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0. Introduction

This paper is devoted to constructing compact ccc non-separable spaces of weight $\leq$ c, which we shall henceforth refer to as C-spaces. In section 2 we verify a conjecture of van Mill [16], namely, that there is a C-space growth of ω. Let L stand for the Stone space of the Boolean algebra of Lebesgue measurable subsets of [0,1] mudulo the nullsets. L is an extremally disconnected C-space. As van Mill points out, assuming CH, L is a growth of ω. Not being able to show in ZFC that L is a growth of ω, we have extracted from L a space X that is a growth of ω.

C-spaces are hard to come by in ZFC. L is a C-space which has a σ-linked base but there also exists a C-space which does not have a σ-linked base, see Galvin and Hajnal [7]. MA + $\neg$CH, see Juhász [10], implies that no C-space of π-weight < c exists. 2^{κ} is a supercompact and dyadic ccc space of weight κ but is non-separable only when c < κ. No dyadic C-spaces exist.

Assuming CH, Kunen [11] has constructed an hereditarily Lindelöf C-space. A Souslin continuum, assuming one exists, is a supercompact hereditarily Lindelöf C-space. Sections 2 and 3 present new C-spaces. In contrast to L which is extremally disconnected, in section 3 we construct a C-space

Y in which each point is the limit of a non-trivial convergent sequence. Furthermore, Y is the union of c dyadic subspaces. Note that because L is extremally disconnected, every dyadic subspace of L is finite and thus L cannot have this property. The problem of whether in ZFC there is a C-space of size c remains unsolved, although we give an affirmative answer assuming MAS.

In section 3, we present, assuming CH, a new example of a 1st countable C-space. This example actually has a σ-linked base. Kunen was the first to construct such an example which moreover was hereditarily Lindelöf which our space is not. Our example even exists under the weaker assumption of MAC + cfc = ω_1. Recall that MA + $\neg$CH implies no 1^{st} countable C-spaces exist (however in ZFC there is a 1^{st} countable σ-compact ccc non-separable space [2]). This is so because of the following theorem in Juhász [10]: MA + $\neg$CH implies that every compact ccc space K with $[t(K)]^+ < c$ is separable. In section 3 we show that $[t(K)]^+ < c$ cannot be weakened to $t(K) < c$ or even $\mathrm{char}(K) < c$. Tall [17] has pointed out that if c is weakly inaccessible, then $\mathrm{char}(K) < c$ suffices in this theorem.

1. Notation and Definitions

If f is a function, then dom f is the domain of f and rng f is the range of f. If A and B are sets, then ${}^{A}B = \{f\colon \mathrm{dom}\ f = A \text{ and } \mathrm{rng}\ f \subseteq B\}$. If $f \in {}^{A}B$ and $C \subseteq A$, then $f \restriction C$ is the restriction of f to C. For two functions f and g, we write $f \subseteq g$ if $\mathrm{dom}\ f \subseteq \mathrm{dom}\ g$ and $g \restriction \mathrm{dom}\ f = f$.

If $\mathcal{A}$ is a collection of sets, then $\mathcal{A}^{\cap} = \{\cap\mathcal{F}: \mathcal{F}$ is a finite subset of $\mathcal{A}\}$ and $\mathcal{A}^{\cup} = \{\cup\mathcal{F}: \mathcal{F}$ is a finite subset of $\mathcal{A}\}$.

The least infinite ordinal is denoted by ω, c is the least ordinal of the same cardinality as ${}^{\omega}2$ and cfc is the least ordinal κ such that $c = \bigcup_{\alpha<\kappa} A_\alpha$ where for each $\alpha < \kappa$, $|A_\alpha| < c$. For a set A and a cardinal κ, define $[A]^\kappa = \{C \subseteq A: |C| = \kappa\}$ and define $[A]^{<\kappa}$ and $[A]^{\leq\kappa}$ analogously.

Let K be a topological space and let $p \in K$. Then char(p,K) = least κ such that there exists a neighbourhood base at p of cardinality κ, and, t(p,K) = least κ such that whenever $p \in \overline{A}$, there exists $C \in [A]^{\leq\kappa}$ with $p \in \overline{C}$. Char(K) = sup{char(p,K): $p \in K$} and t(K) = sup{t(p,K): $p \in K$}.

The Stone-Čech compactification of the discrete space ω is denoted by $\beta\omega$. 2^κ is the Tychonov product of κ copies of the discrete space $\{0,1\}$. A space is dyadic if it is a T_2 continuous image of 2^κ, for some cardinal κ. For a compact T_2 space K, the compactness number of K, cmpn K = least $n \in \omega$ (if one exists) such that there exists an open subbase $\mathcal{S}$ of K for which every cover of K by members $\mathcal{S}$ has a $\leq n$ subcover. If no such $n \in \omega$ exists, then we say that K has infinite compactness number. If cmpn K = 2, then K is said to be supercompact. The compactness number of a space is defined in [4]. Supercompact spaces were introduced by de Groot in [8]. Supercompact spaces are similar to dyadic spaces in that they have lots of convergent sequences. Whether every dyadic space is supercompact is presently unknown. The reader who wants to find out more about supercompact spaces can refer to van Mill [15].

A space is ccc if every disjoint collection of open sets is countable. K has a σ-n-linked (σ-centered) base if there exists an open base $\mathcal{B}$ for K such that $\mathcal{B} = \bigcup_{m\in\omega} \mathcal{B}_m$ where for each $m \in \omega$ and for each $\mathcal{J} \in [\mathcal{B}_m]^n$ (for each $\mathcal{J} \in [\mathcal{B}_m]^{<\omega}$), $\cap\mathcal{J} \neq \phi$.

2. A C-Space Growth of ω

A growth of ω is a remainder $\gamma\omega - \omega$ for some compactification $\gamma\omega$ of the discrete space ω. A weak P-point p of a space K is a point that is not in the closure of any countable subset of K - p . Clearly no separable space without isolated points can have weak P-points. However, as pointed out to me by Alan Dow, there exist compact ccc spaces without isolated points that have weak P-points; L is such an example. Kunen [12] proved that there exist weak P-points in $\beta\omega - \omega$. This supplied a very concrete reason why $\beta\omega - \omega$ is not homogeneous. In [16], Jan van Mill generalized this fact to: every compact F-space of weight $\leq$ c in which non-empty G_δ's have non-empty interiors has weak P-points. He also proved that if there was a C-space growth of ω, then the weight condition was not needed. This and the bottle of Jenever that he offered for the solution supplied the motivation for this paper. We remark that his result has been generalized in Dow and van Mill [6] to: every compact nowhere ccc F-space has weak P-points.

Example 2.1. A compactification $\gamma\omega$ of ω for which $\gamma\omega - \omega$ is a C-space.

Define $P = \{f \in {}^{\omega}\omega\colon 0 \leq f(n) \leq n + 1 \text{ for each } n \in \omega\}$ and $N = \{f \upharpoonright n\colon f \in P \text{ and } n \in \omega\}$. Define $T = \{\pi \in {}^{\omega}N\colon$ $\operatorname{dom} \pi(n) = n+1$ for each $n \in \omega\}$. For ecah $s \in N$, define $C_s = \{t \in N\colon s \subseteq t\}$ and for each $\pi \in T$, define $C_\pi = \bigcup_{n\in\omega} C_{\pi(n)}$. Note that $N - C_\pi$ is infinite for each π. Define $\mathcal{A} = \{C_\pi\colon$ $\pi \in T\} \cup \{N - C_\pi\colon \pi \in T\}$ and $\mathcal{B} = [\mathcal{A}^{\cap f}]^{\cup f}$. $\mathcal{B}$ is a Boolean subalgebra of $\langle \mathcal{P}(N), \cup, \cap, -, \phi, N\rangle$ such that $\{\{s\}\colon s \in N\} \cup \{C_s\colon s \in N\} \subseteq \mathcal{B}$. Let $\gamma\omega$ denote the Stone space of all ultrafilters of $\mathcal{B}$. $\gamma\omega$ is a compactification of the countable discrete subspace $\{\{B \in \mathcal{B}\colon s \in B\}\colon s \in N\}$ which we identify with the discrete space ω. Let $X = \gamma\omega - \omega$. Then X is a C-space growth of ω.

A. X *is not separable.*

Proof. Let $\{p_n\colon n \in \omega\}$ be countably many free ultrafilters of $\mathcal{B}$. For each $n \in \omega$, there exists $\pi(n)$ with $\operatorname{dom} \pi(n) = n + 1$ such that $C_{\pi(n)} \in p_n$. This is so, because $N = \{s \in N\colon \operatorname{dom} s \leq n\} \cup \bigcup\{C_s\colon \operatorname{dom} s = n + 1\}$ for each $n \in \omega$. Thus, $\{p \in X\colon N - C_\pi \in p\}$ is a non-empty open set in X disjoint from $\{p_n\colon n \in \omega\}$.

B. X *has a* σ-2-*linked base and hence is* ccc.

Proof. It suffices to show that $\{B \in \mathcal{B}\colon |B| = \omega\} = \bigcup_{n\in\omega} \mathcal{B}_n$ such that for each n every two members of $\mathcal{B}_n$ have infinite intersection. To this end, for each $j \in \omega$ and for each $s \in N$ with $2j - 1 \leq \operatorname{dom} s$, define $\mathcal{B}(j,s) = \{B \in \mathcal{B}\colon$

there exist $K \in [T]^{<\omega}$ and $L \in [T]^j$ with $s \in \bigcap_{\pi \in K} C_\pi \cap \bigcap_{\pi \in L} N - C_\pi \in [B]^\omega\}$. Since for each $B \in \mathcal{B}$ with $|B| = \omega$, there exists $D \in \mathcal{A}^{\mathfrak{B}}$ with $D \in [B]^\omega$ and any infinite subset of N contains elements of arbitrarily large domain, it follows that $\{B \in \mathcal{B}: |B| = \omega\} = \bigcup\{\mathcal{B}(j,s): j \in \omega$ and $2j - 1 \leq \text{dom } s\}$.

Fix an index j,s with $2j - 1 \leq \text{dom } s$. If $\{B_0, B_1\} \subseteq \mathcal{B}(j,s)$, then there exist $K_i \in [T]^{<\omega}$ and $L_i \in [T]^j$ such that for each $i = 0,1$, $s \in D_i = \bigcap_{\pi \in K_i} C_\pi \cap \bigcap_{\pi \in L_i} N - C_\pi \in [B_i]^\omega$. We now define, by induction on $\text{dom } s \leq n$, an $h \in P$ such that $\{h \upharpoonright n: \text{dom } s \leq n\} \subseteq D_0 \cap D_1$. Stage dom s: Let $h \upharpoonright \text{dom } s = s$. Then $h \upharpoonright \text{dom } s \in D_0 \cap D_1$. Assume we have defined $h \upharpoonright n$ for some $\text{dom } s \leq n$ such that $h \upharpoonright n \in D_0 \cap D_1$. Stage $n + 1$: Define $h \upharpoonright n + 1$ to be some sequence in N of domain $n + 1$ that extends $h \upharpoonright n$ and such that $h \upharpoonright n + 1 \notin \{\pi(n): \pi \in L_0 \cup L_1\}$. This is possible because there are $n + 2$ sequences in N of domain $n + 1$ that extend $h \upharpoonright n$ and $|L_0 \cup L_1| \leq 2j < \text{dom } s + 2 \leq n + 2$. Then $h \upharpoonright n + 1 \in D_0 \cap D_1$.

Remark 1. If we replace $2j - 1$ by $nj - 1$, then we find that X also has a σ-n-linked base. Thus X, like the space L, is a compact space that has a σ-n-linked base for each $n \in \omega$, but does not have a σ-centered base (= separability in compact spaces).

Remark 2. $\langle \mathcal{P}(\omega), \cup, \cap, -, \phi, \omega \rangle$ can be embedded as a Boolean subalgebra of $\mathcal{B}$. This means that $\beta\omega$ is a continuous image of $\gamma\omega$. It follows, see [4], that $\gamma\omega$ has infinite compactness number. X also has infinite compactness number, since it is easy to see that if cmpn $K = n$ and K has a σ-n-linked base, then K is separable. This latter fact was first proved by Eric van Douwen [5]. According to [1], if $\delta\omega$ is a supercompactification of ω, then $\delta\omega - \omega$ is ccc. This was the author's first approach to van Mill's question. Is there a supercompactification $\delta\omega$ of ω with $\delta\omega - \omega$ non-separable?

3. More C-Spaces

In this section, we produce "nicer" examples of C-spaces. Nicer, in the sense that they have more properties in common with the closed unit interval. Our main example requires only ZFC, but its offshoots require Martin's Axiom MA; MA is known to be consistent with $\neg$CH, see Martin and Solovay [14]. For undefined terms, we refer the reader to Jěch [9].

MA asserts that for every ccc poset P and for every collection $\mathcal{D}$ of $< c$ dense subsets of P, there is a $\mathcal{D}$-generic $G \subseteq P$. CH implies MA. MAS is MA restricted to σ-centered posets. MAS is strictly weaker than MA, see Kunen and Tall [13], and is known to be equivalent to the combinatorial principle P(c), see [3]. MAC is MA restricted to countable posets. MAC is strictly weaker than MAS and is known to be equivalent to the statement that the real line cannot be covered by $< c$ nowhere dense sets, see [13]. What is nice about MAC is that it places no cardinal restriction on the continuum c. MAS implies that for each $\omega \leq \kappa < c$, $2^\kappa = c$ and hence that c is regular, whereas MAC is consistent with any permissible cardinal behaviour. MAC holds in any model of ZFC obtained by adding Cohen reals to a model of ZFC + GCH.

The following lemma is the key construction for our C-spaces.

Lemma 3.1. *The Cantor Set* C *has a base* $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ *and a collection* $\mathcal{O}$ *of open sets such that:*

(a) Each $\mathcal{B}_n$ *is finite and if* $B \in \mathcal{B}_n$ *and* $\mathcal{J} \in [\{O \in \mathcal{O}: B \nsubseteq O\}]^{\leq n+1}$, *then* $B - \bigcup\mathcal{J} \neq \phi$.

(b) For each $f \in C$, $\mathcal{O}_f = \{O \in \mathcal{O}: f \notin O\}$ *is independently dense at* f, i.e., *for disjoint finite subsets* $\mathcal{J}$ *and* $\mathcal{G}$ *of* $\mathcal{O}_f$ *we have that* $f \in \overline{\bigcap\mathcal{J} - \bigcup\mathcal{G}}$.

(c) For each $A \in [C]^\omega$, *there exists an* $O \in \mathcal{O}$ *with* $A \subseteq O \subsetneq C$. *Furthermore, assuming* MAC, *we have that for each* $A \in [C]^{<c}$ *there exists an* $O \in \mathcal{O}$ *with* $A \subseteq O \subsetneq C$.

Proof. Let $\{I_m: m \in \omega\}$ be finite subsets of ω such that $n \neq m$ implies $I_n \cap I_m = \phi$, $|I_m| = m + 1$ and $m < \min I_m$. Give each I_m the discrete topology and give $C = \prod_{m\in\omega} I_m$ the product topology. C is a compact metric space homeomorphic to the Cantor Set. For each $n \in \omega$ and $s \in \prod_{m\leq n} I_m$, define $B_s = \{f \in C: s \subseteq f\}$ and $\mathcal{B}_n = \{B_s: s \in \prod_{m\leq n} I_m\}$. Each $\mathcal{B}_n$ is finite and $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ is a base for the open sets of C. Define $T = \{\pi \in {}^\omega\omega: \pi(0) \in I_0$ and $\pi(n + 1) \in I_{\pi(n)}$ for each $n \in \omega\}$. For each $\pi \in T$, define $O_\pi = \{f \in C$: there exists an $n \in \omega$ with $f(\pi(n)) = \pi(n+1)\}$. $\mathcal{O} = \{O_\pi: \pi \in T\}$ is a collection of open sets of C. Let us check that (a), (b) and (c) hold.

(a) Let $B_s \in \mathcal{B}_n$ with $s \in \prod_{m \leq n}$ and let $F \in [\pi \in T: B_s \not\subseteq O_\pi\}]^{\leq n+1}$. For each $n < m$, choose $a_m \in I_m - \{\pi(j): \pi \in F$ and $j \in \omega\}$. This is possible because $n + 1 < m + 1 = |I_m|$ and for each m and π there is at most one j with $\pi(j) \in I_m$. Define $f \in C$ by $f(m) = s(m)$ for $m \leq n$ and $f(m) = a_m$ for $n < m$. Then, $f \in B_s - \bigcup_{\pi \in F} O_\pi$.

(b) Let $f \in B_s$ and let F and G be two disjoint finite subsets of $\{\pi \in T: f \notin O_\pi\}$. We must show that $B_s \cap (\bigcap_{\pi \in F} O_\pi - \bigcup_{\pi \in G} O_\pi) \neq \phi$. Choose dom $s \leq k$ such that $\{\{\pi(n): k \leq n\}: \pi \in F \cup G\}$ is a disjoint family. This is possible because $\{\{\pi(n): n \in \omega\}: \pi \in T\}$ is an almost disjoint family. Let $\ell = \max\{\pi(k): \pi \in F \cup G\}$. Define $g \in C$ so that

$$g(m) = s(m) \text{ for } m \leq \ell$$
$$g(\pi(r)) = \pi(r + 1) \text{ for } \pi \in F \text{ and } \ell < \pi(r)$$
$$g(\pi(r)) \neq \pi(r + 1) \text{ for } \pi \in G \text{ and } \ell < \pi(r)$$
$$g(m) = \min I_m \text{ otherwise.}$$

$g \in B_s$ since dom $s \leq k \leq \ell$. By the definition of g and the O_π's and the facts that $G \subseteq \{\pi \in T: f \notin O_\pi\}$ and $s \subseteq f$, we have that $g \in \bigcap_{\pi \in F} O_\pi - \bigcup_{\pi \in G} O_\pi$ as well.

(c) Let $\{f_n: n \in \omega\} \subseteq C$. Define $\pi \in T$ by induction so that $\pi(0) \in I_0$ and $\pi(n + 1) = f_n(\pi(n))$ for each $n \in \omega$. Then, $\{f_n: n \in \omega\} \subseteq O_\pi \subsetneqq C$.

Now assume MAC and let $\{f_\alpha: \alpha < \kappa < c\} \subseteq C$. Define $P = \{p: p \in {}^n\omega$ for some $n \in \omega$, $p(0) \in I_0$, and $p(i + 1) \in I_{p(i)}$

for each $i \in \omega$ with $i + 1 \in \operatorname{dom} p\}$. Define $\leq$ on P by $p \leq q$ iff $q \subseteq p$. $(P,\leq)$ is a countable poset. For each $\alpha < \kappa$, define $D_\alpha = \{p \in P$: there exists $n + 1 \in \operatorname{dom} p$ with $f_\alpha(p(n)) = p(n + 1)\}$ and for each $n \in \omega$, define $E_n = \{p \in P: n \leq \operatorname{dom} p\}$. $\{D_\alpha: \alpha < \kappa\} \cup \{E_n: n \in \omega\}$ are $< c$ dense subsets of P. Let G be a generic subset of P meeting all of these dense sets. Let $\pi = \cup G$. Then $\pi \in T$ and O_π is our required member of $\mathcal{O}$. Let $\alpha < \kappa$. There exists $p \in G \cap D_\alpha$. There exists $n + 1 \in \operatorname{dom} p$ with $f_\alpha(p(n)) = p(n + 1)$. Thus, $\pi(n + 1) = p(n + 1) = f_\alpha(p(n)) = f_\alpha(\pi(n))$, and therefore $f_\alpha \in O_\pi$.

Example 3.2. A C-space Y that has a σ-n-linked base for all $n \in \omega$ and is the union of c dyadic subspaces.

Let C, $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$, $\mathcal{O}$ and $\{\mathcal{O}_f: f \in C\}$ be as in Lemma 3.1. Define $Y = \bigcup_{f\in C} (\{f\} \times 2^{\mathcal{O}_f})$. Remember that if $\mathcal{O}_f = \phi$, then $2^{\mathcal{O}_f} = \{\phi\}$. For each $B \in \mathcal{B}$, define $B^* = \{(f,p) \in Y: f \in B\}$ and for each $O \in \mathcal{O}$, define $O^{**} = \{(f,p) \in Y: (f \in O)$ or $(f \notin O$ and $p(O) = 1)\}$. Let $\mathcal{S} = \{B^*: B \in \mathcal{B}\} \cup \{O^{**}: O \in \mathcal{O}\} \cup \{Y - O^{**}: O \in \mathcal{O}\}$ serve as an open subbase for the topology on Y. Y has weight $\leq c$ and each $\{f\} \times 2^{\mathcal{O}_f}$ is homeomorphic to $2^{|\mathcal{O}_f|}$

A. Y *is compact*.

Let $\{B^*: B \in a \subseteq \mathcal{B}\} \cup \{O^{**}: O \in b \subseteq \mathcal{O}\} \cup \{Y - O^{**}: O \in d \subseteq \mathcal{O}\}$ be a cover of Y by subbasic open sets where we may assume that $b \cap d = \phi$. If $f \in C - (\bigcup_{B\in a} B \cup \bigcup_{O\in b} O)$, then we define $p \in 2^{\mathcal{O}_f}$, if $\mathcal{O}_f \neq \phi$, by $p(O) = 1$ iff $O \notin b$; if $\mathcal{O}_f = \phi$, then $b = \phi$ and we define $p = \phi$. In either case, we see that (f,p) is not contained in any of $\{B^*: B \in a\} \cup \{O^{**}: O \in b\} \cup \{Y - O^{**}: O \in d\}$. Consequently $C = \bigcup_{b \in a} B \cup \bigcup_{O \in b} O$, and as C is compact, we have finite subsets a' of a and b' of b such that $C = \bigcup_{B \in a'} B \cup \bigcup_{O \in b'} O$. Hence, $\{B^*: B \in a'\} \cup \{O^{**}: O \in b'\}$ is a finite subcover of Y.

B. Y *is* T_2.

Let (f,p) and (g,q) be two distinct points of Y. If $f \neq g$, choose disjoint sets U and V in $\mathcal{B}$ such that $f \in U$ and $g \in V$. Then $(f,p) \in U^*$, $(g,q) \in V^*$ and $U^* \cap V^* = \phi$. If $f = g$ and $p \neq q$, then $\mathcal{O}_f \neq \phi$ and we can choose $O \in \mathcal{O}_f$ such that $p(O) \neq q(O)$. Then O^{**} and $Y - O^{**}$ separate (f,p) and (f,q).

C. Y *is non-separable.*

Let $\{(f_n,p_n): n \in \omega\} \subseteq Y$. Choose $O \in \mathcal{O}$ with $\{f_n: n \in \omega\} \subseteq O \subsetneq C$. $Y - O^{**}$ is a nonempty open set disjoint from $\{(f_n,p_n): n \in \omega\}$.

D. Y *has a* σ-2-*linked base.*

For each $(a,b,d) \in [\mathcal{B}]^{<\omega} \times [\mathcal{O}]^{<\omega} \times [\mathcal{O}]^{<\omega}$, define $\langle a,b,d\rangle = \bigcap_{U \in a} U^* \cap \bigcap_{O \in b} O^{**} \cap \bigcap_{O \in d} Y - O^{**}$. For each $n \in \omega$ and $B \in \mathcal{B}_n$, define $\mathcal{A}(n,B) = \{\langle a,b,d\rangle: B \subseteq \bigcap_{U \in a} U \cap \bigcap_{O \in b} O$, $B - \bigcup_{O \in d} O \neq \phi$ and $|d| \leq \frac{1}{2}(n + 1)\} - \{\phi\}$.

(a) $\bigcup\{\mathcal{A}(n,B): n \in \omega$ and $B \in \mathcal{B}_n\}$ is a base for Y. To see this, let $(f,p) \in \langle a,b,d\rangle$. Therefore $b \cap d = \phi$. Now

$b = b_0 \cup b_1$ where $f \in \bigcap_{O \in b_0} O - \bigcup_{O \in b_1} O$. Since $\mathcal{O}_f$ is independently dense at f and $b_1 \cap d = \phi$, there exists $g \in \bigcap_{U \in a} U \cap \bigcap_{O \in b_0} O \cap (\bigcap_{O \in b_1} O - \bigcup_{O \in d} O)$. Choose $n \in \omega$ and $B \in \mathcal{B}_n$ such that $g \in B \subseteq \bigcap_{U \in a} U \cap \bigcap_{U \in b} O$ and $|d| \leq \frac{1}{2}(n + 1)$. Then $\langle a,b,d \rangle \in \mathcal{A}(n,B)$.

(b) Each $\mathcal{A}(n,B)$ is 2-linked. Choose $\langle a,b,d \rangle$, $\langle a',b',d' \rangle \in \mathcal{A}(n,B)$. Therefore, $d \cup d' \subseteq \{O \in \mathcal{O}: B \nsubseteq O\}$ and $|d \cup d'| \leq n + 1$. Thus, by (a) of Lemma 3.1, there exists $g \in B - \cup\{O: O \in d \cup d'\}$. Define $p \in 2^{\mathcal{O}_g}$, if $\mathcal{O}_g \neq \phi$, by $p(O) = o$ for all $O \in \mathcal{O}_g$; if $\mathcal{O}_g = \phi$, then $d \cup d' = \phi$ and define $p = \phi$. In either case, we have that $(g,p) \in \langle a,b,d \rangle \cap \langle a',b',d' \rangle$.

Replacing $\frac{1}{2}(n + 1)$ by $\frac{1}{j}(n + 1)$ gives us that Y also has a σ-j-linked base for any $j \in \omega$.

E. *For each* $(f,p) \in Y$, $\mathrm{char}((f,p),Y) \leq \omega \cdot |\mathcal{O}_f|$.

Define $\mathcal{S}_f = \{B^*: f \in B \in \mathcal{B}\} \cup \{O^{**}: f \notin O \in \mathcal{O}\} \cup \{Y - O^{**}: f \notin O \in \mathcal{O}\}$. $|\mathcal{S}_f| = \omega \cdot |\mathcal{O}_f|$. Each member of $\mathcal{S}$ containing (f,p) contains a member of $\mathcal{S}_f$ that contains (f,p) so $\mathcal{S}_f$ is a neighbourhood subbasis at (f,p) and our claim follows.

Example 3.3. (MAC) A C-space Z with a σ-n-linked base for all $n \in \omega$, of cardinality $\leq \sum_{k<\mathrm{cfc}} 2^k$ and for which each $p \in Z$ has $\mathrm{char}(p,Z) < \mathrm{cfc}$.

Let $C, \mathcal{B}, \mathcal{O}$ and $\{\mathcal{O}_f: f \in C\}$ be as in Lemma 3.1. Let $C = \bigcup_{\alpha<\mathrm{cfc}} F_\alpha$ where each $|F_\alpha| < c$. For each $\alpha < \mathrm{cfc}$, choose $O_\alpha \in \mathcal{O}$ such that $\bigcup_{\beta<\alpha} F_\beta \subseteq O_\alpha \subsetneqq C$. Define $\mathcal{O}' = \{O_\alpha: \alpha < \mathrm{cfc}\}$ and $\mathcal{O}'_f = \mathcal{O}_f \cap \mathcal{O}'$. For each $f \in C$ we have $|\mathcal{O}'_f| < \mathrm{cfc}$. C, $\mathcal{B}$ and $\mathcal{O}'$ satisfy (a), (b) and (c) of Lemma 3.1 except possibly for the furthermore part of (c). The space Z is to be constructed from C, $\mathcal{B}$ and $\mathcal{O}'$ in the same way that Y was constructed from C, $\mathcal{B}$ and $\mathcal{O}$. Z is compact, T_2, non-separable and has a σ-n-linked base for all $n \in \omega$. Each $p \in Z$ has $\mathrm{char}(p,Z) < \mathrm{cfc}$ according to E of Example 3.3. Also, $|Z| = \sum_{f\in C} 2^{|\mathcal{O}'_f|} \leq \sum_{\kappa<\mathrm{cfc}} 2^\kappa$.

Remark 3. If we assume CH or just MAC + cfc = ω_1, then Z is a 1^{st} countable C-space with a σ-n-linked base for all $n \in \omega$. By previous remarks, it follows that Z has infinite compactness number. This gives an answer, modulo CH or MAC + cfc = ω_1, to a question raised in [4] of whether there exists a 1^{st} countable compact space of infinite compactness number.

Remark 4. If we assume MAS, then Z is a compact ccc non-separable space of size c.

Remark 5. Assuming MA + $\neg$CH, see Juhász [10], every compact ccc space K with $[t(K)]^+ < c$ is separable. If we assume MA + $\neg$CH + c is a successor, then Z is a compact ccc space with char(K) < c that is non-separable. This shows that $[t(K)]^+ < c$ cannot be replaced by $t(K) < c$ in this theorem.

Remark 6. Both Y and Z are growths of ω in a natural way.

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