TOPOLOGY PROCEEDINGS

Volume 5, 1980

Pages 105–110

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN:	0146-4124

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1. Introduction

There is an extensive mathematical literature devoted to the problem of extending a continuous function to a larger space. In this note we consider the problem of extending continuous, increasing functions. We consider functions between two ordered topological spaces in the sense of Nachbin [4]. Nachbin [4] proves a theorem concerning extensions of continuous, increasing functions which is analogous to the Tietze extension theorem. He also constructs a compactification of an ordered topological space which is analogous to the Stone-Cech compactification of a topological space and characterizes this compactification by an extension property involving continuous, increasing functions. Our main result is a generalization of a theorem due to Taimanov [7]. We apply this result to ordered topological spaces which are determined by quasi-proximities. We show that the qp-continuous functions are the only continuous, increasing functions from such a space into a compact ordered topological space that have a continuous, increasing extension to the associated order compactification [1], [3]. This is a generalization of a theorem of Smirnov [6, Theorem 12]. The author is indebted to W. F. Lindgren for suggestions leading to an improvement of an earlier version of this article. For further information on ordered topological spaces and quasi-proximities, the reader is referred to [2] and [4].

2. The Main Result

An ordered topological space is a triple $(X,\mathcal{J},\leq)$ where $(X,\mathcal{J})$ is a topological space, $\leq$ is a partial order on X, and $\{(x,y): x \leq y\}$ is closed in $X \times X$. A subset S of X is said to be increasing if $x \in S$ whenever $x \geq a$ for some $a \in S$. S is decreasing if $X - S$ is increasing. Let $A \subset X$. $Cl_iA = \cap\{H: H$ is closed, increasing, $A \subset H\}$, Cl_dA is defined analogously. Cl_i and Cl_d both define Kuratouski closure operators on X.

Theorem (2.1). *Let* $(X,\mathcal{J},\leq)$ *be an ordered topological space, and let* D *be a dense subset of* X. *Let* Y *be a compact ordered topological space, and let* $f: D \to Y$ *be continuous and increasing. Then* f *has a continuous, increasing extension* $F: X \to Y$ *if and only if* $Cl_if^{-1}(A) \cap Cl_df^{-1}(B) = \emptyset$ *whenever* $Cl_iA \cap Cl_dB = \emptyset$, *where the former closures are taken in* X.

Proof. Let $F: X \to Y$ be a continuous, increasing extension of f. Suppose that A and B are subsets of Y and $Cl_iA \cap Cl_dB = \emptyset$. From

$$Cl_iF^{-1}(A) \subset F^{-1}(Cl_iA), \text{ and}$$
$$Cl_dF^{-1}(B) \subset F^{-1}(Cl_dB),$$

it follows that $Cl_if^{-1}(A) \cap Cl_df^{-1}(B) = \emptyset$.

Conversely, let $x \in X$ and let $\mathcal{N}_x$ be the collection of all neighborhoods of x. Let $\mathcal{F}(x) = \{Cl_i f(D \cap W): W \in \mathcal{N}_x\} \cup \{Cl_d f(D \cap W): W \in \mathcal{N}_x\}$. Since Y is compact, $\cap\mathcal{F}(x) \neq \emptyset$. We now show that $\cap\mathcal{F}(x)$ is a singleton. Suppose $y_1, y_2 \in \cap\mathcal{F}(x)$, $y_1 \neq y_2$. Without loss of generality we may assume that $Cl_d\{y_1\} \cap Cl_i\{y_2\} = \emptyset$. Since Y is a normally ordered space, there exists an open decreasing set G_1 and an open increasing set G_2 such that $Cl_d\{y_1\} \subset G_1$, $Cl_d\{y_2\} \subset G_2$, and $Cl_d G_1 \cap Cl_i G_2 = \emptyset$. Clearly, $Cl_d f^{-1}(G_1) \cap Cl_i f^{-1}(G_2) = \emptyset$. Put

$$U_1 = X - Cl_d f^{-1}(G_1), \text{ and}$$
$$U_2 = X - Cl_i f^{-1}(G_2).$$

Then $x \in U_{k_0}$, $k_0 = 1$ or 2, say $k_0 = 1$. Clearly $G_1 \cap f(D - Cl_d f^{-1}(G_1)) = \emptyset$. Since G_1 is open, decreasing, we have $G_1 \cap Cl_i(f(D - Cl_d f^{-1}(G_1))) = \emptyset$. Therefore $y_1 \notin Cl_i f(D - Cl_d f^{-1}(G_1))$ and since $Cl_i f(D - Cl_d f^{-1}(G_1)) = Cl_i f(D \cap U_1)$ we have $y_1 \notin \cap\mathcal{F}(x)$, a contradiction. Define $F(x) = \cap\mathcal{F}(x)$. Clearly F is an extension of f. We shall now demonstrate that F is continuous. Let V be a neighborhood of F(x) in Y. Since each set of the form $Cl_i f(D \cap W)$ and $Cl_d f(D \cap W)$ is compact, there exists a set $U \in \mathcal{N}_x$ such that

$$Cl_i f(D \cap U) \cap Cl_d f(D \cap U) \subset V.$$

If $y \in U$, then

$$F(y) \in Cl_i f(D \cap U) \cap Cl_d f(D \cap U) \subset V.$$

Hence $F(U) \subset V$. It remains to show that F is increasing. Suppose $F(x) \leq F(y)$ is false. Put $A = \{F(x)\}$ and $B = \{F(y)\}$

Then $Cl_i A \cap Cl_d B = \emptyset$. Since Y is a normally ordered space [4] there exist an open increasing set G and an open decreasing set H such that $Cl_i A \subset G$, $Cl_d B \subset H$ and $Cl_i G \cap Cl_d H = \emptyset$. By hypothesis, $Cl_i f^{-1}(G) \cap Cl_d f^{-1}(H) = \emptyset$. Suppose $x \notin Cl_i f^{-1}(G)$; then $W = X - Cl_i f^{-1}(G) \in \mathcal{N}_x$ and $f(D \cap W) \cap G \neq 0$ so that $F(x) \in Cl_d f(D \cap W) \subset X - G$, a contradiction. Similarly $y \in Cl_d f^{-1}(H)$ so $x \leq y$ is false. This completes the proof.

3. Order Compactifications

A quasi-proximity is a binary relation on $\mathcal{P}(X)$ satisfying most of the axioms used to define a proximity. The only difference is that a quasi-proximity is not assumed to be symmetric. For further information on quasi-proximities, the reader is referred to [1], [2], and [5].

A quasi-proximity δ determines an ordered topological space $(X,\mathcal{J},\leq)$ if the proximity $\delta \vee \delta^{-1}$ is compatible with $(X,\mathcal{J})$ and $\{x\}\delta\{y\}$ if and only if $x \leq y$. In [1] the authors prove that a compact ordered topological space is determined by a unique quasi-proximity. This quasi-proximity is defined by: $A\delta_o B$ if and only if $Cl_i A \cap Cl_d B \neq \emptyset$. For the remaining part of this paper, δ_o will always denote this quasi-proximity. An ordered topological space $(\tilde{X},\tilde{\mathcal{J}},\tilde{\leq})$ is an order compactification of $(X,\mathcal{J},\leq)$, if $(X,\mathcal{J})$ is a dense subspace of the compact space $(\tilde{X},\tilde{\mathcal{J}})$, and the restriction of $\tilde{\leq}$ to $X \times X$ is $\leq$.

Theorem (3.1) [1, Theorem 5.16]. *Let* X *be an ordered topological space.*

(i) X *has an order compactification if and only if it is determined by a quasi-proximity.*

(ii) If δ *determines* X, *then there is an order compactification* $\tilde{X}$ *of* X *such that* δ *is the restriction to* $X \times X$ *of the quasi-proximity* δ_o *on* $\tilde{X}$.

(iii) Two order compactifications of X *are equivalent if and only if they have the same associated quasi-proximity.*

Proposition (3.2). *Let* $(X,\mathcal{J},\leq)$ *be determined by the quasi-proximity* δ_o *and let* $f\colon X \to Y$, *where* Y *is a compact ordered topological space. Then* f *is continuous and increasing if and only if* f *is* qp-*continuous.*

Theorem (3.3). *Let* X *be an ordered topological space determined by a quasi-proximity* δ. *Let* $\tilde{X}$ *be the order compactification corresponding to* δ, *and let* Y *be any compact ordered topological space. Then a continuous, increasing function* $f\colon X \to Y$ *has a continuous, increasing extension* $F\colon \tilde{X} \to Y$ *if and only if* f *is* qp-*continuous.*

Proof. Suppose A, B are subsets of Y such that $Cl_i A \cap Cl_d B = \emptyset$, then $A\bar{\delta}_o B$ and since f is qp-continuous we have $f^{-1}(A)\bar{\delta}f^{-1}(B)$. Since (X,δ) is a subspace of $(\tilde{X},\delta_o)$, we have $f^{-1}(A)\bar{\delta}_o f^{-1}(B)$ and consequently $Cl_i f^{-1}(A) \cap Cl_d f^{-1}(B) = \emptyset$, where the closures are taken in $\tilde{X}$. The result now follows from Theorem 2.1.

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