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WEAKLY FLAT CURVES

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At the Oklahoma State University Topology conference in October (78), Daverman asked some questions concerning weakly flat curves. The following theorem gives an answer to one of those questions.

Theorem. Suppose s is a simple closed curve in E^3 which bounds a cellular disk D_s which is locally flat mod s , and further that each point of s is a piercing point for D_s . Then s is weakly flat.

Proof. We will prove this by applying the Semicullarity Theorem ([5]) to show that s is toroidal. The defining sequence of tori will clearly be concentric and unknotted ([4]) and this will be sufficient to prove the theorem. The first step is to fatten D_s into a 3-cell K with D_s as the equatorial plane and with $Bd K$ locally flat mod s . K is then considered to be the union of two 3-cells K_1 and K_2 which intersect only in D_s . (All of this is possible since D_s is locally flat at interior points.) Squeeze D_s ([2]) to an arc $\alpha \equiv [0,1]$. Let $K'_i = K_i/D_s \approx \alpha$, $i = 1,2$. Note that by the Semicullarity Theorem (S.C.T.), each point of $Bd K'_1$ and $Bd K'_2$ is a piercing point. Therefore, we may choose tame arcs b_1, b'_1 and b_2, b'_2 on $Bd K'_1$ and $Bd K'_2$, respectively, with the following properties:

- (1) $b_1 \cap \alpha = 0 = b_2 \cap \alpha$ and $b'_1 \cap \alpha = 1 = b'_2 \cap \alpha$.
- (2) $b_i \cup b'_i$ is a tame arc on K'_i , $i = 1,2$.

Since the complement does not change when D_s is squeezed to α , it is clear that α is cellular and that we could easily obtain an annulus A (or even a sequence of annuli) with the boundary components a_1 and a_2 in $\text{Bd } K'_1$ and $\text{Bd } K'_2$, respectively, if we could only find a defining sequence with boundary spheres $\{S_i\}$ for α , and hence D_s , which hit b_1, b'_1, b_2 and b'_2 in one point each. The S_i are to be constructed by the same techniques used in the proof of the S.C.T. and such that the A_i 's will have the following properties:

(1) The A_i 's are disjoint.

(2) Except for some subdisks which may be altered by cutting and pasting, $A_i \subset S_i, \forall i$

(3) If T is the solid torus whose boundary is composed of A and the annulus on K which has a_1 and a_2 for its boundary, then the intersection of the T 's is s .

(4) $\text{Int } A_i \subset \text{Ext } K$.

We now provide the essential details for obtaining the S_i 's.

There exists a defining sequence $\{\Sigma_n\}$ for α with the property that each sphere in the sequence intersects b_1 and b'_1 in one point each (since by S.C.T., the image of $b_1 \cup b'_1$ is tame in E^3/α). Furthermore, the Σ_n may be chosen to intersect $\text{Bd } K'_1$ and $\text{Bd } K'_2$ only in circles which link α/α , i.e., the "small" disks on $\text{Bd}(K'_1 \cup K'_2)$ which are bounded by these circles contain α/α in their interiors. Also, we shall show that $b'_1 \cup b'_2$ is tame so that it has a penetration index ([1]) of two at 1, and thus, there is

a defining sequence of spheres for l with the property that each sphere intersects b'_1 and b'_2 in one point each. To see that $b'_1 \cup b'_2$ is tame we need only to consider the simple closed curve $b = b_1 \cup b_2 \cup b'_1 \cup b'_2$ on $K' \cong K/D_S \approx \alpha$. b bounds a disk D_b on K' which is locally tame everywhere except on α . We may approximate $\text{Int } D_b$ so that b becomes the boundary of a 2-cell which is locally tame mod b or in fact, locally tame mod $b_1 \cup b'_1$. Thus by Theorem 1 of [3], we may conclude that b , and hence, $b'_1 \cup b'_2$ is tame.

To complete the argument we choose a defining sequence S_i for l with the following properties (see [5], page 11):

- (1) Each S_i is the union of two tame disks D_0^i and D_1^i .
- (2) $\text{Int } D_0^i \subset \text{Int } K'_1$ and $D_0^i \cap \alpha$ is a single point d_i .
- (3) $D_1^i \cap \text{Bd } K'_1$ is the union of a null sequence of tame simple closed curves $\{C_j^i\}_{j=1}^\infty$ and a totally disconnected set C^i .

(4) All circles of intersection in $D_1^i \cap \text{Bd } K'_1$ hit α (for each i) and $C^i \subset \alpha$.

(5) The S_i 's are pairwise disjoint and only one component of $\alpha - S_i - S_j$, $i \neq j$, hits both spheres.

(6) Each S_i hits $b'_1 \cup b'_2$ in exactly two points.

Order the S_i in the obvious manner (from larger to smaller) and assume that S_1 is small enough that $0 \in \text{Ext } S_1$. Let q_1 and q'_1 denote the points in $\alpha \cap S_1$ with property that $[q_1, 1] \cap S_1 = q_1$ and $[0, q'_1] \cap S_1 = q'_1$. Then a new sequence is chosen by choosing S_2 so that it hits α inside $(q_1, 1)$ only, i.e. $[0, q_1] \subset \text{Ext } S_2$. By induction, we choose q_1, q'_1 for all i .

Let $\epsilon > 0$ be given and much less than $\text{diam}(\alpha)$. We will construct a 3-cell which contains α and lies within ϵ of α and whose boundary intersects $b_1 \cup b_2$ and $b'_1 \cup b'_2$ twice each. To do this, let a be a disk on $\text{Bd } K'_1$ which lies within $N_\epsilon(\alpha)$ and contains α in its interior. Choose yet another subsequence if necessary so that S_1 is an ϵ -sphere within $N_\epsilon(\alpha)$ and $S_1 \cap \text{Bd } K'_1 \subset a$. Consider S_i , $i = 1, \dots, 6$. Choose $q \in (q_6, 1)$ and $\{\Sigma^j\}$, a defining sequence for $[0, q]$. Notice that if we collapse $[0, q]$ to a point $b_1 \cup b_2$ remains a tame arc by the same proof we first used to show that b and $b'_1 \cup b'_2$ are tame. (Our proof will show that $b_1 \cup b_2$ is also tame if we collapse α to a point. However, this does not follow from the previous argument since $D_{b/\alpha}$ is not a disk.) Thus we may choose the defining sequence $\{\Sigma^j\}$ so that each element hits $b_1 \cup b_2$ exactly twice. Then Σ^1 is selected small enough for the following conditions to hold.

- (1) $\Sigma^1 \subset N_\epsilon(\alpha)$.
- (2) $\Sigma^1 \cap (\alpha \cup b_1 \cup b_2) \subset (q_6, 1)$ except for $\Sigma^1 \cap (b_1 \cup b_2)$.
- (3) $\Sigma^1 \cap \text{Bd } K'_1 \subset a$.

We next choose N so large that, if $i \geq N$, Σ_i satisfies several conditions--the first being: Σ_i intersects $b'_1 \cup b'_2$ inside S_6 and $b_1 \cup b_2$ inside Σ^1 . Secondly, Σ_i will be such that $\Sigma_i \subset N_\epsilon(\alpha)$ and $\Sigma_i \subset \text{Bd } K'_1 \subset a$. Let $\{B^i\}_{i=2}^6$ be disks on $\bigcup_{i=2}^6 S_i$ with the following properties:

- (1) Each B^i is the union of two disks B_0^i and B_1^i where $B_0^i \subset D_0^i$ and $B_1^i \subset D_1^i$ and $B_0^i \cap \text{Bd } D_0^i = B_1^i \cap \text{Bd } D_0^i = B_1^i \cap \text{Bd } D_1^i$.

$$(2) \text{Bd}(B_1^i) - B_0^i \subset \text{Ext Bd } K_1^i.$$

$$(3) \text{Int } B^i \text{ contains } d_i.$$

$$(4) (UB^i) \subset \text{Int } \Sigma^1.$$

Note that this requires $\text{Bd } B^i \cap \alpha = \emptyset$. Finally, N is chosen so that for $j \geq N$, $\Sigma_j \cap D_0^i \subset \text{Int } B_0^i$ and $\text{Bd } B^i \cap (3\text{-cell bounded by } \Sigma_j) = \emptyset$. $i = 2, \dots, 6$. We assume the following claim (see [5], claim II).

Claim. $\Sigma_n \cap \text{Bd } K_1^i$ may be assumed to be a finite collection of disjoint circles which bound disjoint disks E_j , $j = 1, \dots, m$ on Σ_n with $\text{Int } E_j \subset \text{Int } \text{Bd } K_1^i$ and which link α on $\text{Bd } K_1^i$, i.e., the small disks on α bounded by $\{\text{Bd } E_j\}_{j=1}^m$ contain α . Further, these circles hit $b_1 \cup b_1'$ and $\text{Bd } D_0^i$, $i = 2, \dots, 6$, in two points each. Also $S_i \cap E_j$ is a simple arc $e_{i,j}$ for each i , $i = 2, \dots, 6$, and j , $j = 1, \dots, m$.

Now we must adjust $\Sigma_n \cap (\cup_{i=2}^6 S_i)$ in several special cases with all necessary adjustments being made within $N_\varepsilon(\alpha)$. We will assume that $\Sigma_n \cap (\cup_2^6 S_i)$ is a finite collection of disjoint circles since each sphere is tame. Let $G \subset \Sigma_n \cap S_4$ be an innermost circle on Σ_n . Let $\Sigma_n(G)$ denote the innermost disk it bounds on Σ_n and $S_4(G)$ the disk it bounds on $S_4 - s_4$. The adjustments are made in exactly the same manner as before ([5]), and will show the existence of a disk $H \subset \Sigma_n$ which hits either $b_1 \cup b_2$, or $b_1' \cup b_2'$ in exactly two points. H is then used along with either Σ^1 or S_1 , depending on whether H comes from Case I or Case II ([5]), to construct a sphere which contains α and hits $b_1 \cup b_2 \cup b_1' \cup b_2'$ in exactly four points. As stated before

this sphere shows the existence of the desired annulus outside K . Since K is a 3-cell, a similar annulus would exist inside K . Furthermore, the annulus inside K could be chosen to have the same boundary components a_1 and a_2 as A_1 , the one on the outside. The union of these two annuli would yield a torus. In like manner, a sequence of pairs of annuli and therefore a sequence of tori could be chosen. By construction, it is obvious that the sequence of tori could be chosen so that their intersection is s and so that they are concentric. This completes the proof.

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