TOPOLOGY PROCEEDINGS
Volume 5, 1980
Pages 59–69
http://topology.auburn.edu/tp/

SEPARABILITY IN NEARLY COMPACT SYMMETRIZABLE SPACES

by

D. K. Burke and S. W. Davis

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

SEPARABILITY IN NEARLY COMPACT SYMMETRIZABLE SPACES

D. K. Burke and S. W. Davis

1. Introduction

In [SS] C. T. Scarborough and A. H. Stone defined a space to be *feebly compact* if every locally finite system of open sets is finite, or equivalently, if every countable open filterbase has a cluster point. It is known that every countably compact space is feebly compact and that every feebly compact space is pseudocompact (= all continuous real valued functions are bounded). Conversely, normal T_1 feebly compact spaces are countably compact, and completely regular T_1 pseudocompact spaces are feebly compact. For T_1 spaces which are only regular, however, the last two implications need not hold.

A space X is called *symmetrizable* provided that there is a function $d: X \times X \to [0,\infty)$ such that for each $(x,y) \in X \times X$, $d(x,y) = d(y,x)$, $d(x,y) = 0$ iff $x = y$, and $A \subseteq X$ is closed iff $d(x,A) > 0$ for each $x \in X \setminus A$. All Moore spaces and all semimetrizable spaces are symmetrizable, and all T_2 first countable symmetrizable spaces are semimetrizable.

Another base condition which we will have occasion to use is the notion of a weakly first countable space or equivalently a space satisfying the g-first axiom of countability [A_1]. A space X is called *weakly first* *countable* iff there is a function B: $X \times \omega \to \mathcal{P}(X)$ such that for each $x \in X$, $B(x,n+1) \subseteq B(x,n)$ for each $n \in \omega$ and $\bigcap_{n\in\omega} B(x,n) = \{x\}$, and $U \subseteq X$ is open iff for each $x \in U$ there exists $n_x \in \omega$ with $B(x,n_x) \subseteq U$. We call the function B a wfc-system for X. It is clear that the "symmetric balls" are a wfc-system for any symmetrizable space. It is also easy to show that any T_2 weakly first countable space is *sequential*, see [F] for definition.

It is shown in [N] that all countably compact symmetrizable T_2 spaces are compact and metrizable. Hence, for normal spaces we have that feebly compact symmetrizable spaces are compact and metrizable. Our purpose here is to explore this question for regular spaces. Exercise 5I of [GJ] provides an example of a feebly compact Moore space which is neither compact nor metrizable. It does retain the separability present in the earlier results. Furthermore, it is shown in [S_1] that all regular feebly compact semimetrizable spaces are separable, and the relations among Moore-closed, semimetric, and Moore spaces are discussed in the feebly compact setting. In the non-first countable case, the question of separability remains open.

In [S_2] a Hausdorff, non-regular example is given of a feebly compact Baire symmetrizable space which is not separable, and the following question is asked.

Question 1.1. [S_2] Is every regular, feebly compact symmetrizable space separable?

A negative answer to this question would also finally lay to rest Michael's question of whether points of a regular symmetrizable space must be G_δ-sets.

The results in this paper were obtained while attempting to solve 1.1. They involve a property which lies between feebly compact and countably compact. Stephenson calls this property e-countably compact.

Definition 1.2. A space X is called e-*countably compact* iff X has a dense conditionally compact subset.

Our main result is that it is consistent with the usual axioms for set theory that every regular, e-countably compact symmetrizable space is first countable and hence, by [S_1], separable.

In this paper we shall follow the custom that an ordinal is equal to the set of its predecessors, and cardinals are initial ordinals. We use MA to denote Martin's Axiom. (For a discussion of MA, the interested reader is referred to [R].) We shall assume that all spaces are at least T_1. The cardinality of the real numbers is denoted by c.

2. Preliminary Results

We now state without proof two results which we shall use later. Proofs may be found in the references given.

Theorem 2.1. [B], [R] (Booth) *Assume* MA. *If* $\mathcal{A}$ *is a family of subsets of* ω, $|\mathcal{A}| < c$, *and for* $\mathcal{B} \subseteq \mathcal{A}$ *with* $|\mathcal{B}| < \omega$ *we have* $|\cap\mathcal{B}| = \omega$, *then there exists* $D \subseteq \omega$ *such that*

$|D| = \omega$ *and* $|D - A| < \omega$ *for each* $A \in \mathcal{A}$.

The statement of Booth's Theorem is sometimes called P(c), and we shall refer to it as such. Being a consequence of MA, it is consistent with the usual axioms and also with CH.

Theorem 2.2. [A_2] (Arhangel'skiǐ) *If a compact Hausdorff space* X *is first countable and* $|X| > \omega$, *then* $|X| = c$.

3. The Main Result

We present the main result following a group of lemmas some of which use certain cardinal functions. Following Juhász we use $\chi(x,X)$ to denote the *character of* x *in* X, i.e.

$$\chi(x,X) = \min\{|\mathcal{B}|: \mathcal{B} \text{ is a local base at } x\}\cdot\omega$$

and $\psi(x,X)$ to denote the pseudocharacter of x in X, i.e.

$$\psi(x,X) = \min\{|\mathcal{U}|: \mathcal{U} \text{ is a collection open sets and } \{x\} = \cap\mathcal{U}\}\cdot\omega.$$

As usual, $\chi(X) = \sup\{\chi(x,X): x \in X\}$ and $\psi(X) = \sup\{\psi(x,X): x \in X\}$.

Lemma 3.1. [J] *If* X *is regular and separable, then* $\chi(X) \leq c$.

Proof. This is an immediate consequence of the fact that for a dense subset $D \subseteq X$ and open $W \subseteq X$ one has $\overline{W} = \overline{W \cap D}$.

The next lemma was proved in a somewhat less general setting in [S_3]. The proof is quite similar here in the more general case.

Lemma 3.2. *If* X *is a symmetrizable* T_2 *space,* κ *is a cardinal with* $cf(\kappa) > \omega$, *and* $x \in X$ *with* $\chi(x,X) \leqq \kappa$, *then* $\psi(x,X) < \kappa$.

Proof. Let d be a symmetric on X which is compatible with the topology. For $n \in \omega$ we denote by $B(x,n)$ the ball $\{y \in X: d(x,y) < \frac{1}{n}\}$. Suppose $\{V_\alpha: \alpha < \kappa\}$ is a local base at x, and, to arrive at a contradiction, $\psi(x,X) \geqq \kappa$.

Choose $x_0 \in V_0\backslash\{x\}$, W_0 open with $x_0 \in W_0$, $x \notin \overline{W}_0$. Suppose $\beta < \kappa$ and we have points $\{x_\alpha: \alpha < \beta\}$ and open sets $\{W_\alpha: \alpha < \beta\}$ such that for each $\gamma < \beta$ we have $x_\gamma \in \bigcap_{\alpha<\gamma} V_\alpha$, $x_\gamma \in W_\gamma$, $x_\gamma \notin \bigcup_{\alpha<\gamma} W_\alpha$, and $x \notin \overline{W}_\gamma$. Since $|\{V_\alpha: \alpha < \beta\} \cup \{X\backslash\overline{W}_\alpha: \alpha < \beta\}| < \kappa$, we may choose $x_\beta \in (X\backslash\{x\}) \cap (\bigcap_{\alpha<\beta} V_\alpha) \cap (\bigcap_{\alpha<\beta} X\backslash\overline{W}_\alpha)$ and choose W_β open with $x \notin \overline{W}_\beta$, and $x_\beta \in W_\beta$. Inductively, we define $\{x_\alpha: \alpha < \kappa\}$ and $\{W_\alpha: \alpha < \kappa\}$. For each $\alpha < \kappa$, choose $k_\alpha \in \omega$ with $B(x_\alpha,k_\alpha) \subseteq W_\alpha$. Since $cf(\kappa) > \omega$, we may choose $A \subseteq \{x_\alpha: \alpha < \kappa\}$ with $|A| = \kappa$ and $k_\alpha = n$ for all $x_\alpha \in A$. Note that A is cofinal in $\{x_\alpha: \alpha < \kappa\}$, so that $x \in \overline{A}\backslash A$.

Suppose $y \notin A$. If $y \notin \bigcup\{W_\alpha: x_\alpha \in A\}$, then $B(y,n) \cap A = \phi$. Otherwise, choose $\beta = \text{first } \{\alpha: x_\alpha \in A, y \in W_\alpha\}$. For $\alpha < \beta$, $x_\alpha \notin B(y,n)$. Choose $m \in \omega$ such that $B(y,m) \subseteq W_\beta$, then $x_\alpha \notin B(y,m)$ for $\alpha > \beta$. Since $y \neq x_\beta$, choose $k \in \omega$ such that $x_\beta \notin B(y,k)$. Let $N = \max\{k,m,n\}$, then $B(y,N) \cap A = \phi$.

Thus for any $y \notin A$, there exists $N \in \omega$ with $B(y,N) \cap A = \phi$, i.e. A is closed. However $x \in \overline{A}\setminus A$, a contradiction.

Corollary 3.2.1. *If* X *is* T_3, *separable and symmetrizable, then for each* $x \in X$, $\psi(x,X) < c$.

Proof. Since $cf(c) > \omega$, simply apply 3.1 and 3.2.

In what follows, we will use the notation $K_1(A) = \{x$: there is a sequence in A which converges to $x\}$. It is clear that $\overline{A} = K_1(A)$ for every $A \subseteq X$ iff X is Frechet. Moreover, if $K_\alpha(A)$ is defined in the obvious recursive way taking unions at limit ordinals, then $\overline{A} = K_{\omega_1}(A)$ for every $A \subseteq X$ iff X is sequential, [AF].

**Lemma* 3.3. *Assume* $P(c)$. *If* X *is sequential,* D *is a countable dense conditionally compact subset of* X, *and* $\{x\} = \bigcap_{\alpha<\beta} \overline{U}_\alpha = \bigcap_{\alpha<\beta} U_\alpha$ *where* U_α *is open,* $\beta < c$, *then* $x \in K_1(D)$.

Proof. Let $D = \{d_n: n \in \omega\}$. For each $\alpha < \beta$, let $F_\alpha = D \cap U_\alpha$, and $\omega(F_\alpha) = \{n: d_n \in F_\alpha\}$. Apply $P(c)$ to the sets $\{\omega(F_\alpha): \alpha < \beta\}$ to obtain an infinite set $F \subseteq D$ such that $F\setminus F_\alpha$ is finite for each $\alpha < \beta$. Now F has a cluster point and X is sequential, so there is a convergent sequence $\langle x_n: n \in \omega \rangle$ in F with infinite range, say $x_n \to y$. Since $F\setminus F_\alpha$ is finite for each α, we have $y \in \overline{F}_\alpha$ for each $\alpha < \beta$. Hence $y \in \bigcap_{\alpha<\beta} \overline{U}_\alpha = \{x\}$. Thus $x_n \to x$.

Lemma 3.4. *If* E *is a dense subspace of a weakly first countable* T_2 *space* X *and* $K_1(A) = \overline{A}$ *for every* $A \subseteq E$, *then for each* $x \in X$ *and* $n \in \omega$ *we have* $x \in \mathrm{Int}(\overline{B(x,n)})$, *where* B *is a* wfc-*system for* X.

*See note added in proof.

Proof. If $x \notin \mathrm{Int}(\overline{B(x,n)})$, then $x \in \overline{X \backslash \overline{B(x,n)}}$. Hence $x \in \mathrm{Cl}[(X \backslash \overline{B(x,n)}) \cap E] = K_1((X \backslash \overline{B(x,n)}) \cap E)$. Choose a sequence $\langle x_k : k \in \omega \rangle$ in $(X \backslash \overline{B(x,n)}) \cap E$ with $x_k \to x$, then $\langle x_k : k \in \omega \rangle$ must be eventually in $B(x,m)$ for every $m \in \omega$, in particular, $\langle x_k : k \in \omega \rangle$ is eventually in $B(x,n)$ which is impossible.

**Theorem* 3.5. *Assume* P(c). *If* X *is sequential and regular,* E *is a dense conditionally compact subset of* X, *and* $\psi(x,X) < c$ *for every* $x \in X$, *then for each* $A \subseteq E$, $K_1(A) = \overline{A}$.

Proof. Suppose $A \subseteq E$ and $x \in K_2(A)$. Choose a sequence $\langle x_n : n \in \omega \rangle$ in $K_1(A)$ with $x_n \to x$, and for each $n \in \omega$ choose a sequence $\langle x_{n,k} : k \in \omega \rangle$ in A with $x_{n,k} \to x_n$. Let $S = \{x_{n,k} : n \in \omega, k \in \omega\}$. Since X is regular, $\overline{S}$ is regular, and since $\psi(x,X) < c$, there exist $\overline{S}$-open sets $\{U_\alpha : \alpha < \beta\}$ for some $\beta < c$ with $\{x\} = \bigcap_{\alpha<\beta} \overline{U}_\alpha = \bigcap_{\alpha<\beta} U_\alpha$. Moreover, since X is sequential, $\overline{S}$ is sequential, and S is conditionally compact in $\overline{S}$. Now by 3.3, $x \in K_1(S) \subseteq K_1(A)$. Now since $K_2(A) = K_1(A)$, we have that $K_1(A) = \overline{A}$.

4. Applications

In this section we present several applications of Theorem 3.5 and certain of the lemmas given in section 3. In particular, we give a consistent solution to the problem of whether e-countably compact symmetrizable spaces must be separable.

**Theorem* 4.1. *Assume* P(c). *If* X *is a regular, symmetrizable,* e*-countably compact space, then* X *is first countable (in fact, a Moore space).*

Proof. Let E be a dense conditionally compact subset of X, and let $A \subseteq E$. As in 3.5, construct S in an attempt to show $K_2(A) = K_1(A)$. Now $\overline{S}$ is a regular symmetrizable space with a countable dense conditionally compact subset S. Thus by 3.2.1, $\psi(x,\overline{S}) < c$ for every $x \in \overline{S}$. Now apply 3.5 in the space $\overline{S}$, and since $\overline{S}$ is closed, we have $K_2(A) = K_1(A)$ in X. Now by 3.4, since X is T_2, each point of X is a G_δ-set. Moreover, if a point is a G_δ-set in a regular feebly compact space, it has countable character. Thus X is first countable.

**Corollary* 4.1.1. *Assume* P(c). *If* X *is a regular, symmetrizable,* e*-countably compact space, then* X *is separable.*

Proof. This is proved for first countable spaces in [S_1] and 4.1 shows that such a space must be first countable.

**Theorem* 4.2. *Assume* P(c). *If* X *is* T_2*, compact, weakly first countable and* $\omega \leq |X| \leq c$*, then* $|X| \in \{\omega, c\}$.

Proof. Suppose $\omega < |X| < c$. Note that X is sequential, regular, $\psi(x) < c$, and by 3.5, $K_1(A) = \overline{A}$ for every $A \subseteq X$. Now by 3.4 and the regularity, $\{\overline{B(x,n)}: n \in \omega\}$ is a neighborhood base at x for each $x \in X$. Hence X is first countable and by 2.2, we have a contradiction.

Remark 4.2.1. In [M], Malýhin has announced that it is consistent that there exist T_2 compact weakly first countable spaces with cardinality lying strictly between ω and c. We have just established the independence of the existence of such spaces.

An examination of the proof of 4.2 leads to the observation that the following is proved.

**Corollary* 4.2.2. *Assume* $P(c)$. *If* X *is* T_3, *countably compact, weakly first countable and* $\psi(x,X) < c$ *for each* $x \in X$, *then* X *is first countable.*

5. Feeble Compactness

In this final section, we give a result which is a first step toward the general problem 1.1.

**Theorem* 5.1. *Assume* $P(c)$. *If* X *is regular, feebly compact, sequential,* $\psi(x,X) < c$ *for every* $x \in X$, *and there is a countable subset* $M \subseteq X$ *with* $\chi(x,X) = \omega$ *for every* $x \in M$, *then* $\overline{M} = K_1(M)$.

Proof. Let $M = \{x_i: i \in \omega\}$ and for each $i \in \omega$ choose an open local base $\{U_n(x_i): n \in \omega\}$ at x_i. Suppose $x \in \overline{M}$, and $\{V_\alpha: \alpha < \beta\}$ is a collection of open sets with $\beta < c$ and $\{x\} = \bigcap_{\alpha<\beta} \overline{V}_\alpha = \bigcap_{\alpha<\beta} V_\alpha$. Index $\omega \times \omega$ by ω and apply $P(c)$ to the sets $F_\alpha = \{(n,i): U_n(x_i) \subseteq V_\alpha\}$ to obtain an infinite set $\mathcal{H} \subseteq \{U_n(x_i): n,i \in \omega\}$ such that for each $\alpha < \beta$, $|\{U \in \mathcal{H}: U \not\subseteq V_\alpha\}| < \omega$. Let $\mathcal{F}$ be the set of unions of co-finite subcollections of $\mathcal{H}$. Now $\mathcal{F}$ is a countable open filterbase; hence $\mathcal{F}$ must have a cluster point. Moreover, if $\mathcal{G}$ is a finer filter than $\mathcal{F}$, we have $\cap\{\overline{G}: G \in \mathcal{G}\} \subseteq \cap\{\overline{F}: F \in \mathcal{F}\} \subseteq \bigcap_{\alpha<\beta} \overline{V}_\alpha = \{x\}$. Now if there is an open set U with $x \in U$ and U contains no member of $\mathcal{F}$, then we choose W open with $x \in W \subseteq \overline{W} \subseteq U$ and $\{F\backslash\overline{W}: F \in \mathcal{F}\}$ is a countable open filterbase which is finer than $\mathcal{F}$, but cannot cluster at x, and hence it cannot cluster at all, a contradiction. (Note that this proves that in a regular feebly compact space, any countable open filterbase with a unique cluster point converges, a fact which may be known to some readers.) Thus $\mathcal{F} \to x$, and so the sequence $\langle y_k: k \in \omega\rangle$, where $\langle F_k: k \in \omega\rangle = \mathcal{F}$ and $y_k \in \bigcap_{i=1}^{k} (M \cap F_i)$, converges to x. Hence $x \in K_1(M)$. Thus $\overline{M} \subseteq K_1(M)$ and clearly $K_1(M) \subseteq \overline{M}$, and the proof is complete.

Remark 5.1.1. The restriction that M be countable is unnecessary in 5.1, since by assuming $x \in K_2(M)$ we obtain a countable set $S \subseteq M$ as in 3.5 with $x \in \overline{S}$, then $x \in K_1(S)$ by 5.1 and so $x \in K_1(M)$.

*Added in Proof: The authors have recently shown that all the results using P(c) which are contained in this paper may be obtained using BF(c).

References

[A_1] A. V. Arhangel'skiǐ, *Mappings and spaces*, Russian Math. Surveys 21 (1966), 115-162.

[A_2] ________, *On the cardinality of bicompacta satisfying the first axiom of countability*, Soviet Math. Dokl. 10 (1969), 951-955.

[AF] ________ and S. Franklin, *Ordinal invariants for topological spaces*, Michigan Math. J. 15 (1968), 313-320.

[B] D. Booth, *Countably indexed ultrafilters*, Ph.D. Thesis, University of Wisconsin, Madison, Wisconsin, 1969.

[F] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. 57 (1965), 107-115.

[GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Graduate Texts in Mathematics, Springer-Verlag, New York, Heidelberg, Berlin, 1976.

[J] I. Juhász, *Cardinal functions in topology*, Math. Centre Tracts, 34, Mathematical Centre, Amsterdam, 1971.

[M] V. I. Malýhin, *On the power of weakly first countable bicompacta*, Abstracts AMS 1 (1980), 247.

[N] S. Nedev, *Symmetrizable spaces and final compactness*, Soviet Math. Dokl. 8 (1967), 890-892.

[R] M. E. Rudin, *Lectures on Set Theoretic Topology*, AMS Regional Conference Series in Mathematics, No. 23, 1975.

[SS] C. T. Scarborough and A. H. Stone, *Products of nearly compact spaces*, Trans. Amer. Math. Soc. 124 (1966), 131-147.

[S_1] R. M. Stephenson, Jr., *Symmetrizable-closed spaces*, Pacific J. Math. 68 (1977), 507-514.

[S_2] ________, *Near compactness and separability of symmetrizable spaces*, Proc. Amer. Math. Soc. 68 (1978), 108-110.

[S_3] ________, *Symmetrizable, $\mathcal{F}$-, and weakly first countable spaces*, Canadian J. Math. 29 (1977), 480-488.

Miami University

Oxford, Ohio 45056