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ON QUASI-METRIZABILITY

by

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Jacob Kofner

0. Introduction

Quasi-metrics have been known for quite a long time as generalized metrics d satisfying the triangle inequality, $d(x,z) \leq d(x,y) + d(y,z)$, but generally not the symmetry axiom, $d(x,y) = d(y,x)$. Naturally, early results in quasi-metric spaces were motivated by the metrization theory (see Theorems 3 and 4). Nowadays quasi-metrizability theory has its own group of problems some of which influence topology outside the theory of generalized metric spaces (see Theorems 19, 22).

In the present paper some old and new problems in the theory of quasi-metric spaces will be discussed and some old and new results will be exhibited. While the paper encompasses work of several people, it certainly is not all-inclusive and probably fits best the interests of the author.

In order to remain in the traditional framework of topological spaces, other topological structures are avoided whenever possible. In particular, in this paper "a quasi-metric space" means just a topological space which has a compatible quasi-metric, and so there is no distinction between quasi-metric spaces and quasi-metrizable spaces. Although there is no mention in this paper of quasi-uniformities and bitopologies, it is well worth pursuing

the relations between these structures and quasi-metrics in a book to appear by P. Fletcher and W. F. Lindgren [FL4] and in a coming work by R. Fox [Fo1].

The discussion is concentrated mainly around the γ-space problem, which seems to be the most important in quasi-metrizability theory. Section 1 is devoted to general problems and results on quasi-metric spaces, non-archimedean quasi-metric spaces and γ-spaces. Section 2 treats some classes of spaces in which the γ-space problem has an affirmative solution, namely ordered spaces, spaces with ortho-bases and developable spaces. The exposition of Sections 1 and 2 uses only the "official" definitions. Technical details are postponed until Section 3. In that section the definitions used earlier are translated into the language of neighbornets, and the quasi-metrizability theory turns into a theory dealing with collections of neighborhoods and binary relations rather than with generalized metrics. Some problems in quasi-metric spaces are related to transitivity, a quite general paracompactness-like property of topological spaces.

Throughout the paper, "space" means "T_1 topological space" and "map" means "continuous surjection."

1. Quasi-Metrics and Related Concepts

(a) Quasi-metrics and non-archimedean quasi-metrics.

Definition 1. Let X be a space. A non-negative function $d: X \times X \to R$ is called here *a generalized metric* provided that for each $x \in X$ *the spheres*, i.e. the sets

σ-interior-preserving base is sufficient for quasi-metrizability of a space and have asked whether this condition is necessary. An affirmative answer would have provided a criterion for quasi-metrizability analogous to one for metrizability; however, the following counter-example has been found [K2].

Example 1. *The space* T. The space is the plane retopologized by letting all open circles along with their "southern" poles be a base. The function

$$d(x,y) = \begin{cases} 0, \text{ when } y = x \\ r, \text{ when } y \text{ the circumference of radius } r \leq 1 \text{ with the "southern pole" } x \\ 1, \text{ otherwise,} \end{cases}$$

is a quasi-metric. The space T is orthocompact, i.e. each open cover of T has an open interior preserving refinement. However, T has no σ-interior preserving base [K2,K4].

Theorem 1. *For a space* X *the following are equivalent.*

(i) X *has a* σ*-interior preserving base,*

(ii) there is a quasi-metric d *on* X *with* $d(x,z) \leq \max\{d(x,y),d(y,z)\}$ [K2].

Definition 3. A quasi-metric d with $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ is called *non-archimedean* and the respective space is called here *non-archimedean quasi-metrizable* or *non-archimedean quasi-metric* [FL1].

Evidently, the class of non-archimedean quasi-metric spaces, like that of quasi-metric spaces, is countably

$d_{S(x,\varepsilon)} = \{y | d(x,y) < \varepsilon\}$, $\varepsilon > 0$, form a base of neighborhoods of x in X. A generalized metric is called *quasi-metric* (*semi-metric*) provided that $d(x,z) \leq d(z,y) + d(y,z)$ ($d(x,y) = d(y,z)$). A space with a quasi-metric (semi-metric) is called here *quasi-metrizable* or *quasi-metric* (*semi-metrizable* or *semi-metric*).

Although spheres of a generalized metric need not be open sets, those of a quasi-metric are always open. Evidently, the class of quasi-metric spaces is countably multiplicative and hereditary.

One of the best known quasi-metric spaces is the *Sorgenfrey line* Z, i.e. the reals retopologized by letting all right open intervals [a,b[be a base. The function

$$d(x,y) = \begin{cases} y - x, \text{ when } y \geq x \\ 1 \quad , \text{ otherwise,} \end{cases}$$

is obviously a quasi-metric on Z. Note that the intersection of any subcollection of $B_r = \{[x,r[| x < r\}$ is an open set in Z and $\cup\{B_r | r \text{ is rational}\}$ is a base in Z (cf. [S]).

Definition 2. A collection of open sets is called *interior-preserving* provided that the intersection of any subcollection is open and σ-*interior preserving*, provided that it is a countable union of interior preserving collections. A space is called (σ-) *orthocompact* if each open cover has an open (σ-) interior preserving refinement.

D. Doichinov and S. Nedev [N] and P. Fletcher and W. F. Lindgren [FL1] have found that the existence of a

multiplicative and hereditary. The condition of being non-archimedean seems to be quite a restrictive one for a quasi-metric and the existence of a quasi-metric space which is not non-archimedean quasi-metrizable may not be surprising. It is surprising, however, that almost all known quasi-metric spaces have a σ-interior preserving base (cf. Theorems 9-11). In fact, the only available regular quasi-metric spaces which are not non-archimedean quasi-metrizable are the space T and spaces built upon T. It would therefore be quite interesting to find an answer to the following question.

Question 1. What (classes of) quasi-metric spaces other than the space T are not non-archimedean quasi-metrizable?

It was mentioned that the space T is orthocompact. Generally, however, a quasi-metric space need not be even σ-orthocompact. The space $T \times Z$, for example, is perfect and sub-paracompact, but not σ-orthocompact. Moreover, this space contains a non-σ-orthocompact subspace, which is the union of countably many closed orthocompact subspaces. Under the continuum hypothesis $T \times Z$ contains a Lindelöf subspace that is not hereditarily σ-orthocompact [K4]. Obviously the space $T \times Z$ shows that σ-orthocompactness is not productive in quasi-metric spaces, and it also shows that σ-orthocompactness is neither summable nor hereditary in quasi-metric spaces.

Non-archimedean quasi-metric spaces, of course, must be σ-orthocompact, however they need not be orthocompact. A "Real, weird" space Γ' by E. K. van Dowen and H. H. Wicke [vDW] is a regular non-archimedean quasi-metric space which is not orthocompact.

We shall see in Section 2 that most of the patology described in two former paragraphs evaporates when we restrict ourselves to developable spaces (cf. Theorem 14).

(b) The γ-space conjecture.

We have seen above that the existence of a σ-interior preserving base is not necessary for quasi-metrizability. Consider the following definition.

Definition 4. A collection $\mathcal{P}$ of pairs $\langle P',P''\rangle$ of subsets of a space X, $P' \subset P''$ is a *pair-base* of X provided that for each $x \in X$ and a neighbourhood $O(x)$ of x there exists $\langle P',P''\rangle$ such that $x \in P' \subset P'' \subset O(x)$. A collection $\mathcal{P}$ of pairs $\langle P',P''\rangle$ is *interior-preserving* provided that $\bigcap_{\mathcal{P}_0} P' \subset \operatorname{int} \bigcap_{\mathcal{P}_0} P''$ for each $\mathcal{P}_0 \subset \mathcal{P}$, and is *$\sigma$-interior-preserving* provided that it is a countable union of interior preserving collections. A space is *(σ)-preorthocompact* provided that for each open cover there exists a (σ-) interior preserving collection $\mathcal{P}$ of pairs, such that $\{P''|\langle P',P''\rangle \in \mathcal{P}\}$ refines the cover, while $\{P'|\langle P',P''\rangle \in \mathcal{P}\}$ covers the space (cf. [Fol]).

The following theorem is due to R. Fox.

Theorem 2. *For a space* X *the following are equivalent.*

(i) X *has a* σ*-interior preserving pair-base.*

(ii) there is a distance function d *on* X *with* $d(x,z_n) \to 0$ *whenever* $d(x,y_n) \to 0$ *and* $d(y_n,z_n) \to 0$ [Fol].

Definition 5. A distance function d with $d(x,z_n) \to 0$ whenever $d(x,y_n) \to 0$ and $d(y_n,z_n) \to 0$ is called a γ*-metric* and the respective space is called γ*-space* [Ho].

The study of γ-spaces has a long history; indeed half a century ago V. V. Niemytzki proved essentially the following theorems.

Theorem 3. *Each compact Hausdorff* γ*-space is metrizable.*

Theorem 4. *A space is metrizable iff there is a* γ*-metric* d *on* X *such that* $d(x,y) = d(y,x)$ (cf. Theorem 5) [Nel,Ne2].

Evidently, the class of γ-spaces is countably multiplicative and hereditary. The γ-spaces were shown in [LF] to be the same as co-Nagata spaces [M], Nagata first countable spaces [Ho], spaces with a co-convergent open neighbourhood assignments [S], locally quasi-uniform spaces with a countable base [LF] and spaces with o-metric satisfying property π [NČ].

The following obvious fact is important.

Proposition 1. Each quasi-metric space is a γ-space.

The γ-*space conjecture* states that every γ-space is quasi-metrizable.

Ribeiro essentially claimed in 1943 that γ-space conjecture is true but his proof is incomplete [R]. Since then, the conjecture has been raised in [LF] and [NČ] and listed as Classic Problem VIII in [T].

Question 2. (*The* γ-*space problem*) Is each γ-space quasi-metrizable?

At present only partial solutions have been obtained. Theorems 3 and 4 may be considered to be such partial solutions. Other partial solutions will be discussed below.

It was known in the very beginning of the study of γ-metrics that the existence of a γ-metric d with

(*) $d(x_n,z_n) \to 0$ whenever $d(x_n,y_n) \to 0$ and $d(y_n,z_n) \to 0$

is sufficient for quasi-metrizability. A recent result of R. Fox relaxes the condition (*) and suggests a very general solution to the γ-space problem.

Theorem 5. *A space* X *is quasi-metrizable iff there exists a* γ-*metric* d *on* X *such that* $p(x,y) = d(y,x)$ *is also a* γ-*metric for some topology* [Fol].

Note that if d is a quasi-metric then $p(x,y) = d(y,x)$ also is a quasi-metric for some topology since $p(x,z) \leq p(x,y) + p(y,z)$. Fox's theorem generalizes Theorem 4 and answers a question raised by P. Fletcher and W. F. Lindgren

[LF]; its proof is elegant. An obvious change in the proof of Theorem 5 provides a new and simple proof of Theorem 4.

As long as the γ-space conjecture is not proved, a number of theorems must be proved separately for γ-spaces and for quasi-metric spaces. S. Nedev and M. Čoban have shown that perfect maps preserve γ-spaces [NČ]. They used that in each space a generalized metric d is a γ-metric iff the "distance" between a compact set C and a closed set F, $d(C,F) = \inf\{d(x,y) \mid x \in C, y \in F\}$ is positive. S. Nedev and M. Čoban have raised the γ-space problem in connection with the question whether perfect maps preserve quasi-metric spaces [NČ]. Regardless of the outcome of the γ-space conjecture, the latter question has now been resolved [K4].

Theorem 6. *Quasi-metric spaces, as well as non-archimedean quasi-metric spaces and* γ*-spaces are preserved under perfect map and even under closed maps with first countable images* [NČ,A,K3,K4,V,K5].

R. W. Heath has pointed out that a closed map between quasi-metric spaces may neither be perfect nor have all fibers with compact boundaries. A counter-example is the space Ψ of [GJ] as the domain and $\Psi/\{\text{non-isolated points of } \Psi\}$ as the range.

Theorem 7. *Quasi-metric spaces as well as non-archimedean quasi-metric spaces and* γ*-spaces are preserved under open finite-to-one maps* [Gi].

In [Gi] R. F. Gittings proves the last result and asks whether it also holds for open compact maps. The question is answered in the negative by Example 2 of [K7], which shows that open compact mappings may not preserve quasi-metrizability even if the domain is separable or metacompact. Pseudo-open two-to-one maps also may not preserve quasi-metrizability [K7].

Theorem 8. *If a space* $X = \cup X_\alpha$, *where* $\langle X_\alpha \rangle$ *is either a closed locally finite collection or open* σ*-point-finite collection and each subspace* X_α *is a quasi-metric (non-archimedean quasi-metric,* γ*-) space, then so is* X [FL4].

Question 3. Let $X = \bigcup_{\alpha \in A} X_\alpha$, and $\langle X_\alpha \rangle$ *dominates* X, i.e. $F \subset \bigcup_{\alpha \in A} X_\alpha$, $A_0 \subset A$, is closed in X whenever for each $\alpha \in A_0$ the set $F \cap X_\alpha$ is closed in X_α. If each subspace X_α is quasi-metric non-archimedean quasi-metric, γ-), is X also?

It is known, though, that a Moore space X may not be a γ-space even if it is a union of countable closure-preserving collection of closed metric subspaces (Example 3).

As long as the following question is not answered, there is a hope that an example of a non quasi-metric γ-space can be obtained as a countable union of quasi-metric spaces.

Question 4. Let X be a γ-space and either $X = \cup_{h=1}^{\infty} X_n$, X_n are closed (non-archimedean) quasi-metric subspaces, or

$X = X_1 \cup X_2$, X_1 is closed, X_1 and X_2 are (non-archimedean) quasi-metrizable. Is X (non-archimedean) quasi-metrizable?

2. Quasi-Metrizability in Different Classes of Spaces

(a) Ordered spaces.

Recall that a space X is *a linearly ordered space (a generalized ordered space)* provided that there is a linear order on X such that with respect to this order for each $x \in X$ all intervals $]a,b[$, $a < x < b$ (either all $]a,b[$, $a < x < b$; or all $]a,x)$, $a < x$; or all $[x,b[$, $x < b$; or the singleton $\{x\}$) form a base of neighborhoods of x [Lu].

The real line R is a linearly ordered space, while the Sorgenfrey line (Section 1) is a generalized ordered space. Note that both spaces are separable non-archimedean quasi-metrizable. There is, however, a generalized ordered quasi-metric space that has no σ-discrete dense sets. One such example is the square $[0,1] \times [0,1]$ retopologized by letting all intervals $[\langle a,b\rangle,\langle c,d\rangle[$ under *lexicographic order* be a base; obviously $[\langle a,b\rangle,\langle c,d\rangle \mid c$ or d rational is a σ-interior preserving base.

It is proved in [K6] that the γ-space conjecture is true for *all* generalized ordered spaces. In fact, quite general results concerning transitivity of generalized ordered spaces yield the following theorem.

Theorem 9. *Each generalized ordered γ-space is non-archimedean quasi-metrizable* [K6].

The theorem answers a question of H. Bennett and D. Lutzer. H. Bennett has essentially proved in [B] that generalized ordered γ-spaces *with σ-discrete dense sets* are quasi-metrizable. Methods of [K6] though, provide a clear description of the generalized ordered γ-spaces with σ-discrete dense sets and their σ-interior preserving bases.

Theorem 10. *Let* X *be a generalized ordered space, and fix a respective linear order on* X. *Let the subspace* X_0 *of the non-isolated points of* X *have a σ-discrete dense set* D. *Then the following statements are equivalent.*

(i) X *is a γ-space,*

(ii) the set $R = \{x | x \in X_0$ *and* $[x,\rightarrow[$ *is open*$\}$ *is a countable union* $R = \bigcup_{i=1}^{\infty} R_i$ *such that each* R_i *is closed under limits of increasing sequences and similarly the set* $L = \{x | x \in X_0$ *and* $]\rightarrow,x]$ *is open*$\}$ *is a countable union* $L = \bigcup_{i=1}^{\infty} L_i$ *such that each* L_i *is closed under limits of decreasing sequences.*

(iii) X *is non-archimedeanly quasi-metrizable and* $\{\{x\} | x \in X - X_0\} \cup \{]x,y[| x,y \in D,\ x < y\} \cup \{[x,y] | x \in R,\ y \in D,\ x < y\} \cup \{]x,y] | x \in D,\ y \in L,\ x < y\}$ *is a σ-interior preserving base.*

It follows immediately from the theorem that, for example, the Sorgenfrey line is non-archimedean quasi-metrizable, while the Engelking-Lutzer line, which is the reals, retopologized by letting all $[x,y[$, x is rational, and all $]x,y]$, y is irrational, be a base, is not a

γ-space (cf. [ELu,B]).

Notice, that the result similar to Theorem 10 is not true for all generalized ordered spaces, since there is an example of a linearly ordered paracompact space with a point countable base which is not quasi-metrizable [Gr] (cf. Remarks to Theorem 11). On the other hand, a generalized ordered quasi-metric space even with no isolated points may have no σ-discrete dense set: see, for example, a quasi-metrizable Souslin space with a point countable base by R. W. Heath [H2], or the above mentioned space on $[0,1] \times [0,1]$. In that sense Theorem 9 is essentially more general than Theorem 10.

(b) Spaces with ortho-bases and ortho-pair-bases.

Recall that a base $\mathcal{B}$ of a space X is an *ortho-base* provided that for each $\mathcal{B}_0 \subset \mathcal{B}$ either $\cap\mathcal{B}_0$ is an open set in X or $\mathcal{B}_0$ is a base of neighborhoods of a point $x \in X$. W. F. Lindgren and P. Nyikos have introduced the notion of the ortho-base in [LNi] and raised the question whether a γ-space with an ortho-base is (non-archimedean) quasi-metrizable.

The question has been answered recently. A general result concerning transitivity of spaces with ortho-bases (Theorem 22) yields the following theorem.

Theorem 11. *Each γ-space with an ortho-base is non-archimedean quasi-metrizable* [K7].

Note that G. Gruenhage has proved in [Gr] that *paracompact* γ-spaces with ortho-bases are non-archimedean quasi-metrizable. He has also shown that an example due to P. Nyikos of a first countable paracompact linearly ordered space with a point-countable ortho-base fails to be quasi-metrizable. The example is even a non-archimedean space [Ny], i.e. a space with a base $\mathcal{B}$ such that for each $B, B' \in \mathcal{B}$ either $B \cap B' = \emptyset$, or $B \subset B'$, or $B' \subset B$.

The analogy between ortho-bases, σ-interior-preserving bases and σ-interior preserving pair-bases suggests the following definition.

Definition 5. A pair-base $\mathcal{P}$ of a space X is an *ortho-pair-base* provided that for each $\mathcal{P}_0 \subset \mathcal{P}$ either $\cap\{P' | \langle P', P''\rangle \in \mathcal{P}_0\} \subset \operatorname{int}\{P'' | \langle P', P''\rangle \in \mathcal{P}_0\}$ or there is a point $x \in X$ such that for each neighborhood $O(x)$ there is $\langle P', P''\rangle \in \mathcal{P}_0$ such that $x \in P' \subset P'' \subset O(x)$. (cf. Definition 4).

The next theorem follows from a result concerning pretransitivity in spaces with ortho-pair-bases in much the same way that Theorem 11 is established using transitivity of spaces with ortho-bases.

Theorem 12. *Each γ-space with an ortho-pair-base is quasi-metrizable.*

Question 5. Is each quasi-metric space with an ortho-pair-base non-archimedean quasi-metrizable? Does each such space have an ortho-base?

We shall see below that developable spaces are non-archimedean quasi-metric (are γ-spaces) iff they have an ortho-(pair-) base. (Theorems 14 and 16), and it follows from Theorem 15 that each developable γ-space is quasi-metrizable. A positive answer to Question 5 will also provide a positive answer to the corresponding open question concerning developable spaces (Question 6).

(c) Developable spaces.

Recall that a space X is *developable* provided that there is a sequence $\langle \omega_n \rangle$, $n = 1,2,\cdots$ of open covers of X such that for each point $x \in X$ and each sequence of sets $O_n \in \omega_n$ with $x \in O_n$, $\{O_n | n \in \mathbf{N}\}$ is a base for the neighborhoods of x. Each developable space is semi-metrizable (cf. Definition 1), and hence *semi-stratifiable*. A space is semi-stratifiable provided that each open set O can be presented as a union of closed sets $F_n(O)$, $n = 1,2,\cdots$ in a way that for each pair of open sets O, O', $O \subset O'$ and for each $n = 1,2,\cdots$ $F_n(O) \subset F_n(O')$. A space is semi-metrizable if and only if it is semi-stratifiable and first countable [C,Kl]. While a semi-stratifiable space is not necessarily developable, S. Nedev and M. Čoban have shown that in the theory of quasi-metrizability semi-stratifiable spaces are as good as developable spaces.

Theorem 13. *Each semi-stratifiable* γ*-space is developable* [NČ].

The following result, due to P. Fletcher, W. F. Lindgren and P. Nyikos, gives a clear description of the developable non-archimedean quasi-metric spaces.

Theorem 14. *For a developable space* X *the following are equivalent.*

(i) X *is non-archimedean quasi-metrizable,*

(ii) X *is hereditary orthocompact,*

(iii) X *is* σ*-orthocompact,*

(iv) X *has an ortho-base* [FL3,LNi].

This theorem certainly simplifies the relations between orthocompactness, ortho-bases and at least non-archimedean quasi-metrizability (cf. remarks on orthocompactness in Section 1a). The theorem, however, leaves open the following question raised by H. Junnila.

Question 6. Is each developable quasi-metric space non-archimedean quasi-metrizable?

The γ-space conjecture, however, is true in developable spaces.

Theorem 15. *Each developable* γ*-space is quasi-metrizable.*

This important theorem was first proved by H. Junnila [J1][1] by essentially using the notion of pretransitivity, which will be discussed in the last section and the implication (i) $\Rightarrow$ (ii) of the following analogue of Theorem 14.

[1] A similar theorem has been obtained independently by the author (cf. [J1]); recently an elegant proof of this theorem was found by R. Fox [Fo2].

Theorem 16. *For a developable space* X *the following are equivalent.*

(i) X *is a* γ*-space.*

(ii) X *is hereditarily preorthocompact.*

(iii) X *is* (σ-) *pre-orthocompact.*

(iv) X *has an ortho-pair-base.*

Since we know now that developable γ-spaces are quasi-metrizable, and since we do not know whether developable quasi-metric spaces are non-archimedean quasi-metrizable it is not known whether Theorems 14 and 16 are essentially different.

It is a question of R. Stoltenberg whether each developable space is quasi-metrizable [St]. The following example, which was defined independently in [K2] and [H1] answers this question as well as some other questions on quasi-metrizability.

Example 2. *The space* H. The space is a subspace $A \cup B$ of the plane R^2 where $A = R \times \{0\}$ and $B = \{\langle x,y\rangle | x,y$ are rational $y > 0\}$. The set B is open in H and inherits the Euclidean topology from R^2. For each point $a \in A$ the singleton $\{a\}$ along with the set of all points in B that belong to the interior of an equilateral triangle above A, having vertex a and one side parallel to A, is a basic neighborhood of a. H is a Moore space and it is not quasi-metrizable as may be proved using that all basic neighborhoods in each point a A have the same "angle," and that A is of the second category in Euclidean topology.

The space H is a union of the two metrizable subsets A and B, one of which is closed and discrete; it is also a countable and closure preserving union of metrizable subspaces $A \cup \{\langle x,y\rangle \in B | y \geq \frac{1}{n}\}$, $n = 1,2,\cdots$ (cf. Questions 3 and 4).

The space H is an open compact image of a separable non-archimedean quasi-metric Moore space; and a modification of the space H which is also a non-quasi-metrizable is an open compact image of a metacompact Moore space and hence a non-archimedean quasi-metric space [K7] (cf. remarks to Theorem 7).

Some other examples of non-quasi-metric Moore spaces have been obtained in [HL].

3. Quasi-Metrizability and Transitivity

Recall that a *binary relation* on a set X is any subset U of $X \times X$. Given a binary relation U on X we define $U\{x\} = \{y \in X | \langle x,y\rangle \in U\}$. Conversely, we can construct a binary relation U on X by specifying each of the sets $U\{x\}$ and then letting $U = \cup\{\{x\} \times U\{x\} | x \in X\}$. Throughout the rest of this paper, we will systematically confuse binary relations as subsets U of X^2 and binary relations which are obtained by specifying subsets $U\{x\}$ of X, as above.

For a binary relation U on X and any subset $A \subset X$, we define $UA = U(A) = \cup\{U\{x\} | x \in A\}$. Given two binary relations U and V on X, a new binary relation $V \circ U$ is defined by $V \circ U = \{\langle x,z\rangle |$ for some $y \in x$, $\langle x,y\rangle \in U$ and $\langle y,z\rangle \in V\}$. Equivalently, $V \circ U$ can be obtained by defining $(V \circ U)\{x\} = V(U\{x\})$. If U is a binary relation,

then U^k denotes $U \circ \cdots \circ U$ k times. A binary relation U is *transitive* if $U^2 \subset U$, i.e. if $z \in U\{x\}$ whenever $z \in U\{y\}$ and $y \in U\{x\}$.

A binary relation U on a space X is called *neighbornet in* X provided that each $U\{x\}$ is a neighborhood of x [J2]. If U is a neighborhood of the diagonal in $X \times X$, then it is a neighbornet in X, but the converse need not be true.

A sequence of neighbornets $\langle U_n \rangle$, $n = 1,2,\cdots$, is called *basic* if for each $x \in X$, $\{U_n\{x\} \mid n \geq 1\}$ is a base of neighborhoods of x. A neighbornet U is *normal* if there exists a sequence $\langle U_n \rangle$ of neighbornets, $n = 1,2,\cdots, U_{n+1}^2 \subset U_n$ and $U_1 = U$. Note that if d is a quasi-metric (a non-archimedean quasi-metric, a γ-metric) and $U_n = \{\langle x,y \rangle \in X^2 \mid d(x,y) < \frac{1}{2^n}\}$ then $\langle U_n \rangle$ is a basic sequence of normal neighbornets ($\langle U_n \rangle$ is a basic sequence of transitive neighbornets, $\langle U_n^2 \rangle$ is a basic sequence).

In fact the usefulness of neighbornets in the study of quasi-metrizability theory is based on the following characterization theorem.

Theorem 17.

(i) a space X *is quasi-metrizable iff there is a basic sequence of normal neighbornets in* X.

(ii) a space X *is non-archimedean quasi-metrizable iff there is a basic sequence of transitive neighbornets in* X.

(iii) a space X *is a* γ*-space iff there is a sequence* $\langle U_n \rangle$ *of neighbornets in* X *such that* $\langle U_n^2 \rangle$ *is basic* (cf. [J2]).

Concerning (iii) we remark, that if $\langle U_n^2 \rangle$ is basic then so is $\langle U_n^k \rangle$ for each fixed $k \geq 1$.

It follows immediately from Theorem 17 that in order to show that a γ-space is (non-archimedean) quasi-metrizable it is enough to prove, for example, that for each neighbornet U there exists a normal (transitive) neighbornet V such that $V \subset U^2$, and even $V \subset U^k$ for any fixed $k \geq 1$. This suggests the following definition due to P. Fletcher and W. F. Lindgren.

Definition 6. A space X is called k-*pretransitive* (k-*transitive*), $k \geq 1$, provided that for each neighbornet U in X there exists a normal (transitive) neighbornet $V \subset U^k$; and it is called *transitive* if for each normal neighbornet U there exists a transitive neighbornet $V \subset U$ [FL2]. Obviously, each k-(pre-) transitive space is m-(pre-) transitive, $m \geq k$, and each k-transitive space is transitive.

Theorem 18. *Each* k-*pretransitive (*k-*transitive) γ-space is (non-archimedean) quasi-metrizable. Each transitive quasi-metric space is non-archimedean quasi-metrizable* [FL4].

It follows that in order to show that each γ-space from a certain class of spaces is (non-archimedean) quasi-metrizable, it is enough to show that each space in this class is k-pretransitive (k-transitive), $k \geq 1$.

It was shown in [K6] that generalized ordered γ-spaces are non-archimedean quasi-metrizable by proving that *each* generalized ordered space is 3-transitive. If we suppose that a generalized ordered space is a γ-space, it can be proved that it is even 2-transitive.

Theorem 19.

(i) Each generalized ordered space is 3-transitive.

(ii) Each generalized ordered γ-space is 2-transitive (cf. Theorem 9).

The γ-space conjecture for developable spaces was proved in [J1] by essentially showing that each developable γ-space is 2-pretransitive. In fact, the following results have been proved in [J1,J2].

Theorem 20.

(i) Each semi-stratifiable (σ-) orthocompact space is 3-transitive.

(ii) Each developable non-archimedean quasi-metric space is 2-transitive.

Theorem 21.

(i) Each semi-stratifiable (σ-) pre-orthocompact space is 3-pretransitive.

(ii) Each developable γ-space is 2-pretransitive (cf. Theorem 15; cf. also Theorems 14 and 16).

Recently it was proved that all γ-spaces with ortho-(pair-) bases are non-archimedean quasi-metrizable

(quasi-metrizable) by showing that *each* space with an ortho-(pair-) base is 2-(pre-) transitive [K8].

Theorem 22. *Each space with an ortho-base is* 2-*transitive.*

Theorem 23. *Each space with an ortho-pair-base is* 2-*pretransitive* (cf. Theorems 11 and 12).

Since each developable γ-space has an ortho-pair-base (Theorem 16), the γ-space conjecture for developable spaces can be also proved using Theorem 23.

It is not known whether Theorems 20 and 21 and Theorems 22 and 23 have essentially different domains; in particular, the answers to the following questions are not known.

Question 7. Is each developable (quasi-metric) space transitive?

Question 8. Is each (quasi-metric) space with an ortho-pair-base transitive?

The positive answer to Questions 7 and 8 will provide the positive answer to questions 5 and 6 as well.

The real line is not 1-pretransitive. The Engelking-Lutzer line (cf. Remarks to Theorem 10) is not 2-pretransitive (cf. [K6]). There is also an example by H. Junnila of a semi-stratifiable orthocompact space which is not 2-pretransitive [J2]. In this sense Theorems 19-23 cannot be improved.

The theorems show that transitivity serves as a useful tool in the theory of quasi-metrizability. For example, the γ-space conjecture in generalized ordered spaces and in spaces with ortho-bases proved first in [B] and [Gr] with quite a heavy use of restrictive assumptions (the existence of a σ-discrete dense set and the paracompactness) and without use of neighbornets (cf. Section 2, a and b), now have quite logical and very general, while still not simple, proofs in terms of transitivity [K6,K8].

Transitivity is a kind of compactness or rather paracompactness property in which neighbornets play the role of covers. Although it has been studied for quite a short time, what is known so far suggests that the transitivity of topological spaces may be of some intrinsic interest.

In many cases it is not easy to show that a particular space is transitive. In fact, the only classes of k-(pre-) transitive spaces available thus far are those described in Theorems 19-23. (It is also known that each P-*space*, i.e. a space in which countable intersections of open sets are open is transitive [FL4].) There are several different proofs of 2-transitivity of the real line, although they are not much simpler than those of Theorems 19, 20, 22 (cf. [FL4]).

The following theorem contains results due to P. Fletcher, W. F. Lindgren and H. Junnila [FL4].

Theorem 24.

(i) Each open or closed subspace of a transitive space is transitive.

(ii) If a space is a union of two transitive subspaces, one of which is closed, then the space is transitive.

(iii) If a space is a countable union of closed or σ-point finite union of open transitive subspaces then the space is transitive.

(iv) Each closed image of a transitive space is transitive.

Some of the results (i)-(iv) can also be stated for k-(pre)-transitivity (cf. Theorem 8 and Questions 3 and 4).

Although little is known about which spaces are transitive, almost nothing is known about which spaces are not transitive. The space T of Example 1 is quasi-metric but is not non-archimedean quasi-metric, and hence is not transitive (cf. Theorem 18) and at present all available Tychonoff spaces that are not transitive are built upon the space T. T is first countable, and it follows that open maps do not preserve transitivity, since each metric space is 2-transitive by Theorem 20; T is real compact, hence, as observed in [FL4], transitivity is not uncountably multiplicative. If one assumes the continuum hypothesis, there is a regular Lindelöf non-transitive space (cf. Section 1a); it is not known, however, if each compact space is transitive.

Most of the following questions were raised in [FL4].

Question 9. Is each open compact image of a transitive space transitive? Is each product of two transitive spaces transitive? (cf. Section 1b)

Question 10. Is each non-archimedean quasi-metric space transitive or even k-transitive for some k?

Question 11. Is each quasi-metric space k-pretransitive for some k?

The positive answer to the similar question for γ-spaces constitutes a proof of the γ-space conjecture.

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Added in proof: Recently R. Fox solved the γ-space problem by constructing a non-regular Hausdorff γ-space which is not quasi-metrizable. The counterexample answers in the negative not only Question 2 but also Questions 4, 11 and the weak version of Question 10.

References

[B] H. R. Bennett, *Quasi-metrizability and the γ-space property in certain generalized ordered spaces*, Topology Proceedings 4 (1979), 1-12.

[C] G. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. 32 (1970), 47-54.

[vDW] E. K. van Douwen and H. H. Wicke, *A real weird topology on the reals*, Houston J. Math. 3 (1977), 141-152.

[ELu] R. Engelking and D. Lutzer. *Paracompactness in ordered spaces*, Fund. Math. 94 (1976).

[FL1] P. Fletcher and W. F. Lindgren, *Transitive quasi-uniformities*, J. Math. Anal. Appl. 39 (1972), 397-405.

[FL2] _____, *Quasi-uniformities with a transitive base*, Pacific J. Math. 43 (1972), 619-631.

[FL3] _____, *Orthocompactness and strong Cech compactness in Moore spaces*, Duke Math. J. 39 (1972), 753-766.

[FL4] _____, *Quasi-uniform spaces*, Marcel-Dekker (to appea

[Fol] R. Fox, *On metrizability and quasi-metrizability* (to appear).

[Fo2] _____, *A short proof of H. Junnila quasi-metrization theorem* (to appear).

[GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Springer-Verlag, Berlin and New York, 1976.

[Gi] R. F. Gittings, *Finite one-to-one maps of generalized metric spaces*, Pacific J. Math. 59 (1975), 33-41.

[Gr] G. Gruenhage, *A note on quasi-metrizability*, Can. J. Math. 29 (1977), 360-366.

[Hl] R. W. Heath, *A postscript to a note on quasi-metric spaces*, Notices Amer. Math. Soc. 19 (1972), A-338.

[H2] _____, *A construction of a quasi-metric Souslin space with a point countable base*, Set theoretic topology (Inst. Med. Math., Ohio Univ.), Academic Press (1977), 219-224.

[HL] _____ and W. F. Lindgren, *On generating non-ortho-compact spaces*, Set theoretic topology, Academic Press (1970), 225-237.

[Ho] R. E. Hodel, *Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points*, Duke Math. J. 39 (1972), 253-263.

[Jl] H. Junnila, *Covering properties and quasi-uniformities of topological spaces*, Ph.D. thesis, Virginia Polytech. Inst. and State Univ., 1978.

[J2] _____, *Neighbornets*, Pacific J. Math. 76 (1978), 83-108.

[Kl] J. Kofner, *On a new class of spaces and some problems of symmetrizability theory*, Soviet Math. Dokl. 10 (1969), 845-848.

[K2] _____, *On Δ-metrizable spaces*, Math. Notes 13 (1973), 168-174.

[K3] _____, *Semi-stratifiable spaces and spaces with generalized metrics*, D.Sc. thesis, Technion, Haifa, 1975.

[K4] _____, *Quasi-metrizable spaces*, Pacific J. Math. 88 (1980), 81-89.

[K5] _____, *Closed mappings and quasi-metrics*, Proc. Amer. Math. Soc. 80 (1980), 333-336.

[K6] _____, *Transitivity and the γ-space conjecture in ordered spaces*, Proc. Amer. Math. Soc. (to appear).

[K7] _____, *Open compact mappings, Moore spaces and orthocompactness*, Rocky Mount. Math. J. (to appear).

[K8] _____, *Transitivity and ortho-bases*, Canadian J. Math. (to appear).

[LF] W. F. Lindgren and P. Fletcher, *Locally quasi-uniform spaces with countable bases*, Duke Math. J. 41 (1974), 231-240.

[LN] W. F. Lindgren and P. Nyikos, *Spaces with bases satisfying certain order and intersection properties*, Pacific J. Math. 66 (1976), 455-476.

[Lu] D. Lutzer, *On generalized ordered spaces*, Dissertationes Math. 89 (1971).

[M] H. Martin, *Metrization and submetrization of topological spaces*, Ph.D. thesis, Univ. of Pittsburg, 1973.

[N] S. I. Nedev, *Generalized-metrizable spaces*, C. R. Acad. Bulgare Sci. 20 (1967), 513-516.

[NČ] _____ and M. M. Čoban, *On the theory of 0-metrizable spaces III*, Vestnik Mosk Univ., Ser. I Mat. Men. 27 (1972), 10-15.

[Nel] V. V. Niemytzki, *On the third axiom of metric spaces*, Trans. Amer. Math. Soc. 29 (1927), 507-513.

[Ne2] _____, *Uber die Axioms des metrischen Raumes*, Math. Ann. 106 (1931), 666-671.

[Ny] P. J. Nyikos, *Some surprising base properties in topology*, Studies in Topology, New York, Academic Press (1975).

[R] H. Ribeiro, *Sur les espaces a metrique faible*, Port. Math. 4 (1943).

[S] R. Sabella, *Convergence properties of neighboring sequences*, Proc. Amer. Math. Soc. 38 (1973), 405-409.

[St] R. A. Stoltenberg, *On quasi-metric spaces*, Duke Math. J. 36 (1969), 65-71.

[T] Topology Proceedings, 2 (1977), 687.

[V] N. V. Veličko, *Quasi-uniform sequential spaces*, C. R. Acad. Bulgare Sci. 25 (1972), 589-591.

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