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by

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1. Introduction

The biological problem of finding the evolutionary distance between two DNA sequences has led to the abstract problem of defining and computing the distance between two finite sequences, e.g., see [1], [2], [3], [4]. One approach to this problem is to define the distance as the least number of interchanges or deletions which would transform one sequence into another, reflecting the number of mutations and deletions separating two DNA sequences. This approach suffers the defect that it treats all genes as equally important. In [4], P. Sellers remedied this defect by first assigning a weight to each mutation and deletion and then computing the least weighted distance between two sequences. We now give a formal description of Seller's procedure.

Let (X,r,e) be a pointed metric space, that is, a metric space $X = (X,r)$ with a distinguished or neutral element e. An *evolutionary sequence* is any sequence in X which is eventually constant with the constant value being the neutral element e. Let $P(X)$ denote the set of all evolutionary sequences. Define a metric s on $P(X)$ in the following way. If $x = \{x_n\}$ and $y = \{y_n\}$ are elements of $P(X)$, then let

$$s(x,y) = \sum_{n=1}^{\infty} r(x_n,y_n).$$

We call the metric space $(P(X),s)$ the *evolutionary product* of (X,r,e).

Define an equivalence relation $\sim$ on $P(X)$ as follows:

$\{x_n\} \sim \{y_n\}$ iff the finite subsequence of all non-neutral elements of $\{x_n\}$is identical to the finite subsequence of all non-neutral elements of $\{y_n\}$. If $x \in P(X)$, let $\overline{x} = \{y \in P(X): x \sim y\}$. Let $E(X) = \{\overline{x}: x \in P(X)\}$. Define a distance function d on E(X) by

$$d(\overline{x},\overline{y}) = \min\{s(a,b): a \in \overline{x} \text{ and } b \in \overline{y}\}.$$

That d is a metric for E(X) was established in [4]. We call the metric space (E(X),d) the *evolutionary metric space* associated with (X,r,e).

Observe that E(X) is the quotient image of P(X) and that P(X) is a subspace of a product of countably many copies of X. The questions of what kind of a product topology and what kind of a quotient map are in use here are answered in Sections 2 and 3. In Section 2 it will be seen that the product topology usually lies strictly between the Tychonoff and the box, a feature in common with the familiar sequence space ℓ_1 in the sense that neither the Tychonoff topology nor the box topology on the set of all real sequences relativizes to the usual norm topology for ℓ_1. Also in Section 2 we shall note a few properties of X that carry over to P(X), but a detailed study of the impact of X upon P(X) is not carried out. In Section 3 we shall establish the interesting fact that the quotient map from P(X) onto E(X) is always a local isometry. This, of course, implies that many properties of X which carry over to P(X) must also carry over to E(X).

We shall also see in Sections 2 and 3 that the nature of the neutral element e is crucial, for if e is an isolated point, then complete topological descriptions of P(X) and

E(X) are easily given in terms of X, while no such simple descriptions of P(X) and E(X) are apparent if e is not isolated.

In Section 4 we look at a generalization of the construction of P(X) from X, namely, the box metric product. We are motivated by two goals. One is to remove the reliance upon the distinguished element e. The second objective in constructing a generalization of P(X) is to replace the product of countably many copies of X by a product of countably many (possibly distinct) metric spaces. These considerations lead naturally to the idea of a box metric product, which will be defined and studied in Section 4.

2. Evolutionary Products

Let (X,r,e) be a pointed metric space with neutral element e, let P(X) be the set of all evolutionary sequences in (X,r,e) and let s be the metric on P(X) defined by

$$s(x,y) = \sum_{n=1}^{\infty} r(x_n,y_n)$$

If T_s denotes the topology generated on P(X) by the metric s and T denotes the relativized Tychonoff product topology on $P(X) \subset \prod_{n=1}^{\infty}\{X_n\}$ where each space X_n is a copy of (X,r), then it is easy to show that $T \subset T_s$. That the inclusion $T \subset T_s$ may be proper is seen in the following example.

2.1 *Example.* Let X = {0,1} and let r(0,1) = 1. Let 0 be the distinguished element. For each natural number n, let $\{x_n\}$ denote the evolutionary sequence with $x_n = 1$ and $x_i = 0$ if $i \neq n$. Let x_0 denote the sequence $\{x_i\}$ with $x_i = 0$ for all i. Then $s(x_n,x_0) = 1$ for $n = 1,2,\cdots$, but $x_n \to x_0$ in the relative Tychonoff topology on P(X), so that in this case T and T_s are distinct.

The following example shows that if (X,r_1,e) and (X,r_2,e) are pointed metric spaces such that r_1 and r_2 are equivalent metrics for the set X, and if $(P(X),s_1)$ and $(P(X),s_2)$ are the associated evolutionary products, then s_1 and s_2 are not necessarily equivalent metrics for P(X).

2.2 *Example.* Let $X = \{0,1,2^{-1},\cdots,n^{-1},\cdots\}$ have the usual relative topology and let 0 be the neutral element. Let $r_1(a,b) = |a-b|$ for a and b in X and define a metric r_2, equivalent to r_1, as follows: $r_2(0,n^{-1}) = 2^{-n}$ and $r_2(n^{-1},m^{-1}) = |2^{-n} - 2^{-m}|$.

Let $q_n = \{x_i\} \in P(X)$ be defined inductively as follows: $q_1 = \{1,0,0,\cdots\}$; $q_2 = \{0,2^{-1},3^{-1},4^{-1},0,\cdots\}$; for general n,

$$q_n = \{0,\cdots,0,n^{-1},(n+1)^{-1},\cdots,(2^n)^{-1},0,0,\cdots\}$$

where the first non-zero term occurs in the n^{th} position. Let $q_0 = \{0,0,0,\cdots\}$. Note that $s_1(q_n,q_0) > 2^{-1}$ for all n and that $s_2(q_n,q_0) \to 0$ as $n \to \infty$, so that s_1 and s_2 are not equivalent metrics for P(X).

If the distinguished point e is isolated, then the phenomenon of Example 2.2 cannot occur and the topological structure of P(X) is completely determined by the topological structure of X.

2.3 *Theorem. Let* (X,r,e) *be a pointed metric space and suppose that* {e} *is an open set. Let* N *denote the set of all positive integers and let* $F = \{A_0,A_1,A_2,\cdots\}$ *be the set of all finite subsets of* N*; let* A_0 *be the empty set. Let* Y_0 *be a one point space and, for* $n > 0$*, let* Y_n *be a space homeomorphic with the Tychonoff product of* m *copies of*

(X,r), *where* $m = \text{card}(A_n)$. *Then the evolutionary product* $(P(X),s)$ *is homeomorphic to*

$$Y_0 \oplus Y_1 \oplus Y_2 \oplus \cdots$$

where $\oplus$ *denotes the disjoint or topological sum.*

Proof. Let Y_0 be a singleton consisting of the constant sequence $\{e,e,e,\cdots\}$. Then Y_0 is an open and closed subspace of $(P(X),s)$. For $n > 0$, let $Y_n = \{\{x_n\} \in P(X): x_n = e$ iff $n \notin A_n\}$. Because $\{e\}$ is open in (X,r), we have $r(e,X-\{e\}) > 0$. From this it follows that Y_n is an open and closed subspace of $P(X)$. Moreover, Y_n is clearly homeomorphic to the Tychonoff product of m copies of X, where $m = \text{card}(A_n)$, completing the proof.

The space X may be embedded as a closed subspace of $P(X)$. Define a map $\phi: X \to P(X)$ by $\phi(x) = (x,e,e,\cdots)$. It is not difficult to show that ϕ is a homeomorphism from X onto $\phi[X]$ and that $\phi[X]$ is closed in $P(X)$.

For each natural number n, define

$$Z_n = \{\{x_i\} \in P(X): x_i = e \text{ if } i > n\}.$$

Each set Z_n is closed in $P(X)$, $Z_1 \subset Z_2 \subset \cdots$, and $P(X) = \bigcup_{n=1}^{\infty} \{Z_n\}$.

Theorem 2.3 together with the observations of the preceding two paragraphs lead to an easy proof of the following.

2.4 *Corollary. Let* X *be a pointed metric space with neutral element* e *and* $P(X)$ *be the evolutionary product of* X. *Then the following hold:*

a) The space X *is discrete iff* $P(X)$ *is discrete.*

b) The space X *is separable iff* $P(X)$ *is separable.*

c) The space X *is* σ*-compact iff* P(X) *is* σ*-compact. Furthermore, if* {e} *is open, then the following also hold:*

d) The space X *is locally connected iff* P(X) *is locally connected.*

e) The space X *is locally compact iff* P(X) *is locally compact.*

f) The space X *is completely metrizable iff* P(X) *is completely metrizable.*

3. Evolutionary Metric Spaces

Let (P(X),s) be the evolutionary product for the pointed space (X,r,e). Recall that $\{x_n\} \sim \{y_n\}$ iff the subsequence of non-neutral terms of $\{x_n\}$ is identical to the subsequence of non-neutral terms of $\{y_n\}$ and that if $x = \{x_n\}$, then $\overline{x} = \{y \in P(X): x \sim y\}$. Let $E(X) = \{\overline{x}: x \in P(X)\}$ and define

$$d(\overline{x},\overline{y}) = \min\{s(a,b): a \sim x \text{ and } b \sim y\}.$$

Then (E(X),d) is the *evolutionary metric space* associated with X and d is called the *evolutionary metric*.

Let $F: P(X) \to E(X)$ be defined by $F(x) = \overline{x}$. We shall now show that the mapping F is a local isometry.

3.1 *Theorem. Given* x *in* P(X), *there exists* $\epsilon > 0$ *such that if* $y \in P(X)$ *satisfies* $s(x,y) < \epsilon$, *then* $s(x,y) = d(\overline{x},\overline{y})$.

Proof. First suppose that $x = \{x_n\}$ is such that $x_n = e$ for all n. Then $s(x,y) = d(\overline{x},\overline{y})$ for any y in P(X).

Now suppose that there exists a positive integer i such that $x_n = e$ if $n \neq i$ and $x_i \neq e$. Let $\epsilon = 2^{-1}\cdot r(x_i,e)$. If $y = \{y_n\}$ satisfies $s(x,y) < \epsilon$, then $r(x_i,y_i) < \epsilon$, and in particular, $r(e,y_i) > \epsilon$. Therefore, if $b = \{b_n\} \sim y$ and if

$b_1 \neq y_i$, then $s(x,b) > \epsilon$; for if $b_i \neq y_i$, then $y_i = b_k$ for some $k \neq i$ so that $s(x,b) \geq f(x_k,b_k) = r(e,y_i) > \epsilon$. If $b_i = y_i$, then $s(x,b) = s(x,y)$ since the non-neutral terms of $\{b_n\}$ are identical to the non-neutral terms of $\{y_n\}$ and $x_n = e$ for all n except i. It follows that $s(x,y) = d(\overline{x},\overline{y})$.

Suppose that there exist two positive integers, i and j, with $x_i \neq e$ and $x_j \neq e$ but $x_n = e$ if $n \neq i$ and $n \neq j$. If $x_i = x_j$, let $\epsilon = 2^{-1} \cdot r(x_i,e)$. If $x_i \neq x_j$, let

$$\epsilon = 2^{-1} \cdot \min\{r(x_i,x_j),\ r(x_i,e),\ r(x_j,e)\}.$$

Assume that $y = \{y_n\}$ satisfies $s(x,y) < \epsilon$. Then $r(x_i,y_i) < \epsilon$, $r(x_j,y_j) < \epsilon$, $r(e,y_i) > \epsilon$ and $r(e,y_j) > \epsilon$. It follows that if $b = \{b_n\} \sim y$ and if it is not the case that both $y_i = b_i$ and $y_j = b_j$, then either $y_i = b_k$ or $y_j = b_k$ for some k for which $x_k = e$, that is, $s(x,b) \geq r(e,y_i) > \epsilon$ or $s(x,b) \geq r(e,y_j) > \epsilon$. In the case that both $y_i = b_i$ and $y_j = b_j$, then clearly $s(x,b) = s(x,y)$. It follows that $s(x,y) = d(\overline{x},\overline{y})$.

In the general case, let $\{w_1,w_2,\cdots,w_m\}$ be the subsequence of all non-neutral terms of $\{x_n\}$. Set $w_0 = e$ and define

$$\epsilon = 2^{-1} \cdot \min\{r(w_i,w_j) : w_i \neq w_j\}.$$

An analysis similar to that for the case in which $\{x_n\}$ had two non-neutral terms shows that if y satisfies $s(x,y) < \epsilon$, then $d(\overline{x},\overline{y}) = s(x,y)$, completing the proof.

From Theorem 3.1 and the nature of the map F, we have the following simple description of $E(X)$ in the case in which the distinguished point e is isolated.

3.2 *Theorem. Let* $E(X)$ *denote the evolutionary metric* *space associated with a metric space* X *with distinguished point* e. *Suppose that* {e} *is open and let* $X_0 = \{e\}$. *For each positive integer* n *let* X_n *be a space which is homeomorphic to the Tychonoff product of* n *copies of the space* X. *Then* E(X) *is homeomorphic to the topological sum* $X_0 \oplus X_1 \oplus X_2 \oplus \cdots$.

In Section 2 we remarked that X may be embedded as a closed subspace of P(X) by means of the map $\phi: X \to P(X)$, where $\phi(x) = (x,e,e,\cdots)$. The composition $F \circ \phi$ is a homeomorphism from X into E(X) and $F[\phi[X]]$ is a closed subspace of E(X), so that X may also be regarded as a closed subspace of E(X). This fact together with previous results in this section and Section 2 yield the following.

3.3 *Corollary. Corollary* 2.4 *remains true if* "P(X)" *is replaced by* "E(X)" *throughout.*

4. Box Metrics

Let $\{(X_a,d_a): a \in A\}$ be a non-empty family of metric spaces. Let $X = \prod_{a\in A} \{X_a\}$ and define an extended real-valued metric t on X by

$$t(x,y) = \sum_{a\in A} d_a(x_a,y_a)$$

where $x = (x_a)$, $y = (y_a)$, and $t(x,y) = \infty$ if it is not finite. Using a common technique of constructing an equivalent bounded metric from an unbounded metric, define

$$d(x,y) = \frac{t(x,y)}{1+t(x,y)}$$

where we set $d(x,y) = 1$ if $t(x,y) = \infty$. We call d a *box metric* and (X,d) the *box metric product* of the family $\{(X_a,d_a)\}$.

For a non-empty family $\{X_a: a \in A\}$ of metrizable spaces, let T denote the box topology on $\Pi\{X_a\}$. If for each a in A, d_a is a compatible metric for X_a, then let $T(\{d_a\})$ denote the box metric topology for the family $\{(X_a,d_a)\}$. It is not difficult to show that the union of all such box metric topologies $T(\{d_a\})$ forms a base for T, so there exist a great many non-equivalent box metrics on $\Pi\{X_a\}$. Here is a simple example.

4.1 *Exmaple.* Let $X_n = \{0,1\}$ and $d_n(0,1) = 2^{-n}$ for $n = 1,2,\cdots$. Then the box metric product in this case is homeomorphic to the Tychonoff product. If $d_n(0,1) = 1$ for all n, then the resulting box metric product is homeomorphic to the box product. And, if $d_n(0,1) = n^{-1}$ for all n, then the resulting box metric topology lies strictly between the Tychonoff and the box topologies.

Generalizing Example 4.1, let $\{X_n\}$ denote a sequence of metrizable spaces. For each n, let r_n and s_n be equivalent compatible metrics for the space X_n. Let r be the box metric associated with the family $\{(X_n,r_n)\}$ and s be the box metric associated with the family $\{(X_n,s_n)\}$. The general problem arises of finding necessary and sufficient conditions on the sequences $\{r_n\}$ and $\{s_n\}$ in order to ensure that r and s be equivalent metrics on $\Pi\{X_n\}$. We first solve this problem for the case in which $X_n = \{0,1\}$ for all n. Although this is the simplest non-trivial case, it is interesting since $\Pi\{X_n\}$ with the Tychonoff topology is homeomorphic to the Cantor set.

Let $\{x_n\}$ be a sequence of positive real numbers. Define

$d_n(0,1) = x_n$ for each n and let $r(\{x_n\})$ denote the box metric for the family $\{(X_n,d_n)\}$ where $X_n = \{0,1\}$. The problem is to find necessary and sufficient conditions on the sequences $\{x_n\}$ and $\{y_n\}$ in order to ensure that the metrics $r(\{x_n\})$ and $r(\{y_n\})$ be equivalent. First define an extended metric $d = d(\{x_n\})$ as follows: let $a = \{a_n\}$ and $b = \{b_n\}$ be elements of $\Pi\{X_n\}$, that is, a_n and b_n are either 0 or 1 for all n. Define d by

$$d(a,b) = \Sigma\{x_n\colon a_n \neq b_n\}.$$

Note that

$$r(a,b) = \frac{d(a,b)}{1+d(a,b)}$$

and that the metric r and the extended metric d generate the same topology. Therefore $r(\{x_n\})$ and $r(\{y_n\})$ are equivalent iff $d(\{x_n\})$ and $d(\{y_n\})$ are equivalent.

4.2 *Theorem. The box metrics* $r(\{x_n\})$ *and* $r(\{y_n\})$ *are equivalent iff whenever* $\{n_i\}$ *is any subsequence of the natural numbers, then* $\sum_{i=1}^{\infty} x_{n_i}$ *and* $\sum_{i=1}^{\infty} y_{n_i}$ *are either both finite or both infinite.*

Proof. Assume that $r(\{x_n\})$ and $r(\{y_n\})$ are equivalent. Then $d(\{x_n\})$ and $d(\{y_n\})$ are equivalent. Let $\{n_i\} = A$ be a subsequence of the natural numbers and assume that $\Sigma\{x_m\colon m \in A\}$ is finite while $\Sigma\{y_m\colon m \in A\} = \infty$. For $m = 0,1,2,\cdots$, let $b(m) = \{b_n(m)\} \in \Pi\{X_n\}$ be defined inductively as follows: $b_n(0) = 0$ for all n. For $m = 1$, $b_n(1) = 0$ if $n \notin A$ and $b_n(1) = 1$ if $n \in A$. For $m = 2$, $b_n(2) = 0$ if $n \notin A$ or $n = n_1 \in A$, and $b_n(2) = 1$ otherwise. Generally, for $j \geq 2$, define $b_n(j) = 0$ if $n \notin A$ or $n \in \{n_1,n_2,\cdots,n_{j-1}\}$, and $b_n(j) = 1$ if $n \in \{n_j,n_{j+1},\cdots\}$. Let $d_x = d(\{x_n\})$ and

$d_y = d(\{y_n\})$. Observe that since $\Sigma\{x_m: m \in A\}$ is finite, we have $d_x(b(0), b(m)) \to 0$ as $m \to \infty$, and since $\Sigma\{y_m: m \in A\} = \infty$, we have $d_y(b(0), b(m)) = \infty$ for all m. This contradicts the equivalence of the extended metrics for d_x and d_y so that $\Sigma\{x_m: m \in A\}$ and $\Sigma\{y_m: m \in A\}$ must both be finite or both be infinite.

To prove the converse, assume that $r(\{x_n\})$ and $r(\{y_n\})$ are not equivalent. Then $d(\{x_n\}) = d_x$ and $d(\{y_n\}) = d_y$ are not equivalent, so there exists a sequence $\{c_n\}$ and a point c such that $d_x(c_n,c) \to 0$ and $d_y(c_n,c) \not\to 0$ or a sequence $\{b_n\}$ and a point b such that $d_y(b_n,b) \to 0$ and $d_x(b_n,b) \not\to 0$. Without loss of generality we may suppose that $d_x(c_n,c) \to 0$ and $d_y(c_n,c) \not\to 0$. There exists an $\epsilon > 0$ and a subsequence $\{a_n\}$ of $\{c_n\}$ such that the following hold, where $c = a_0$:

$$d_y(a_n,a_0) > 2\epsilon \text{ for all } n > 0, \text{ and}$$
$$d_x(a_n,a_0) < 2^{-n} \text{ for all } n > 0.$$

For $n = 0,1,2,\cdots$, let $a_n = (a(n,1),a(n,2),\cdots)$. For $n \geq 1$, define $S(n) = \{i: a(n,i) \neq a(0,i)\}$. Set $S = \cup\{S(n): n = 1,2,\cdots\}$. Then S is a subset of the natural numbers and clearly $\Sigma\{x_n: n \in S\} \leq 1$. We shall complete the proof by showing that $\Sigma\{y_n: n \in S\} = \infty$.

Assume that there exists a subsequence $\{n_i\}$ of the natural numbers such that $a(n_i,1) \neq a(0,1)$ for $i = 1,2,\cdots$. Then $d_x(a_{n_i},a_0) \geq x_1$ for $i = 1,2,\cdots$, a contradiction. So there must exist a natural number n_1 such that $a(m,1) = a(0,1)$ for all $m \geq n_1$. Similarly, there exists $n_2 > n_1$ such that $a(m,1) = a(0,1)$ and $a(m,2) = a(0,2)$ for all $m \geq n_2$. There exists $n_3 > n_2$ such that $a(m,1) = a(0,1)$, $a(m,2) = a(0,2)$ and $a(m,3) = a(0,3)$ for all $m \geq n_3$, etc.

Pick i(1) so large that

$$\Sigma\{y_i: i \in S(1) \text{ and } 1 \leq i \leq i(1)\} > \epsilon.$$

Now $n_{i(1)}$ has the property that if $m > n_{i(1)}$, then $a(m,1) = a(0,1)$, $a(m,2) = a(0,2),\cdots,a(m,i(1)) = a(0,i(1))$. Choose $i(2) > n_{i(1)}$ such that

$$\Sigma\{y_i: i \in S(n_{i(1)}) \text{ and } 1 \leq i \leq i(2)\} > \epsilon.$$

Note that $\Sigma\{y_i: i \in S(n_{i(1)}) \text{ and } 1 \leq i \leq i(2)\} = \Sigma\{y_i: i \in S(n_{i(1)}) \text{ and } i(1) < i \leq i(2)\} > \epsilon$. The integer $n_{i(2)}$ has the property that if $m > n_{i(2)}$, then $a(m,1) = a(0,1)$, $a(m,2) = a(0,2),\cdots,a(m,i(2)) = a(0,i(2))$. Choose $i(3) > n_{i(2)}$ such that

$$\Sigma\{y_i: i \in S(n_{i(2)}) \text{ and } 1 \leq i \leq i(3)\} > \epsilon.$$

Note that $\Sigma\{y_i: i \in S(n_{i(2)}) \text{ and } 1 \leq i \leq i(3)\} = \Sigma\{y_i: i \in S(n_{i(2)}) \text{ and } i(2) < i \leq i(3)\} > \epsilon$. Continuing the process inductively, we get a sequence $\{i(j)\}$, $j = 1,2,\cdots$, such that for each j,

$$\Sigma\{y_i: i \in S(n_{i(j)}) \text{ and } i(j) < i \leq i(j+1)\} > \epsilon.$$

It follows that $\Sigma\{y_i: i \in S\} = \infty$, completing the proof.

Using Theorem 4.2 together with an argument similar to that used in proving that theorem, one may also establish the following general result.

4.3 *Theorem. For each natural number* n, *let* X_n *be a metrizable space and let* s_n *and* t_n *be equivalent compatible metrics for* X_n. *Let* d_s *be the box metric for* $\Pi\{(X_n,s_n)\}$ *and* d_t *be the box metric for* $\Pi\{(X_n,t_n)\}$. *Then,* d_s *and* d_t *are equivalent metrics iff given any two points* $a = \{a_n\}$ *and* $b = \{b_n\}$ *in* $\Pi\{X_n\}$, *and any subset* M *of the natural numbers, the sums*

$$\sum_{i\in M} s_i(a_i,b_i) \text{ and } \sum_{i\in M} t_i(a_i,b_i)$$
are either both finite or both infinite.

In the remainder of this section we shall give several examples and establish two interesting theorems on box-metric products of separable metric spaces. First we look at the box-metric product of countably many copies of the reals with the usual metric; this space turns out to be locally homeomorphic to the sequence space ℓ_1.

4.4 *Example.* For each natural n, let X_n denote the reals with the usual metric. Let $X = \Pi\{X_n\}$ and let d denote the extended box-metric given by

$$d(x,y) = \Sigma|x_n - y_n|.$$

For arbitrary points $x = \{x_n\}$ and $y = \{y_n\}$ of X, define $x \sim y$ iff $d(x,y) < \infty$. It is easy to show that $\sim$ is an equivalence relation and that each equivalence class is an open and closed subspace which is homeomorphic to ℓ_1.

It follows from Theorem 4.7 below that the space of Example 4.4 is not separable. Note also that this space is not locally compact. In fact, the box-metric product of a countable family of compact metric spaces may fail to be locally compact.

4.5 *Example.* Let X denote the box-metric product of countably many copies of the closed unit interval with the usual metric. Let $w = \{w_n\}$ be given by $w_n = 0$ for all n. Given any $\epsilon > 0$, choose a real number b with $0 < b < \epsilon$. Then define for each natural number n a point $x(n) = \{x(n,i)\}$ by $x(n,n) = b$ and $x(n,i) = 0$ if $i \neq n$. The set $\{x(n):$

$n = 1,2,\cdots\}$ is discrete and is contained in the ϵ-sphere about w so that X is not locally compact. In fact, the space X is nowhere locally compact.

The following theorem shows that box-metric products of countable families of separable metric spaces are reasonably well-behaved.

4.6 *Theorem. The box-metric product of countably many separable metric spaces is locally separable.*

Proof. Let $\{(X_n,d_n)\}$ be a countable family of separable metric spaces and let (X,d) denote the box-metric product, where d is the extended box-metric. As in Example 4.4, for arbitrary points $a = \{a_n\}$ and $b = \{b_n\}$ of X, define $a \sim b$ iff $\sum d_n(a_n,b_n) < \infty$. It is easy to establish that $\sim$ is an equivalence relation on X and that the equivalence classes are sets which are both open and closed in (X,d). The proof shall be completed by showing that each equivalence class is a separable subspace.

Let $a = \{a_n\}$ be an arbitrary point of X and $A = \{x \in X: x \sim a\}$. For each positive integer n, let $\{x(n,i): i = 1,2,\cdots\}$ be a countable dense subset of (X_n,d_n); now define $A_n = \{y = \{y_j\}: y_j \in \{x(j,i)\}$ for $1 \le j \le n$ and $y_j = a_j$ for $j > n\}$. The set $D = \cup\{A_n: n = 1,2,\cdots\}$ is dense in A. For let $b = \{b_n\}$ be an arbitrary point of A. Then $d(a,b) = \sum d_i(a_i,b_i) < \infty$ and, given $2\cdot\epsilon > 0$, we may pick m so large that

$$\sum_{i=m}^{\infty} d_i(a_i,b_i) < \epsilon.$$

For each j satisfying $1 \le j \le m$, pick $y_i \in \{x(j,i)\}$ so that $d_j(y_j,b_j) < \epsilon\cdot m^{-1}$. Define $p = \{p_j\}$ by $p_j = y_j$ for $1 \le j \le m$

and $p_j = a_j$ for $m < j$. The point p belongs to D and $d(p,b) < \epsilon$. It follows that D is dense in A, completing the proof.

Using Theorem 4.3, it can be shown that the only separable box-metric products are Tychonoff products.

4.7 *Theorem. Let* $\{(X_n,d_n)\}$ *be a countable family of separable metric spaces. Then the box-metric product* X *is separable iff* X *is homeomorphic to the Tychonoff product of the family* $\{X_n\}$.

Proof. Let (X,d) denote the box-metric product where d is the extended box-metric. Assume that (X,d) is not homeomorphic to the Tychonoff product. We shall show that (X,d) is not separable. By Theorem 4.3, there exist two points $a = \{a_n\}$ and $b = \{b_n\}$ such that $\sum d_n(a_n,b_n) = \infty$. Let M be the infinite set of positive integers such that $a_n \neq b_n$ if $n \in M$. Let $x_n = d_n(a_n,b_n)$ for each n in M. Then (X,d) contains a copy of the box-metric product $T = \Pi\{(T_n,x_n): n \in M\}$ where $T_n = \{0,1\}$ and x_n is the distance between 0 and 1 in T_n. The proof will be completed by showing that T is not separable.

Let $E = \{e_1,e_2,\cdots\}$ be any countable subset of T. Set $e_1 = \{e_1(i)\}$, $e_2 = \{e_2(i)\}$, $\cdots$. We shall show that E is not dense in T. Choose n_1 so that

$$\sum_{i=1}^{n_1} x_i > 1.$$

For $1 \leq i \leq n_1$, define $y(i) = 0$ if $e_1(i) = 1$ and $y(i) = 1$ if $e_1(i) = 0$. Now choose $n_2 > n_1$ such that

$$\sum_{i=n_1+1}^{n_2} x_i > 1.$$

Define $y(i) = 0$ if $e_2(i) = 1$ and $y(i) = 1$ if $e_2(i) = 0$ for all i satisfying $n_1 < i \leq n_2$. Continuing this process, we get an element $y = \{y(i)\}$ in T such that the box-metric distance in T from y to any member of E is greater than one, completing the proof.

We have seen that the box metric product of a countable family of separable metric spaces must be locally separable. The following example shows that the box-metric product of a countable family of locally separable spaces need not be locally separable, even if the spaces are all discrete.

4.8 *Example*. The box-metric product of a countable family of discrete spaces which is not locally separable.

Consider the set $Y = \{(x,n): x \in R$ and $n \in N\}$ where R denotes the reals and N the natural numbers. Define a metric t on Y by

$$t((x,n),(y,m)) = \begin{cases} |n^{-1} - m^{-1}| \text{ if } x = y \text{ and } n \neq m \\ \text{and} \\ n^{-1} + m^{-1} \text{ if } x \neq y. \end{cases}$$

The space (Y,t) is discrete. Let $(X_n,d_n) = (Y,t)$ for each n in N and let (X,d) denote the box-metric product of the family $\{(X_n,d_n)\}$ where d is the extended box-metric. Let $a = \{a_n\}$ where $a_n = (n,n)$ for $n = 1,2,\cdots$. Let $\epsilon > 0$ be given arbitrarily. Choose m so large that $2 < m\cdot\epsilon$. For each irrational real number x, define $p_x = \{b_n\}$ by $b_n = a_n = (n,n)$ if $n \neq m$ and $b_m = (x,m)$. Then $d(p_x,a) < \epsilon$ for each irrational x, and if x and y are distinct irrationals, then $d(p_x, p_y) = 2 \cdot m^{-1}$. The ϵ-sphere about the point a contains an uncountable discrete set and, since ϵ was arbitrary, it follows that the point a has no separable neighborhood.

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