) and

(2) $F_n(B(n)) \supset A(j)$ for all $j < n$.

If such a sequence exists, define F: $\mathbf{R} \times (-2,2) \to \mathbf{R}^2$ by $F(x) = \lim_{n\to\infty} F_n(x)$; then F extends $f|B(1)$ and is a diffeomorphism onto an open set which contains every 1-simplex of the triangulation T of $\mathbf{R}^2 - f(\mathbf{R} \times 0)$ and hence by simple-connectivity every point of $\mathbf{R}^2$. It follows easily that there is a diffeomorphism of $\mathbf{R}^2$ onto itself extending the original closed embedding $\mathbf{R} \to \mathbf{R}^2$.

The claim is proved by induction. Assume F_n has been obtained satisfying (2).

If $A(n) \cap F_n(B(n)) = \emptyset$, it is easy to construct a diffeomorphism f of $\mathbf{R}^2$ onto itself so that g is the identity on $F_n(B(n))$ and $g(A(n)) \subset F_n(B(n+1))$. In this case, take $F_{n+1} = g^{-1}F_n$. If $A(n) \cap F_n(B(n)) \neq 0$, there is a finite sequence of closed subintervals $\{C_1, C_2, \cdots, C_r\}$ so that $A(n) - \cup\{C_i | i \leq r\}$ is contained in $F_n(B(n+\frac{1}{2}))$ and so that $C_i \cap F_n(B(n)) = \emptyset$ for $i \leq r$. By a preliminary diffeomorphism, if necessary, we may assume the set of endpoints of C_i is a subset of $F_n(B(n+\frac{1}{2}))$ for $i \leq r$. For each C_j let C_j' be an arc lying in $F(B(n+\frac{1}{2})) - F(B(n))$ so that $C_j' \cup C_j'$ is a simple closed curve so that $C_j' \cap C_i = \emptyset$ for all $i \neq j$. Let $M = \{i \leq r | \text{if } j \neq i, C_i$ is not a subset of the bounded domain of $C_j \cup C_j'$. For each $i \in M$, let C_i'' be an arc so that $C_i \cup C_i' \cup C_i''$ is a θ-curve with C_i as the cross-arc such that if $i \neq j$ are in M, then the 2-cells bounded by $C_i' \cup C_i''$ and $C_j' \cup C_j''$ are mutually exclusive and the 2-cells bounded by $C_i' \cup C_i''$ does not intersect $F_n(B(n))$. Let $M = \{i(1), i(2), \cdots, i(t)\}$. For each $i \in M$, let h_i be a diffeomorphism which is the identity on the complementary domain of $C_i' \cup C_i''$ and so that h_i takes the 2-cells bounded by $C_i' \cup C_i$ into Int $F_n(B(n+1))$. Let $h = h_{i(1)} \circ h_{i(2)} \circ \cdots \circ h_{i(t)}$ and let $F_{n+1} = h^{-1} \circ F_n$.

Notation. Throughout Lemma 2 and Theorem 3, we let

$H = \{(0,y) | y \leq 0\}$.

Definition. We will say that the set K is enveloped by the open set U if $K \subset \text{int } \overline{U}$.

Lemma 3. Suppose that $\{U(j)\}_{j\in\omega}$ *is a sequence of open and connected subsets of* $\mathbf{R}^2$, $\text{cl } U(j+1) \subset U(j)$ *and* $\cap_{j\in\omega} U_{(j)} = \phi$. *Suppose further that:*

A. $\{p(j)\}_{j\in\omega}$ *is a sequence of points so that* $p(j) \in U(n)$ *with* $\{|p(j)|\}_{j\in\omega}$ *increasing and unbounded.*

B. $\{N(j)\}_{j\in\omega}$ *is a family of disjoint, infinite subsets of* ω.

Then there is a diffeomorphism g *of* $\mathbf{R}^2$ *onto an open subset of* $\mathbf{R}^2$ *such that*

(1) $\mathbf{R}^2 - g(\mathbf{R}^2)$ *is* H.

(2) each point of H *is a limit point of* $\{g(p(n)) | n \in N(j)\}$ *for each* $j \in \omega$.

(3) $g(U_n)$ *envelopes* H *for each* $n \in \omega$.

Proof. We construct G in several steps:

Step 1. Let h_0 be a diffeomorphism from $\{(x,0) | x \in \mathbf{R}\}$ into $\mathbf{R}^2$ so that $h_0(n,0) = p(n)$ and $h_0(\{(x,0) | x > n\}) \subset U(n)$. Let h_1 be the extension of h_0 taking $\mathbf{R}^2$ onto $\mathbf{R}^2$ given by Lemma 2. Let $h = h_1^{-1}$.

Step 2. Let f be a diffeomorphism from $\mathbf{R}^2$ onto $\mathbf{R}^2$ which leaves the set $\{(x,0) | x \geq 0\}$ fixed and so that $\{(x,y) | x > n\} \subset f(h(U(n)))$.

Step 3. Let $S = \{s_i | i \in \omega\}$ be a countable dense subset of $\mathbf{R}$. Let ϕ be a diffeomorphism from $\mathbf{R}^2$ into $\mathbf{R}^2$ so that

(a) $\phi(x,y) = (x,y')$ (i.e., ϕ is fixed on its first coordinate).

(b) If $N(j) = \{j(1), j(2), \cdots\}$, then $\phi(j(i) + 1, 0) = (j(i) + 1, s_i)$.

Thus, $j(i)$ is the i^{th} number in $N(j)$ and $\phi \circ f \circ h$ takes $p(j(i))$ onto $(j(i) + 1, s_i)$.

Step 4. Let $\beta: \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $\beta(x,y) = (e^{-x}, y)$.

Step 5. Let $\gamma: \{(x,y) | x > 0\} \to \mathbf{R}^2 - H$ be defined by $\gamma(x,y) = (\sqrt{x^2 + y^2} \cos(\pi/2 + 2 \arctan(y/x)), \sqrt{x^2 + y^2} \sin(\pi/2 + 2 \arctan(y/x)))$. Finally $g = \gamma \circ \beta \circ \phi \circ f \circ h$ is the desired diffeomorphism.

Theorem 2. Assuming the continuum hypothesis, there is a hereditarily separable, perfectly normal, analytic manifold that is not metrizable.

Proof. We will build a C^∞-manifold; the existence of an analytic manifold will then follow from [K,P]. The construction is simply a "careful" version of the construction developed in [RZ]. Let $D = D(1) = \{x \in \mathbf{R}^2 | \ |x| \leq 1\}$ and let $D^0 = \text{int } D$. Let $\{x_\alpha | \alpha \in \omega_1\}$ be an indexing of $D - D^0$ (using CH). Let $\{H_\alpha | \alpha \in \omega_1\}$ be a collection of mutually exclusive copies of H. Let $X_0 = \mathbf{R}^2$ and let $X_\alpha = X_0 \cup [\cup_{\beta<\alpha} H_\beta]$ and using CH, let $\{A_\alpha | \alpha \in \omega_1\}$ be an indexing of the countable subsets of X so that $A_\alpha \subset X_\alpha$. Let f_0 be diffeomorphism from $\mathbf{R}^2$ onto D^0 and let F be the function defined by

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in \mathbf{R}^2 \\ x_\alpha & \text{if } x \in H_\alpha \end{cases}$$

and let $f_\alpha = f | X_\alpha$. We will inductively construct a differentiable structure $\mathcal{D}_\alpha$ on X_α such that:

1. $(X_\alpha, \mathcal{D}_\alpha)$ is diffeomorphic to $\mathbf{R}^2$: i.e. $\mathcal{D}_\alpha$ contains a chart (X_α, ϕ_α) with $\phi_\alpha(X_\alpha) = \mathbf{R}^2$.

2. If $\beta < \alpha$, then $(X_\beta, \phi_\beta) \in \mathcal{D}_\alpha$.

3. If $\gamma \leq \beta < \alpha$, $x \in H_\beta$ and x_β is a limit point of $f(A_\alpha)$ in D, then x is a limit point of A_α in (X_α, T_α), where T_α is the topology on X_α given by $\mathcal{D}_\alpha$.

Let $\mathcal{D}_0$ be the usual differential structure on $X_0 = \mathbf{R}^2$ generated by the atlas consisting of the single chart $(X_0$, identity map).

Suppose we have $\mathcal{D}_\alpha$ satisfying (1)-(3) for all $\alpha < \lambda < \omega_1$.

Case I. λ is a *limit ordinal*: Let $\mathcal{D}_\lambda$ be the differential structure generated by $\{(x_\theta, \mathcal{D}_\theta) | \theta < \lambda\}$. That $(X_\lambda, \mathcal{D}_\lambda)$ is diffeomorphic to $\mathbf{R}^2$ is given by Theorem 1.

Case II. $\lambda = \alpha + 1$: For each $n \in \omega$, let $U_n = f_\alpha^{-1}(D_{1/n}(x_\alpha))$, where $D_{1/n}(x_\alpha) = \{x \in D | d(x, x_\alpha) < 1/n\}$.

Then $\{U_n\}$ is a nested sequence of open sets in X_α such that $\cap_{n\in\omega}\overline{U}_n = \phi$. Let $\{N_j\}_{j\leq\omega}$ be a disjoint family of infinite subsets of ω and fix a 1-1 map $i: \alpha + 1 \to \omega$. For each $n \in \omega$, choose $p_n \in U_n$ so that if $\beta \leq \alpha$ and x_α is a limit point of $f(A_\beta)$ in D, then $p_n \in A_\beta \cap U_n$ for all $n \in N_{i(\beta)}$.

Let ϕ be the diffeomorphism from $(X_\alpha, \mathcal{D}_\alpha)$ onto $\mathbf{R}^2$ given by our induction and let g be the diffeomorphism given by Lemma 3 from $\mathbf{R}^2$ into $\mathbf{R}^2$ so that (1) $\mathbf{R}^2 - g(\mathbf{R}^2)$ is H, (2) each part of H is a limit point of $\{g(\phi(p(k))) | k \in N_j\}$ for each $j \in \omega$, and (3) $g(U_n)$ envelopes H for each $n \in \omega$. Let $\mathcal{D}_{\alpha+1}$ be the differential structure on $X_{\alpha+1}$ generated by the atlas $\mathcal{D}_\alpha \cup \{(X_{\alpha+1},\phi_{\alpha+1})\}$ where $\phi_{\alpha+1}|X_\alpha = g \circ \phi_\alpha, \phi_{\alpha+1}|H_\alpha$ is the identification of H_α with H.

As in [RZ], the construction of $\mathcal{D}_{\alpha+1}$ is such that $f_{\alpha+1}$ is continuous and our induction is complete. We will let $\mathcal{D}$ be the atlas on X generated by $\cup_{\alpha<\omega_1}\mathcal{D}_\alpha$ and let T be the topology on X given by $\mathcal{D}$. The argument that (X,T) is hereditarily separable, perfectly normal, but not Lindelöf follows exactly as in [R,Z].

Note. As with the Rudin-Zenor manifold, we can, using ♦, obtain a differentiable, perfectly normal, countably compact, hereditarily separable, non-metrizable manifold. It remains an open question if there is a complex analytic, perfectly normal, non-metrizable manifold.

References

[A] P. Alexandroff, *On local properties of closed sets*, Ann. Math. 36 (1935), 1-35.

[B] M. Brown, *The monotone union of open* n-*cells is an open* n-*cell*, Proc. Amer. Math. Soc. 12 (1961), 812-814.

[K,P] W. Koch and D. Puppe, *Differenzierbare Structuren auf Mauigfaltigkeiten ohne abzahlbare Basis*, Archiv der Math. 19 (1968), 95-102.

[R,Z] M. E. Rudin and P. Zenor, *A perfectly normal non-metrizable manifold*, Houston J. Math. 2 (1976), 129-134.

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