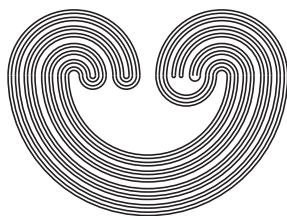

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SYMMETRIZABLE SPACES AND SEPARABILITY

R. M. Stephenson, Jr.

For several years topologists have been interested in discovering properties of feebly compact symmetrizable spaces and in learning if they must all be separable and e -countably compact (defined below), as is the case with the well-known feebly compact Moore space ψ in [GJ, 5I] (see also [M3, p. 66 and p. 381]). Some of the recent results which have been obtained will be given here, and a examples will be constructed in order to prove that (1) there exists a feebly compact Moore space which is not e -countably compact, and (2) there exists a feebly compact semimetrizable Hausdorff space which is not separable.

Recall that a topological space is called *feebly compact* if every locally finite system of open sets is finite. It is well-known that in completely regular spaces feeble compactness and pseudocompactness (every continuous real valued function is bounded) are equivalent concepts, and in normal spaces feeble compactness and countable compactness are equivalent.

A topological space X is said to be *symmetrizable* [A1] if there exists a mapping $d: X \times X \rightarrow [0, \infty)$ such that:

- i) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- ii) $d(x, y) = 0$ if and only if $x = y$; and
- iii) for every set $V \subset X$, V is open if and only if for each point $v \in V$, there exists $\epsilon > 0$ such that $V \supset B(v, \epsilon) \equiv \{x: d(x, v) < \epsilon\}$.

In case d can be chosen so that, in addition, each $B(v, e)$ is a neighborhood of v , then X is called *semimetrizable*.

Let us first consider separability and then e -countable compactness.

1. Separability and Feeble Compactness

The first result I know of linking these concepts for symmetrizable spaces is the following.

Theorem 1.1 (Reed [R]). Every Moore-closed space is separable.

Recall that if P is a property of topologies, a P -space X is called *P-closed* if X is a closed subspace of every P -space in which it can be embedded.

By a result of J. Green [G] a Moore space is Moore-closed if and only if it is feebly compact.

Several years after Theorem 1.1 was discovered, R. W. Heath obtained the following result.

Theorem 1.2 (Heath). Every regular, feebly compact semimetrizable space is a Moore space.

Thus, the hypothesis of 1.1 could be at least formally weakened to regular, feebly compact, and semimetrizable. Then in 1977 the next result was obtained.

Theorem 1.3 ([S2]). Every Baire, feebly compact semimetrizable space is separable.

Because every regular, feebly compact space is Baire

([M2], see also [CN]), Theorem 1.3 is an extension of Theorem 1.1. A construction given in [S3] showed that "semetrizable" could not be replaced by "symmetrizable" in 1.3. It was proved there that *for every cardinal number n , there exists a Baire, feebly compact, symmetrizable Hausdorff space having no dense subset of cardinality $\leq n$* . One direction in which 1.3 can be extended is indicated by the following result concerning the family $\mathcal{J}$ studied in [D], [DGN], [DS], and [HS].

Theorem 1.4 ([DS]). Let X be a Baire, feebly compact neighborhood $\mathcal{J}$ -space, and let I be the set of isolated points of X . Then X has a dense subset D with $|D| \leq \max\{|I|, \aleph_0\}$.

We will now give an example which provides a negative answer to the question [S2]: Is every feebly compact semimetrizable space separable?

Theorem 1.5. Let n be an infinite cardinal number. Then there exists a Hausdorff, feebly compact developable space X which has no dense subset of cardinality $< n$.

Construction. Let Q be the set of rational numbers, with its usual topology, let the cardinal number n have the discrete topology, and let $Y = Q \times n$ have the product topology. Let N denote the set of natural numbers and list the members of Q as $\{q_i : i \in N\}$. For each $i \in N$ let $F_i = \{q_k : k \leq i\} \times n$. Let $\mathcal{S}$ be the set of all countable clopen filter bases $\mathcal{V}$ on Y such that:

- i) $\mathcal{V}$ has no adherent point in Y , i.e., $\bigcap \{V : V \in \mathcal{V}\} = \emptyset$;
- and

ii) the sets comprising $\mathcal{V}$ can be labeled $\{V_i: i \in \mathbb{N}\}$, where for each i , $V_i \supset V_{i+1}$ and $V_i \cap F_i = \emptyset$. Next, let $\underline{T}$ be a maximal subset of $\underline{S}$ such that whenever $V, W \in \underline{T}$ with $V \neq W$, then there exist disjoint sets $V' \in \mathcal{V}$ and $W' \in \mathcal{W}$. Select distinct points $p_V \notin Y$ for each $V \in \underline{T}$, and let $X = Y \cup \{p_V: V \in \underline{T}\}$, topologized as follows: each open subset of Y is open in X ; and a neighborhood of a point p_V is any set $U \subset X$ such that for some $V' \in \mathcal{V}$, one has $U \supset \{p_V\} \cup V'$.

Proof. Because no member of $\underline{T}$ has an adherent point in Y , and because two distinct members of $\underline{T}$ contain at least two disjoint sets, it is easy to see that X is a Hausdorff space.

A development for X will now be given. For each $i \in \mathbb{N}$, $m \in n$, and point $q_k \in Q$ let

$$B((q_k, m), i) = \{(q, m) \in Y: |q - q_k| < 1/i\}.$$

Label the members of $\mathcal{V}$ as in ii) and for each $i \in \mathbb{N}$ and $V \in \underline{T}$ let

$$B(p_V, i) = \{p_V\} \cup V.$$

Define $\mathcal{D}_i = \{B((q_k, m), i): (q_k, m) \in Y\} \cup \{B(p_V, i): V \in \underline{T}\}$, $i \in \mathbb{N}$. That $\mathcal{D}_i$, $i \in \mathbb{N}$, is a development for the space X is clear except possibly at the points of Y . For a point (q_k, m) and an open neighborhood U of (q_k, m) , find $j \in \mathbb{N}$ with $B((q_k, m), j) \subset U$, and note that for $i \geq \max\{2, j\}$, one has $\text{star}((q_k, m), \mathcal{D}_i) \subset U$, since $(q_k, m) \in F_i$ and hence $(q_k, m) \notin B(p_V, i)$ for any $V \in \underline{T}$.

Suppose X fails to be feebly compact. Then [BCM] there exists a pairwise disjoint, locally finite, countably

infinite family $\{U_i: i \in \mathbb{N}\}$ of nonempty open subsets of X . For each $i \in \mathbb{N}$, F_i is a closed nowhere dense subset of Y and Y is a dense subset of X , so there exists a nonempty clopen subset C_i of Y with $C_i \subset U_i \setminus F_i$. Let $V_i = \bigcup \{C_k: k \geq i\}$ and $\mathcal{V} = \{V_i: i \in \mathbb{N}\}$ and note that $\mathcal{V}$ has no adherent point in X since $\{C_k: k \in \mathbb{N}\}$ is locally finite. But clearly i) and ii) are satisfied and so $\mathcal{V} \in \underline{S}$. By the maximality of $\underline{T}$, there must exist $\mathcal{W} \in \underline{T}$ such that $V \cap W \neq \emptyset$ for all sets $V \in \mathcal{V}$ and $W \in \mathcal{W}$. From the latter, however, it would follow that $p_{\mathcal{W}}$ is an adherent point of $\mathcal{V}$, in contradiction of the fact that $\mathcal{V}$ has no adherent point in X .

To complete the proof, it suffices to observe that Y is an open subset of X having no dense subset of cardinality $< n$, and so X cannot have a dense subset of cardinality $< n$.

Let us conclude this section with a question.

Question 1.6. Is every regular, feebly compact symmetrizable space separable?

As noted in [S3], an example providing a negative answer to 1.6 would also provide a negative answer to the question of E. Michael: *Is every point of a regular symmetrizable space a G_δ ?* (Because, by a result of I. Glicksberg, every G_δ point of a regular feebly compact space has a countable neighborhood base, and it is known that a Hausdorff first countable symmetrizable space is actually semimetrizable.)

The reader interested in countable chain and completeness type axioms which imply that a Moore space be separable is referred to [A2], [H], [M1], [M4], and [R].

2. e-Countable Compactness and Feeble Compactness

A topological space X is called *e-countably compact* with respect to a dense subset D if every infinite subset of D has a limit point in X . A space X is called *e-countably compact* if there exists a dense subset of X with respect to which X is e-countably compact.

Like every feebly compact space having a dense set of isolated points, the noncompact Moore-closed space ψ of [GJ, 5I] must be e-countably compact (with respect to its set of isolated points). An interesting question raised by J. Green is the following.

Question 2.1 (Green [G]). Does every noncompact Moore-closed space contain a noncompact, e-countably compact subspace?

The construction below provides a partial answer

Example 2.2. There exists a locally compact, zero-dimensional Moore-closed space X which fails to be e-countably compact.

Description of X . Let C be the Cantor set, $N =$ the set of natural numbers, and $Y = C \times N$, with the product topology. Let $c = 2^{\aleph_0}$ and $\{M_a : a < c\}$ be a 1-1 listing of the members of a maximal infinite family $\mathcal{M}$ of almost disjoint infinite subsets of N . Let $\mathcal{D} = \{D_a : a < c\}$ be the family of all countable dense subsets of Y , and for each ordinal $a < c$ and natural number $n \in M_a$, choose one point

$$d_{a,n} \in D_a \cap (C \times \{n\}),$$

and let

$$J_a = \{d_{a,n} : n \in M_a\}.$$

Next, for each $a < c$, let $\underline{T}_a$ be a maximal family such that:

- i) each $J \in \underline{T}_a$ is a countably infinite, pairwise disjoint, locally finite collection of compact open subsets of $C \times M_a$;
- ii) if J_1 and $J_2 \in \underline{T}_a$ and $J_1 \neq J_2$, then there exist finite sets $J_i \subset J_1$, $i = 1, 2$, so that $(\cup(J_1 \setminus J_1)) \cap (\cup(J_2 \setminus J_2)) = \phi$; and
- iii) each $J \in \underline{T}_a$ satisfies $J_a \cap (\cup J) = \phi$. Finally, select distinct points $p_{J,a} \notin Y$ and let

$$X = Y \cup \{p_{J,a} : a < c \text{ and } J \in \underline{T}_a\},$$

topologized as follows: each open subset of Y is open in X ; and a neighborhood of a point $p_{J,a}$ is any set $V \subset X$ such that for some finite set $J \subset J$, one has

$$V \supset \{p_{J,a}\} \cup (\cup(J \setminus J))$$

Proof. To see that X is Hausdorff, consider distinct points $x, y \in X$. If $x, y \in Y$, it follows from the openness of Y in X that x, y have disjoint neighborhoods. Suppose $x \in Y$ and $y = p_{J,a}$; then by the local finiteness of J in Y there exist an open neighborhood V of x in Y and a finite subset J of J such that $V \cap (\{p_{J,a}\} \cup (\cup(J \setminus J))) = \phi$. If $x = p_{J,a}$ and $y = p_{J',a}$, where $J \neq J'$, then by ii), one can easily find disjoint neighborhoods of x and y . Finally, consider the case $x = p_{J,a}$ and $y = p_{J',b}$, where $a \neq b$. The set $F = M_a \cap M_b$ is finite, so $C \times F$ is compact and there must exist finite subsets J of J and G of J' such that

$$(\cup(J \setminus J)) \cap (C \times F) = \phi = (\cup(J' \setminus G)) \cap (C \times F).$$

Then $\{x\} \cup (U(\mathcal{J} \setminus \mathcal{J}))$ and $\{y\} \cup (U(\mathcal{V} \setminus \mathcal{G}))$ are disjoint neighborhoods of x and y .

Since Y is a locally compact and zero-dimensional subspace of the Hausdorff space X , and since each point of $X \setminus Y$ clearly has a neighborhood base consisting of compact open subsets of X , the space X is locally compact and zero-dimensional.

A development $\mathcal{G}_n, n \in \mathbb{N}$, will now be defined for X .

For each $n \in \mathbb{N}$, point $(x, k) \in Y$, and point $p_{\mathcal{J}, a} \in X \setminus Y$, let

$$B((x, k), n) = \{(y, k) \in Y: |x - y| < \frac{1}{n}\}$$

and

$$\begin{aligned} B(p_{\mathcal{J}, a}, n) &= \{p_{\mathcal{J}, a}\} \cup (U\{T \in \mathcal{J}: T \cap (C \times \{1, 2, \dots, n\}) \\ &= \emptyset\}). \end{aligned}$$

It suffices to take, for each $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{G}_n &= \{B((x, k), n): (x, k) \in Y\} \cup \{B(p_{\mathcal{J}, a}, n): \\ &p_{\mathcal{J}, a} \in X \setminus Y\}. \end{aligned}$$

To verify that X fails to be ϵ -countably compact, note that if D is any dense subset of X , then $D \cap Y$ is a dense subset of the second countable space Y (since Y is open in X) and so for some $a < c$, $D_a \subset D$. Then J_a is an infinite subset of D which by i) and iii) has no limit point in X .

To complete the proof, suppose there exists an infinite locally finite family $\mathcal{C}$ of open subsets of X . Then, because Y is dense in X , one can find a set $M_a \in \mathcal{M}$, an infinite set $H \subset M_a$, and a 1-1 mapping $f: H \rightarrow \mathcal{C}$ such that for each $n \in H$, $(C \times \{n\}) \cap f(n) \neq \emptyset$. Further, for each $n \in H$ one can find a compact open set $K_n \subset (C \times \{n\}) \cap (f(n) \setminus \{d_{a, n}\})$. Then $\mathcal{K} = \{K_n: n \in H\}$ is a countably infinite, pairwise disjoint family of compact open subsets of $(C \times M_a) \setminus J_a$ which is

locally finite in X and Y . But then, by the maximality of $\underline{T}_a$, there must exist $J \in \underline{T}_a$ such that for every finite subset $\mathcal{J}$ of J , one has $(\cup(\mathcal{J} \setminus J)) \cap K_n \neq \emptyset$ for infinitely many $n \in H$. Thus K and, hence, C fail to be locally finite at the point $p_{J,a}$, which is a contradiction.

We will conclude by stating some results which relate e -countable compactness and separability.

Theorem 2.3 ([S1]). (i) Every e -countably compact semimetrizable space is separable.

(ii) If a symmetrizable space is e -countably compact with respect to a set D , then every discrete subspace of D is countable.

Theorem 2.4 (Nedev [N]). Every countably compact (Hausdorff) symmetrizable space is compact (and metrizable).

Question 2.5. Is every e -countably compact symmetrizable space separable?

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