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0. Introduction

This paper is a survey of results and techniques, extending those of [S1] and [S2], in the theory of orthocompactness in products. Section 1 deals with products with a compact factor; the main results have already appeared in [S1], but I think it worthwhile to make readily available the improved techniques developed in (the unpublished) [S3]. A similar comment can be made about Section 2, on infinite products, which has appeared only in [S3]. In Section 3 the main result of [S2], namely, that normality and orthocompactness are equivalent in products of finitely many locally compact LOTS's, is extended on the one hand to products of finitely many locally compact GO-spaces (= suborderable spaces), and on the other to certain infinite products of LOTS's. Most of the results in Section 4, on products with a metric-like factor, and in Section 5, on mapping theorems, are new.

Attention has been drawn in previous papers ([S1] and [S2]) to the existence of a sort of analogy between orthocompactness and normality, in which metacompactness takes the space of paracompactness; or, as I facetiously expressed it in the subtitle of [S3],

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orthocompact:metacompact::normal:paracompact (a.e.). In discussing the new material in Section 4 I shall continue from [S1] the practice of quoting, for purposes of comparison, the corresponding theorems concerning normality in products with metric factor, and of pointing out where the analogy breaks down.

Notation and terminology generally follow that of [S1]. All topological spaces are T_1. Ordinals are denoted by lower-case Greek letters; an ordinal is the set of smaller ordinals, and cardinals, denoted by κ, λ, and μ, are initial ordinals. The collection of subsets of a set X is denoted by $\mathcal{P}(X)$, $[X]^{\kappa} = \{A \subseteq X: |A| = \kappa\}$, $[X]^{\leq\kappa} = \{A \subseteq X: |A| \leq \kappa\}$, etc., where $|A|$ denotes the cardinality of the set A. (Thus, for example, $[X]^{>\omega} = [X]^{\geq\omega_1}$ is the set of uncountable subsets of X.) If X is some set, $x \in X$, and $\mathcal{C} \subseteq \mathcal{P}(X)$, then we define $ST(x,\mathcal{C}) = \{C \in \mathcal{C}: x \in C\}$, $st(x,\mathcal{C}) = \bigcup ST(x,\mathcal{C})$, $c(x,\mathcal{C}) = \bigcap ST(x,\mathcal{C})$, and $ord(x,\mathcal{C}) = |ST(x,\mathcal{C})|$; and $\mathcal{D} \subseteq \mathcal{P}(X)$ is a *refinement* of $\mathcal{C}$ iff for each $D \in \mathcal{D}$ there is a $C \in \mathcal{C}$ such that $D \subseteq C$. (We do not require $\mathcal{C}$ or $\mathcal{D}$ to cover X.) If $\mathcal{C}$ is indexed in some manner, $\mathcal{D}$ is a *precisely-indexed* refinement of $\mathcal{C}$ iff $\mathcal{D}$ is indexed by the same index set, and each member of $\mathcal{D}$ is contained in the corresponding member of $\mathcal{C}$. Proper inclusion is denoted by $\subset$.

Let X be a topological space. The topology of X is denoted by τX; and if, as a set, X is contained in some Euclidean space, E^n, τX will always denote the Euclidean topology on X, others being given other names as the occasion arises. X is *orthocompact* (*metacompact*, resp.) iff every open cover of X has a Q-refinement (point-finite open refinement, resp.) covering X, where $\mathcal{C} \subseteq \mathcal{P}(X)$ is a Q-*collection* (*point-finite*, resp.) iff for every $x \in X$, $c(x,\mathcal{C})$ is open ($\text{ord}(x,\mathcal{C}) < \omega$, resp.). $F \subseteq X$ is Q-*embedded* in X iff whenever $\mathcal{V} \subseteq \tau X$ covers F there is a Q-collection $\mathcal{R} \subseteq \tau X$ also covering F and refining $\mathcal{V}$. It is easy to see that X is orthocompact iff every closed subset of X is Q-embedded in X iff every open cover, $\mathcal{V}$, of X has an open refinement, $\mathcal{R}$, covering X, with the property that for any $\mathcal{U} \subseteq \mathcal{R}$, $\cap\mathcal{U} \in \tau X$, and that X is hereditarily orthocompact iff every subset of X is Q-embedded in X iff every open subset of X is Q-embedded in X. X is countably metacompact iff every countable open cover of X has a point-finite open refinement covering X, and it is also easy to see that X is orthocompact ((countably) metacompact, resp.) iff every indexed open cover of X has a precisely-indexed Q-refinement (point-finite open refinement, resp.) covering X.

If $\mathcal{S}$ is a class of spaces closed under taking homeomorphic images, a space X is said to be $\mathcal{S}$-metacompact ($\mathcal{S}$-paracompact, resp.) iff $X \times Y$ is orthocompact (normal, resp.) for every $Y \in \mathcal{S}$; and by $\mathcal{S}(\lambda)$ we denote the class of $Y \in \mathcal{S}$ of weight (= minimum cardinality of an open base) at most λ. The classes of orthocompact spaces, normal spaces, compact spaces, metacompact and developable spaces, and metrizable spaces will be denoted $\mathcal{O}$, $\mathcal{N}$, $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{M}_0$, respectively. For any such class $\mathcal{S}$ we also set $\mathcal{S}' = \{X \in \mathcal{S}: X = X'\}$, where X' is the set of non-isolated points of X.

Given a topological property P, a space X is said to be a *test space* (OC) (*test space* (N), resp.) *for* P iff for any orthocompact (normal, resp.) space Y, Y×X is orthocompact (normal, resp.) iff Y has property P. Among such properties are (κ,λ)-metacompactness, (κ,λ)-paracompactness, and (κ,λ)-compactness for cardinals κ and λ with $\kappa \geq \lambda$: X is (κ,λ)-compact iff every open cover of X of cardinality at most κ has an open refinement of cardinality less than λ covering X; X is (κ,λ)-metacompact iff every open cover, $\mathcal{U}$, of X of cardinality at most κ has an open refinement, $\mathcal{V}$, covering X and such that ord $(x,\mathcal{V}) < \lambda$ for each $x \in X$; and the other properties are defined similarly. (Thus, for example, (ω,ω)-compactness is just countable compactness.) A space is (∞,λ)-compact iff it is (κ,λ)-compact for all $\kappa \geq \lambda$, and similarly for the other properties.

Finally, an ordinal α as a space always bears the order-topology; if α is to be viewed with the discrete topology, it will be denoted D_α. For any cardinal $\kappa \geq \omega$, P_κ denotes the set $\kappa+1$ retopologized by isolating each $\alpha \in \kappa$. $\underset{\sim}{I}$ is the closed unit interval. And for any space X, X* is the one-point compactification of X, the point at infinity being denoted p_X^*.

Maps are continuous surjections.

1. Products with a Compact Factor

In this section we isolate some of the ideas underlying the results in Section 1 of [S1].

1.0. *Definition.* [S3]. A pair $\langle\kappa,\lambda\rangle$ of regular cardinals, where $\kappa \geq \lambda \geq \omega$, is a P-*type* for a space X at a point $x \in X$ iff there is a $\mathcal{V} \subseteq \tau X$ such that

(i) $|\mathcal{V}| = \kappa$,

(ii) $\mathcal{V} \subseteq ST(x,\tau X)$, and

(iii) if $\mathcal{W} \in [\mathcal{V}]^{\geq\lambda}$, then $x \notin \text{Int} \cap \mathcal{W}$.

Note that if $\langle\kappa,\lambda\rangle$ is a P-type for X at x, and $\kappa \geq \mu \geq \lambda$, with μ regular, then $\langle\kappa,\mu\rangle$ and $\langle\mu,\lambda\rangle$ are also P-types for X at x. However, it is not necessarily possible to find cardinals κ_0 and λ_0 such that $\langle\kappa,\lambda\rangle$ is a P-type for X iff $\kappa_0 \geq \kappa \geq \lambda \geq \lambda_0$ and κ and λ are regular, as may be seen by considering the quotient of the discrete union of P_ω and P_{ω_1} obtained by identifying the two non-isolated points: this space has as P-types $\langle\omega_1,\omega_1\rangle$ and $\langle\omega,\omega\rangle$, but not $\langle\omega_1,\omega\rangle$. There is, neverhteless, a bound on κ: if $\langle\kappa,\lambda\rangle$ is a P-type for X at x, then $\kappa \leq \chi(x,X)$, the minimum cardinality of a local base at x. The utility of the concept of a P-type arises from the following lemma.

1.1. *Lemma. Let* Y *have* $\langle\kappa,\lambda\rangle$ *as a* P-*type, and suppose that* X×Y *is orthocompact; then* X *is* (κ,λ)-*metacompact.*

Proof. Let $\langle\kappa,\lambda\rangle$ be a P-type for Y at the point y, and let $\mathcal{V} \in [\tau Y]^\kappa$ be as specified in the definition of P-type. Suppose that X is not (κ,λ)-metacompact, and let $\mathcal{W} = \{W_\alpha: \alpha \in \kappa\}$ be an open cover of X having no open refinement $\mathcal{R}$ covering X and with the property that $\text{ord}(x,\mathcal{R}) < \lambda$ for each $x \in X$. Index $\mathcal{V}$ as $\{V_\alpha: \alpha \in \kappa\}$, and let

$\mathcal{H} = \{W_\alpha \times V_\alpha : \alpha \in \kappa\}$, an $(X\times Y)$-open cover of $X\times\{y\}$. $X\times\{y\}$ is closed and therefore Q-embedded in $X\times Y$, so there is a precisely-indexed Q-collection, $\mathcal{R} = \{R_\alpha : \alpha \in \kappa\}$, refining $\mathcal{H}$ and covering $X\times\{y\}$. Since $X\times\{y\}$ is homeomorphic to X, there must, by the choice of $\mathcal{W}$, be an $x \in X$ such that $\mathrm{ord}(\langle x,y\rangle,\mathcal{R}) \geq \lambda$; let $A \in [\kappa]^{\geq\lambda}$ be such that for every $\alpha \in A$, $\langle x,y\rangle \in R_\alpha$. Then, since $\mathcal{R}$ is a Q-collection, $\langle x,y\rangle \in \bigcap\{R_\alpha : \alpha \in A\} = \mathrm{Int}\bigcap\{R_\alpha : \alpha \in A\} \subseteq X\times\mathrm{Int}\bigcap\{V_\alpha : \alpha \in A\}$, which contradicts the fact that $y \notin \mathrm{Int}\bigcap\{V_\alpha : \alpha \in A\}$.

Clearly every infinite compact space and every non-discrete first countable space has $\langle\omega,\omega\rangle$ as a P-type. Thus, we have immediately the following theorem from [S1].

1.2. *Theorem. If* $Y \in \mathcal{K}$ *and* $|Y| \geq \omega$, *or if* $Y \in \mathcal{M}$ *and* Y *is not discrete, and if further* $X\times Y$ *is orthocompact, then* X *is orthocompact and countably metacompact.*

The following unwieldy technical lemma is the other main ingredient in this stew. ($\lambda^{\underline{\kappa}} = \sup\{\lambda^\mu : \mu < \kappa\}$.)

1.3. *Lemma. Let* X *and* Y *be orthocompact, and suppose that* $w(Y) = \lambda$, Y *is* (∞,κ)-*compact,* X *is* $(\lambda^{\underline{\kappa}},\omega)$-*metacompact, and the canonical projection* $\pi_X\colon X\times Y \to X$ *is closed; then* $X\times Y$ *is orthocompact.*

Proof. Let $\mathcal{B}$ be an open base for Y of cardinality λ, and let $\mathcal{U}$ be an open cover of $X\times Y$ by sets of the form $V\times B$, where $V \in \tau X$ and $B \in \mathcal{B}$. Since Y is (∞,κ)-compact and π_X is closed, we further assume that for each $x \in X$ there is a

$V(x) \in \tau X$ and a $\mathcal{B}(x) \in [\mathcal{B}]^{<\kappa}$ such that $\mathcal{U}(x) = \{V(x)\times B: B \in \mathcal{B}(x)\}$ is a subcollection of $\mathcal{U}$ covering $\{x\}\times Y$.

Let $\mathcal{R} = \{R(x): x \in X\}$ be a precisely-indexed Q-refinement of $\mathcal{V}$ covering X, and, for each $\mathcal{A} \in [\mathcal{B}]^{<\kappa}$, let $G(\mathcal{A}) = \cup\{R(x) \in \mathcal{R}: \mathcal{B}(x) = \mathcal{A}\}$. Then $\mathcal{G} = \{G(\mathcal{A}): \mathcal{A} \in [\mathcal{B}]^{<\kappa}\}$, being an open cover of X of power at most $\lambda^{\underline{\kappa}}$, has a precisely-indexed, point-finite open refinement, $\mathcal{H}$, which covers X. If, for each $\mathcal{A} \in [\mathcal{B}]^{<\kappa}$ that covers Y, $\mathcal{A}'$ is a Q-refinement of $\mathcal{A}$ also covering Y, it is easy to see that $\{(R(x) \cap H(\mathcal{A}))\times A: x \in X, \mathcal{B}(x) = \mathcal{A}, \text{ and } A \in \mathcal{A}'\}$ is the desired Q-refinement of $\mathcal{U}$.

1.4. *Theorem.* [S1]. *If* X *is orthocompact and* (λ,ω)*-metacompact, and* $Y \in \mathcal{K}(\lambda)$, *then* $X\times Y$ *is orthocompact and* (λ,ω)*-metacompact.*

Proof. The first assertion is immediate from Lemma 1.3, with $\kappa = \omega$; the second follows from the observations that $Y\times D_2^\lambda \in \mathcal{K}(\lambda)$, so that $(X\times Y)\times D_2^\lambda$ is orthocompact, and that $\langle\lambda,\omega\rangle$ is a P-type for D_2^λ.

1.5. *Corollary.* [S1]. *If* X *is orthocompact,* $|Y| \geq \omega$, *and* $Y \in \mathcal{K}(\omega)$, *then* $X\times Y$ *is orthocompact iff* X *is countably metacompact.*

From Theorem 1.4 and the fact that if $\lambda \geq \omega$, $\langle\lambda,\omega\rangle$ is a P-type for D_2^λ, $\underset{\sim}{I}^\lambda$, and D_λ^*, it follows that each of these spaces is a test space (OC) for (λ,ω)-metacompactness. From this we deduce

1.6. *Theorem*. [S1]. *If* $\lambda \geq \omega$, *and* X *is orthocompact, then* X *is* $\mathcal{K}(\lambda)$*-metacompact iff* X *is* (λ,ω)*-metacompact.*

It is worth mentioning that the converse to Theorem 1.4 is definitely false. Let X be the lexicographically ordered square. Then $w(X) = 2^{\omega} \geq \omega_1 = w(\omega_1)$, so, since ω_1 is not metacompact, neither is it $(w(X),\omega)$-metacompact. However, it follows from the main result of [S2] (see Section 3) or, because ω_1 is countably compact, from the following theorem, that $\omega_1 \times X$ is orthocompact.

1.7. *Theorem*. [S3]. *If* X *is orthocompact and* (κ,ω)*-compact*, $\chi(Y) \leq \kappa$, *and* Y *is metacompact, then* $X \times Y$ *is orthocompact.*

Proof. Use the fact that $\pi_Y\colon X \times Y \to Y$ is closed.

Finally, Lemma 1.1 also yields the analogue of the result, announced by Mary Ellen Rudin at the 1978 Spring Topology Conference in Norman, Oklahoma, that a space X is $\mathcal{N}$-paracompact iff X is discrete.

1.8. *Theorem*. *A space is* $\mathcal{O}$*-metacompact iff* X *is discrete.*

Proof. Suppose that X is not discrete, and let p be a non-isolated point of X. There is a least cardinal κ for which there exists an $A \in [X \setminus \{p\}]^{\kappa}$ with $p \in \mathrm{cl}\, A$. Choose such an A, and enumerate it as $\{x_\alpha\colon \alpha \in \kappa\}$. For each $\alpha \in \kappa$, let $A_\alpha = \mathrm{cl}\{x_\xi\colon \xi < \alpha\}$, so that $\langle A_\alpha\colon \alpha \in \kappa\rangle$ is a non-decreasing κ-sequence of closed sets not containing p. Let $\lambda = \mathrm{cf}\, \kappa$, and take $\langle F_\alpha\colon \alpha \in \lambda\rangle$ to be any strictly increasing, cofinal λ-subsequence of $\langle A_\alpha : \alpha \in \kappa \rangle$. Clearly λ is infinite and regular, and $\langle \lambda, \lambda \rangle$ is a P-type for X at p: use the family $\{X \setminus F_\alpha : \alpha \in \lambda\}$ of open nbhds of p.

Now, if $\lambda > \omega$ is regular, it is well known (and an easy consequence of the Pressing-Down Lemma) that λ is not (λ, λ)-metacompact. Thus, for $\lambda > \omega$ we have from Lemma 1.1 that $X \times \lambda$ is not orthocompact, though λ, being a LOTS, is (see [S1]), and hence that X is not $\mathcal{O}$-metacompact. And if $\lambda = \omega$, we take Y to be any orthocompact space which is not countably metacompact ([S4]), so that $X \times Y$ is not orthocompact, and again infer that X is not $\mathcal{O}$-metacompact.

The converse is obvious.

More concerning the relationship of these results to the product theory of normality can be found in [S1]. Briefly, Corollary 1.5 and Theorem 1.6 are analogues of theorems due, respectively, to Dowker [D] and to Morita [Mo1] and, as Starbird showed in his Ph.D. Thesis [St], D_2^λ and $\underset{\sim}{I}^\lambda$ (though not D_λ^*) are test spaces (N) for (λ, ω)-paracompactness.

2. Infinite Products

We begin by establishing some useful notation. If $X = \Pi\{X_i : i \in I\}$, and $S \subseteq I$, define $X_S = \Pi\{X_i : i \in S\}$, and let $\pi_S : X \to X_S$ be the canonical projection. (However, we write X_i and π_i for $X_{\{i\}}$ and $\pi_{\{i\}}$, respectively.) And if $\kappa \geq \omega$, $(X)_\kappa$ denotes the product X retopologized by taking the set of all $\Pi\{V_i : i \in S\} \times X_{I \setminus S}$ as a base, where S ranges over $[I]^{<\kappa}$ and, for $i \in S$, V_i ranges over τX_i.

2.0. *Theorem.* [S3]. *Let* $X = \Pi\{X_\alpha : \alpha \in \kappa\}$ *be* (κ,ω)*-compact, and suppose that each* X_F $(F \in [\kappa]^{<\omega})$ *is orthocompact; then* X *is orthocompact.*

Proof. Let $\mathcal{V}$ be a basic open cover of X, and, for $V \in \mathcal{V}$, let $s(V) = \{\alpha \in \kappa : \pi_\alpha[V] \neq X_\alpha\}$. Also, for each $S \in [\kappa]^{<\omega}$, let $\mathcal{V}(S) = \{V \in \mathcal{V}: s(V) = S\}$, and let $W(S) = \cup\mathcal{V}(S)$. Then $\mathcal{W} = \{W(S): S \in [\kappa]^{<\omega}\}$ is an open cover of X of cardinality at most κ, so there is a finite $\mathcal{W}' = \{W(S_0), \cdots, W(S_n)\} \subseteq \mathcal{W}$ that covers X. Let $S = S_0 \cup \cdots \cup S_n$, and let $\mathcal{V}' = \mathcal{V}(S_0) \cup \cdots \cup \mathcal{V}(S_n)$, so that $\mathcal{V}'$ is a basic open cover of X, and $s(V) \subseteq S$ for any $V \in \mathcal{V}'$. Finally, let $\mathcal{H} = \{\pi_S[V]: V \in \mathcal{V}'\}$, a basic open cover of X_S.

Now, X_S is orthocompact, so $\mathcal{H}$ has a Q-refinement, $\mathcal{R}$, covering X_S. The desired Q-refinement of $\mathcal{V}$ can then be obtained as $\{R \times X_{\kappa\setminus S}: R \in \mathcal{R}\}$.

By taking $\kappa = \lambda^+$ (for some $\lambda \geq \omega$) and $X_\alpha = \kappa$ for each $\alpha \in \kappa$, we see that the compactness requirement on X in the previous theorem cannot be relaxed. Each $A \in [\kappa]^{\leq\lambda}$ has compact closure in κ, so $X = \kappa^\kappa$ is (λ,ω)-compact; but, since κ is obviously not (κ,ω)-compact, neither is X. By results of [S1], κ^n is orthocompact for all $n \in \omega$, so we might still hope that X is orthocompact. However, X contains a closed copy of $X \times D_2^\kappa$, so orthocompactness of X would imply (κ,ω)-metacompactness of X, by the remarks preceding 1.6; and it follows *mutatis mutandis* from the proof of the Arens-Dugundji Theorem that a (λ^+,ω)-metacompact, (λ,ω)-compact space is (λ^+,ω)-compact, which, as noted, X is not.

The argument used above to show that X is not orthocompact generalizes slightly to give the following result.

2.1. *Theorem.* [S3]. *If* $\kappa \geq \omega$, *and* X^κ *is orthocompact, then* X^κ *is* (κ,ω)*-metacompact.*

It would be nice to have some sort of approximate converse to 2.0. The best one could hope for is the following, which I optimistically state as a conjecture. (Compare with [No, Thm. 2.1].)

2.2. *Conjecture.* Let $X = \Pi\{X_\alpha: \alpha \in \kappa\}$ be orthocompact, where $\kappa > \omega$, and $|X_\alpha| \geq 2$ for each $\alpha \in \kappa$; then there is an $A \in [\kappa]^{<\omega}$ for which $X_{\kappa\setminus A}$ is (κ,ω)-compact.

(To see why 2.2 could not be improved by decreasing the size of A, let $X_n = E^1$ for $n \in \omega$, and let $X_\alpha = \omega$ for $\alpha \in \omega_1\setminus\omega$; it follows from 2.0 that $\omega_2{}^{\omega_1}$ is orthocompact, and thence from 1.7 that $E^\omega \times \omega_2{}^{\omega_1}$ is also orthocompact, although it is not even countably compact.) Unfortunately, I am unable to prove more than the following.

2.3. *Theorem.* [S3]. *Under the hypotheses of* 2.2, *there is an* $A \in [\kappa]^{<\omega}$ *such that* $X_{\kappa\setminus A}$ *is countably metacompact and, for each* $B \in [\kappa\setminus A]^{<\omega}$, X_B *is* (κ,ω)*-compact.*

Theorem 2.3 is too unsatisfactory to warrant giving its proof, which proceeds by throwing away "bad" elements of $[\kappa]^{<\omega}$; the next result is used to show that not too many factors are thrown out.

2.4. *Theorem.* [S3]. *If* $\kappa \geq \omega$ *is regular, then* $(D_\kappa^{\kappa^+})_\kappa$ *is not orthocompact.*

Proof. Let $X = (D_\kappa^{\kappa^+})_\kappa$, fix a point $p \in D$, and let $F = \{x \in X: \forall q \in D\setminus\{p\}(|\{\alpha \in \kappa^+: x(\alpha) = q\}| \leq 1)\}$, a closed subset of X; it suffices to show that F is not Q-embedded in X.

For each $\alpha \in \kappa^+$, let $V_\alpha = \{x \in X: x(\alpha) = p\}$, and let $\mathcal{V} = \{V_\alpha: \alpha \in \kappa^+\}$, an X-open cover of F. Also, for each $S \in [\kappa^+]^{<\kappa}$ and $x \in X$, let $B(x,S) = \{y \in X: y\restriction S = x\restriction S\}$, a basic, clopen nbhd of x in X. Finally, suppose that $\mathcal{R} = \{R_\alpha: \alpha \in \kappa^+\}$ is a precisely-indexed Q-refinement of $\mathcal{V}$ covering F.

Recursively construct a sequence, $\langle x_\xi: \xi \in \kappa\rangle$, of points of F as follows: Set $x_0(\alpha) = p$ for all $\alpha \in \kappa^+$, and choose $S_0 \in [\kappa^+]^{<\kappa}$ so that $B(x_0,S_0) \subseteq c(x_0,\mathcal{R})$. If $\eta \in \kappa$, and $x_\xi \in F$ and $S_\xi \in [\kappa^+]^{<\kappa}$ have been constructed for each $\xi < \eta$ in such a way that

(i) $\cup\{S_\zeta: \zeta < \xi\} \subseteq S_\xi$ if $\xi < \eta$;

(ii) $B(x_\xi,S_\xi) \subseteq c(x_\xi,\mathcal{R})$ if $\xi < \eta$;

(iii) $x_\xi\restriction S_\zeta = x_{\zeta+1}\restriction S_\zeta$ if $\zeta < \xi < \eta$; and

(iv) $x_\xi(\alpha) \neq p$ if $\xi < \eta$ and $\alpha \in \cup\{S_\zeta: \zeta < \xi\}$,

let $S'_\eta = \cup\{S_\xi: \xi < \eta\}$, and construct $x_\eta \in F$ as follows. If $\mathrm{cf}(\eta) \geq \omega$, simply define x_η by

$$x_\eta(\alpha) = \begin{cases} p, & \text{if } \alpha \in \kappa^+\setminus S'_\eta \\ x_{\xi+1}(\alpha), & \text{if there is a } \xi < \eta \text{ such that } \alpha \in S_\xi. \end{cases}$$

(By (iii), x_η is well-defined.) Otherwise, $\eta = \xi + 1$ for some $\xi \in \kappa$; choose any $x_\eta \in F$ such that $x_\eta(\alpha) = p$ if

$\alpha \in \kappa^+ \setminus S'_\eta$, $x_\eta(\alpha) = x_{\zeta+1}(\alpha)$ if there is a $\zeta < \xi$ such that $\alpha \in S_\zeta$, and $x_\eta \upharpoonright S'_\eta \colon S'_\eta \to D_\kappa \setminus \{p\}$ is 1-1. (This is possible because $|S'_\eta| < \kappa = |D_\kappa|$.) Note that $B(x_\eta, S'_\eta) \not\subseteq c(x_\eta, \mathcal{R})$: for if $x_\eta \in R_\alpha \in \mathcal{R}$, then $\alpha \in \kappa^+ \setminus S'_\eta$. Now choose $S_\eta \in [\kappa^+]^{<\kappa}$ so that $S'_\eta \subset S_\eta$ and $B(x_\eta, S_\eta) \subseteq c(x_\eta, \mathcal{R})$. Clearly (i)-(iv) are still satisfied with η replaced by $\eta + 1$, so the recursion goes through to κ.

Now let $S = \cup\{S_\eta \colon \eta \in \kappa\}$, and define $x \in F$ by

$$x(\alpha) = \begin{cases} p, & \text{if } \alpha \in \kappa^+ \setminus S \\ x_{\eta+1}(\alpha), & \text{if there is an } \eta \in \kappa \text{ for which } \alpha \in S_\eta. \end{cases}$$

Let $T \in [\kappa^+]^{<\kappa}$ be such that $B(x,T) \subseteq c(x,\mathcal{R})$, and let $T_0 = T \cap S$. Clearly $T_0 \subseteq S_\eta$ for some $\eta \in \kappa$; but then $x_{\eta+1} \in B(x,T) \subseteq c(x,\mathcal{R})$, so that $c(x_{\eta+1},\mathcal{R}) \subseteq c(x,\mathcal{R})$. Fix any $R_\alpha \in ST(x,\mathcal{R})$; clearly $B(x_{\eta+1}, S_{\eta+1}) \subseteq c(x_{\eta+1},\mathcal{R}) \subseteq R_\alpha$, so $\alpha \in S_{\eta+1}$. But $x(\alpha) = p$, whereas $x \upharpoonright S \colon S \to D_\kappa \setminus \{p\}$ and hence $\alpha \notin S \supseteq S_{\eta+1}$, which is the desired contradiction.

A pretty consequence of Theorem 2.4 is the following result (in which, as is well known, normality can replace orthocompactness throughout).

2.5. *Theorem.* [S3]. *For a space* X, *the following are equivalent:*

(a) X *is compact;*

(b) X^κ *is orthocompact for all* κ; *and*

(c) X^{κ_0} *is orthocompact, where* $\kappa_0 = \max\{\omega_1, w(X)\}$.

Proof. Obviously we need only show that (c) implies (a) and may assume that $|X| \geq 2$. Thus, X^{κ_0} contains a closed copy of $X^{\kappa_0} \times D_2^{\kappa_0}$, so that X is (κ_0,ω)-metacompact. But X is (∞,κ_0^+)-compact, so X is in fact metacompact.

By the Arens-Dugundji Theorem X is compact unless it contains a closed copy of D_ω, which is impossible: for then X^{κ_0} contains a closed copy of the non-orthocompact space $D_\omega^{\omega_1}$.

It is worth noting that in 2.4 and 2.5 orthocompactness can be replaced by normality. It suffices to do so in 2.4; the resulting theorem is due essentially to A. H. Stone [Sn] and Borges [B], and a proof can be found in [B].

3. Products of Linearly Ordered Topological Spaces

As usual, we refer to a linearly ordered topological space as a LOTS. Subspaces thereof are called suborderable spaces or GO-spaces (for "generalized orderable"). (Properly speaking, a GO-space is a triple, $\langle X,\leq,\mathcal{J}\rangle$, such that $\leq$ is a linear order on X and $\mathcal{J}$ is a topology on X having at each point a base of intervals of some kind. See [L] as a general reference for this section.) In [S1] was presented a proof, due to Lutzer, of a result of Fleischman, namely, that every LOTS is hereditarily orthocompact. Thus, in fact, every GO-space is normal, orthocompact, and countably paracompact hereditarily, a fact that will be of use to us in Section 4.

As was shown in [S2], locally compact LOTS's are particularly well-behaved; specifically, it was shown that the product of finitely many locally compact LOTS's is orthocompact iff it is normal. (More precisely, necessary and sufficient conditions were found for such a product to be orthocompact, conditions identical to those already found by Conover [C] for such a product to be normal. These conditions are too technical to justify reproducing them here.) This result extends *verbatim* to products of locally compact GO-spaces.

3.0. *Theorem. The product of finitely many locally compact* GO-*spaces is orthocompact iff it is normal.*

(The proof is immediate, given that every locally compact GO-space is a discrete union of locally compact LOTS's. This follows easily from the observation that in a locally compact GO-space, the points at which the topology is *not* induced by the order form a closed, discrete set; which observation is in turn a trivial consequence of the fact that a compact GO-space is a LOTS.)

It is necessary to assume local compactness in Theorem 3.0, as is seen by considering the product of the Michael Line and the irrationals, which, by results of Section 4, is orthocompact, though, as is well known, it is not normal [M]. Indeed, since the Michael Line can be embedded as a closed subspace of some LOTS [L], local compactness is a necessary assumption even for a product of LOTS's.

3.1. *Problem.* If X and Y are GO-spaces, and $X \times Y$ is normal, must $X \times Y$ be orthocompact?

I conclude this section with a result that by rights should have gone into [S2], from which the following definitions should be recalled.

3.2. *Definition.* A locally compact LOTS, $\langle X, \leq \rangle$, is a *fat* κ (for some $\kappa \geq \omega$) iff X contains a closed, cofinal copy of κ whose first element is the first element of X. If $\chi(x,X) < \lambda$ for all $x \in X$, X is said to be λ-*attenuated.*

3.3. *Theorem. Let* $X = \Pi\{X_\alpha : \alpha \in \lambda\}$ *be a product of fat* κ*'s. If* $\kappa > \omega$, *then* X *is orthocompact (normal, resp.) iff* $\lambda < \kappa$ *and each* X_α *is* κ*-attenuated. If* $\kappa = \omega$ *and* $\lambda > \omega$, *then* X *is neither orthocompact nor normal.*

Proof. The first assertion follows from Theorem 3.0, the analogous result of Noble [No] for normality, and the detailed form of the main result of [S2]; the second follows from Theorem 2.4 and the result of Borges [B] cited at the end of Section 3.

Other results concerning GO-spaces are more appropriately treated in the next section.

4. Products with a Metric-Like Factor

We begin by proving a special case of a theorem from [S3].

4.0. *Theorem.* [S3]. *Suppose that* X *is hereditarily orthocompact,* M *has a* σ*-point-finite base, and* $G \subseteq X \times M$ *is countably metacompact; then* G *is orthocompact.*

Proof. Let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \omega\}$ be a base for M such that each $\mathcal{B}_n$ is point-finite, and let $\mathcal{V}$ be a G-open cover of G, each member of which has the form $G \cap (W \times B)$ for some $W \in \tau X$

and $B \in \mathcal{B}$. For each $B \in \mathcal{B}$, let $\mathcal{W}(B) = \{W \in \tau X: G \cap (W \times B) \in \mathcal{V}\}$ (where for each member of $\mathcal{V}$ only one $W \in \tau X$ is chosen), and let $\mathcal{R}(B)$ be a Q-refinement of $\mathcal{W}(B)$ whose union is $\cup\mathcal{W}(B)$. For each $n \in \omega$, let $\mathcal{H}_n = \{G \cap (R \times B): B \in \mathcal{B}_n$ and $R \in \mathcal{R}(B)\}$, and note that $\mathcal{H}_n$ is a Q-collection in G. Thus, $\mathcal{H} = \cup\{\mathcal{H}_n: n \in \omega\}$ is a σ-Q-collection covering G which refines $\mathcal{V}$; it follows from [FL, Prop. 3.1] that G is orthocompact.

In the converse direction we have the following analogue of [RS, Thm. 1].

4.1. *Theorem. For any* $M \in \mathcal{M}$, *if* $X \times M$ *is orthocompact, then* $X \times M$ *is countably metacompact iff* X *is countably metacompact.*

Proof. If M is discrete, the result is trivial, so assume that M is not discrete. Since $M \in \mathcal{M}$, M has a base $\mathcal{B} = \cup\{\mathcal{B}_n: n \in \omega\}$ such that for each $n \in \omega$,

(a) $\mathcal{B}_n$ is point-finite and covers M, and

(b) $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$;

moreover, $\mathcal{B}$ can be so chosen that each finite $B \in \mathcal{B}$ is a singleton and

$$\bigcap_{n\in\omega} \cup \{B \in \mathcal{B}_n: |B| > 1\} = M'.$$

By Theorem 1.2, X is countably metacompact; suppose that $X \times M$ is not. Then [D] $X \times M$ has a strictly increasing open cover $\mathcal{U} = \{U_n: n \in \omega\}$ with no point-finite open refinement covering $X \times M$. If $y \in M \setminus M'$, let $\mathcal{V}(y) = \{U_n \cap (X \times \{y\}): n \in \omega\}$; using the countable metacompactness of X, let $\mathcal{R}(y)$ be a point-finite open refinement of $\mathcal{V}(y)$ covering $X \times \{y\}$.

If $n \in \omega$, $B \in \mathcal{B}_n$, and $|B| \geq \omega$, let $V(B) = (\bigcup\{W \in \tau X: W\times B \subseteq U_n\})\times B$, and let $\mathcal{V}_n = \{V(B): B \in \mathcal{B}_n \text{ and } |B| \geq \omega\}$. Let $\mathcal{V} = \bigcup\{\mathcal{V}_n: n \in \omega\}$; then $\mathcal{V}$ covers $X\times M'$. For if $\langle x,y\rangle \in X\times M'$, there is an $n \in \omega$ such that $\langle x,y\rangle \in U_n$, and there are a $W \in \tau X$, an $m \in \omega$, and an infinite $B \in \mathcal{B}_m$ such that $\langle x,y\rangle \in W\times B \subseteq U_n$. If $m \geq n$, then, since $U_n \subseteq U_m$, $\langle x,y\rangle \in W\times B \subseteq V(B) \in \mathcal{V}_m$, and if $m < n$, there is certainly a $k \geq n$ and an infinite $B_0 \in \mathcal{B}_k$ such that $y \in B_0 \subseteq B$, whence, as above, $\langle x,y\rangle \in V(B_0) \in \mathcal{V}$. Thus, $\mathcal{V} \cup \bigcup\{\mathcal{R}(y): y \in M\setminus M'\}$ is an open refinement of $\mathcal{U}$ covering $X\times M$ *which is already point-finite at each point of* $X\times(M\setminus M')$, and therefore there must be a precisely-indexed Q-refinement $\mathcal{R} = \{R(B): B \in \mathcal{B} \text{ and } |B| \geq \omega\}$ of $\mathcal{V}$ such that $\mathcal{R} \cup \bigcup\{\mathcal{R}(y): y \in M\setminus M'\}$ is a Q-cover of $X\times M$ refining $\mathcal{U}$.

Now, $\mathcal{R}$ cannot be point-finite, so there is a point $p = \langle x,y\rangle \in X\times M'$ such that $\operatorname{ord}(p,\mathcal{R}) \geq \omega$. For each $n \in \omega$, let $\mathcal{R}_n = \{R(B): V(B) \in \mathcal{V}_n\}$, and observe that $\mathcal{R}_n$ is point-finite; it follows that there is an $A \in [\omega]^\omega$ such that for each $n \in A$ there is a $B_n \in \mathcal{B}_n$ for which $p \in R(B_n)$. But $R(B_n) \subseteq X\times B_n$, so $p \in c(p,\mathcal{R}) \subseteq X\times \operatorname{Int}_M \bigcap\{B_n: n \in A\} = X\times\operatorname{Int}_M\{y\} = \emptyset$, (since $y \in M'$), a contradiction which establishes the theorem.

Theorem 4.1 can be generalized exactly as Przymusiński generalized [RS, Thm. 1] in [P].

4.2. *Theorem. Let* $M \in \mathcal{M}$, *and suppose that* $G \in \tau(X\times M)$ *is orthocompact. If for each* $y \in M\setminus M'$, $\{x \in X: \langle x,y\rangle \in G\}$ *is countably metacompact, then* G *is countably metacompact.*

4.3. *Corollary. If* $M \in \mathcal{M}'$, $G \in \tau(X \times M)$, *and* G *is orthocompact, then* G *is countably metacompact.*

4.4. *Corollary. If* $M \in \mathcal{M}$, $G \in \tau(X \times M)$, X *is hereditarily countably metacompact, and* G *is orthocompact, then* G *is countably metacompact.*

4.5. *Corollary. If* $M \in \mathcal{M}'$, *or if* $M \in \mathcal{M}$ *and* X *is hereditarily countably metacompact, and if* $X \times M$ *is hereditarily orthocompact, then* $X \times M$ *is hereditarily countably metacompact.*

Regarding the proofs of 4.2-4.4: 4.2 is proved almost exactly like 4.1 and immediately implies 4.3 and 4.4, which in turn imply 4.5. And by using 4.1 we can almost reverse the implications in 4.2-4.5.

4.6. *Corollary. In* 4.2-4.5, *if* X *is assumed to be hereditarily orthocompact, then the implications reverse.*

Since ([L],[Sl]) every GO-space is hereditarily both orthocompact and countably metacompact, we have the following useful special case.

4.7. *Corollary. If* X *is a* GO-*space,* $M \in \mathcal{M}$, *and* $G \in \tau(X \times M)$, *then* G *is orthocompact iff* G *is countably metacompact.*

For our last result of this kind we strengthen the requirements on X.

4.8. *Theorem.* *If* X *is hereditarily metacompact,* $M \in \mathcal{M}$, *and* $G \in \tau(X\times M)$, *then* G *is orthocompact iff* G *is countably metacompact iff* G *is metacompact.*

Proof. By 4.6, it suffices to show that metacompactness of G follows from countable metacompactness of G, which may be done by imitating the proof of 4.0.

4.2-4.8 are exact analogues of results in [P]; the corresponding theorems about normality may be obtained by the substitution throughout of $\mathcal{M}_0$ for $\mathcal{M}$, normal for orthocompact, and paracompact for metacompact. In the special case in which $G = X\times M$, Morita has proved [Mo2] a strong form of the analogue of 4.0: if $M \in \mathcal{M}_0$, $X\times M$ is countably paracompact, and X is normal, then $X\times M$ is normal. The following example shows that no such strengthening of the corresponding case of 4.0 is possible even if $M \subseteq E^2$ and X is regular and paracompact.

4.9. *Example.* Let $H_0 = \{\langle x,y\rangle \in E^2\colon y > 0$ and x and y are rational$\} \cup \{\langle x,0\rangle \in E^2\colon x$ is irrational$\}$, and let $\mathcal{U}$ be that topology on H_0 (described in [FL, Example 4.2]) making $\langle H_0,\mathcal{U}\rangle$ a non-orthocompact Moore space; $\mathcal{U}$ refines τH_0. Let p be any point not in H_0, and let $H = H_0 \cup \{p\}$. Finally, let $\mathcal{T}$ be the topology on H generated by the base $\mathcal{U} \cup \{X\backslash F\colon F$ is paracompact and clopen in $\langle H_0,\mathcal{U}\rangle\}$. $\langle H_0,\mathcal{U}\rangle$ is 0-dimensional and locally metrizable, so $\langle H,\mathcal{T}\rangle$ is 0-dimensional and Hausdorff, and clearly $\langle H,\mathcal{T}\rangle$ is paracompact. Thus, we now have the paracompact Hausdorff space $\langle H,\mathcal{T}\rangle$ and the metric subspace $\langle H_0,\tau H_0\rangle$ of E^2.

Let X be the product of $\langle H, \mathcal{J}\rangle$ and $\langle H_0, \tau H_0\rangle$. $H_0\times H_0$ is a closed subspace of X homeomorphic to $\langle H_0, \mathcal{U}\rangle$ (since $\mathcal{U} \supseteq \tau H_0$), so X is not orthocompact. However, X is countably metacompact. To see this, note first that $H_0\times H_0$ is an open, Moore subspace of X and as such is countably metacompact. Let $L = X\setminus(H_0\times H_0) = \{p\}\times H_0$. Any X-open cover of L can be refined to one consisting entirely of sets of the form $V\times W$, where $V \in \mathcal{J}$ and $W \in \tau H_0$. The sets W clearly cover the metric space $\langle H_0, \tau H_0\rangle$, so we may assume that they form a locally finite collection. Thus, any X-open cover of L has an X-open, locally finite refinement that still covers L, and it follows from the fact that $X\setminus L$ is open and countably metacompact that X is countably metacompact.

It is also not the case that the hypotheses of 4.1 imply that X is hereditarily orthocompact, even if M is the Cantor set. Take X to be $[(\omega_1+1)\times(\omega_1+1)]\setminus[\{\omega_1\}\times\omega_1]$; then, as is easily checked, X is orthocompact and countably compact. But X contains a copy of $\omega_1\times(\omega_1+1)$ (of which it is, in a natural way, the "one-point orthocompactification"), which ([S1]) is not orthocompact. Finally, that $X\times M$ is orthocompact follows from 1.5.

The above results all involve conditions on $X\times M$. By suitably conditioning X or M, however, we can get nice positive results.

4.10. *Theorem. Let* X *be a* GO-*space, and let* $M \in \mathcal{M}$*; then* $X\times M$ *is orthocompact and countably metacompact.*

Proof. By 4.7, it suffices to prove that $X\times M$ is countably metacompact. Moreover, Lutzer has shown in [L] how to embed a GO-space as a closed subspace of a LOTS, so we may assume that X is in fact a LOTS.

Let $\mathcal{B} = \cup\{\mathcal{B}_n: n \in \omega\}$ be such a base for M as was described in the proof of 4.1, and let $\mathcal{U} = \{U_n: n \in \omega\}$ be an open cover of X×M. For each $n \in \omega$ and $B \in \mathcal{B}$ let $V_n(B) = \cup\{W \in \tau X: W\times B \subseteq U_n\}$. For each $B \in \mathcal{B}$, let $\mathcal{V}(B) = \{V_n(B): n \in \omega\}$, and, X being hereditarily countably metacompact, let $\mathcal{R}(B) = \{R_n(B): n \in \omega\}$ be a precisely-indexed, point-finite open refinement of $\mathcal{V}(B)$ covering $\cup\mathcal{V}(B)$. For each $n \in \omega$, $\mathcal{B}_n$ is point-finite, so $\{R\times B: B \in \mathcal{B}_n \text{ and } R \in \mathcal{R}(B)\}$ is a point-finite open refinement of $\mathcal{U}$; moreover, the union over $n \in \omega$ of these collections covers X×M. For each $B \in \mathcal{B}$, let $R(B) = \cup\mathcal{R}(B)$, and, for $n \in \omega$, let $R_n = \cup\{R(B)\times B: B \in \mathcal{B}_n\}$. Let $\mathcal{R} = \{R_n: n \in \omega\}$; it is clear that it suffices to find a precisely-indexed, point-finite open refinement, $\mathcal{H} = \{H_n: n \in \omega\}$, of $\mathcal{R}$ covering X×M, since $\{H_n \cap (R_m(B)\times B): n,m \in \omega \text{ and } B \in \mathcal{B}_n\}$ will then be a point-finite open refinement of $\mathcal{U}$ covering X×M.

We shall construct each H_n to have the form $\cup\{H(B)\times B: B \in \mathcal{B}_n\}$, where, for $B \in \mathcal{B}_n$, H(B) is X-open and contained in R(B). Note that if $n < m < \omega$, $B_0 \in \mathcal{B}_n$, $B_1 \in \mathcal{B}_m$, and $B_1 \subseteq B_0$, then, by construction, $R(B_0) \subseteq R(B_1)$.

We begin by letting H(B) = R(B) for each $B \in \mathcal{B}_0$. We next set up some useful notation. For each $B \in \mathcal{B}$, let $\ell(B) = \inf\{n \in \omega: B \in \mathcal{B}_n\}$; and, for $B \in \mathcal{B}$ and $n < \ell(B)$, let $\mathcal{B}_n(B) = \{B_0 \in \mathcal{B}_n: B \subseteq B_0\}$, a finite set. Finally, for $B \in \mathcal{B}$ and $n < \ell(B)$, let $G_n(B) = \cup\{R(B_0): \exists i \leq n\ (B_0 \in \mathcal{B}_i(B))\}$, and note that for any $B \in \mathcal{B}\backslash\mathcal{B}_0$, $G_0(B) \subseteq \cdots \subseteq G_{\ell(B)-1}(B) \subseteq R(B)$.

Now suppose that $B \in \mathcal{B}\backslash\mathcal{B}_0$. $R(B)$ is an open subset of X, so it has a partition, $\mathcal{C}(B)$, into order-components, i.e., into maximal, convex, open subsets. Consider a $C \in \mathcal{C}(B)$. If C is an order-component of $G_{\ell(B)-1}(B)$ also, let $H(C) = \emptyset$, and if $C \cap G_{\ell(B)-1}(B) = \emptyset$, let $H(C) = C$. Otherwise, $G_{\ell(B)-1}(B)$ has an order-component, D, such that $D \subset C$. If D is an order-component of $G_0(B)$, let $H(C) = C$; otherwise, let m be the greatest integer less than $\ell(B)-1$ such that $D \cap G_m(B) \subset D$, and let $E = D \cap G_m(B)$. Note that E is either empty or an order-component of $G_m(B)$, and that in any case E is order-convex and $E \subset D \subset C$. Thus, there is an $H(C) \in \tau X$ such that $E \cap H(C) = \emptyset$ but $D \cup H(C) = C$. Having in this manner defined $H(C)$ for all $C \in \mathcal{C}(B)$, we let $H(B) = \cup\{H(C): C \in \mathcal{C}(B)\}$ and observe that $G_{\ell(B)-1}(B) \cup H(B) = R(B)$. This completes the construction of $\mathcal{H}$.

We next show that $\mathcal{H}$ covers $X\times M$. Let $\langle x,y\rangle \in X\times M$. There is certainly a $B \in \mathcal{B}$ with $\ell(B)$ minimal such that $\langle x,y\rangle \in R(B)\times B$. Let $C \in \mathcal{C}(B)$ be such that $x \in C$. If $x \in H(C)$, then clearly $\langle x,y\rangle \in \cup\mathcal{H}$, so suppose that $x \notin H(C)$. It follows from the construction that $G_{\ell(B)-1}(B)$ has an order-component, D, such that $x \in D \subset C$. But $x \in D$ implies that $x \in B_0$ for some $B_0 \in \mathcal{B}$ with $\ell(B_0) < \ell(B)$, which is impossible. Thus, $x \in H(C) \subseteq H(B)$, and $\mathcal{H}$ covers $X\times M$.

To complete the proof, we show that $\mathcal{H}$ is point-finite. If not, pick $p = \langle x,y \rangle \in X \times M$ such that $\mathrm{ord}(p,\mathcal{H}) = \omega$; since each $\mathcal{B}_n$ is point-finite, there must be a strictly increasing $\langle n_i : i \in \omega \rangle \in {}^{\omega}\omega$ such that for each $i \in \omega$ there is a $B_i \in \mathcal{B}_{n_i}$ with $p \in H(B_i) \times B_i$. Moreover, we may assume that $\ell(B_i) = n_i$ for each $i \in \omega$. (This would be impossible only if some $B_i = \{y\}$, in which case the family $\{H(B_i) \times B_i : i \in \omega\}$ would be finite.) Finally, since $\{B_i : i \in \omega\}$ is a local base at $y \in M'$, we may assume that $B_0 \supset B_1 \supset \cdots$.

For each $i \in \omega \setminus 2$, let D_i be the order-component of $G_{n_0+1}(B_i)$ containing x, if $x \in G_{n_0+1}(B_i)$, and let $D_i = \emptyset$ otherwise; similarly, let C_i be the order-component of $G_{n_i-1}(B_i)$ containing x, or $\emptyset$, according as $x \in G_{n_i-1}(B_i)$ or not. If, for some $i \in \omega \setminus 2$, $x \in D_i \subset C_i$, then $x \in R(B_0) \cap H(B_i) \subseteq G_{n_0+1}(B_i) \cap H(B_i) = \emptyset$, which is absurd, so, for all $i \in \omega \setminus 2$, either $D_i = \emptyset$, or $D_i = C_i$. Now, $x \in H(B_0) \subseteq R(B_0) \subseteq G_{n_0+1}(B_2)$, so $D_2 \neq \emptyset$; and since clearly $D_2 \subseteq D_3 \subseteq \cdots$, it follows that $x \in D_i = C_i$ for all $i \in \omega \setminus 2$. Moreover, $\mathrm{ord}(y,\mathcal{B}_{n_0+1}) < \omega$, so there is a $k \in \omega \setminus 2$ such that for all $i \in \omega \setminus k$, $G_{n_0+1}(B_i) = G_{n_0+1}(B_k)$, and hence also $D_i = D_k$. Let E be the order-component of $R(B_k)$ containing x; then $E \subseteq C_{k+1} = D_{k+1} = D_k = C_k$ (since $R(B_k) \subseteq G_{n_{k+1}-1}(B_{k+1})$). But clearly $C_k \subseteq E$, so $E = C_k$, and by construction $H(E) = \emptyset$, which is the desired contradiction.

4.11. *Theorem. If* X *is orthocompact and countably* *metacompact, and if* $M \in \mathcal{M}_0$ *is locally compact, then* $X \times M$ *is orthocompact and countably metacompact.*

Proof. If $K \subseteq M$ is compact, then, by 1.5, $X \times K$ is orthocompact. Since M is locally compact and paracompact, M has a locally finite cover, $\mathcal{C}$, by compact sets. Thus, $\{X \times K: K \in \mathcal{C}\}$ is a locally finite cover of $X \times M$ by closed, orthocompact subsets, and the result follows from [S2, Thm. 2.4].

Theorem 4.10 is the only example I can think of of a situation in which orthocompactness behaves better than normality: the product of the Michael line, which is a GO-space, and the irrationals is *not* normal [M].

In quite a different direction we have the following generalization of [S1, Thm. 2.3].

4.12. *Definition.* [S3]. Let $\langle T, \leq \rangle$ be a tree. A T-*cover* of a space X is a family $\mathcal{V} = \{V(t): t \in T\} \subseteq \tau X$ such that

(a) for every $s,t \in T$, $s \leq t$ implies that $V(s) \subseteq V(t)$;

(b) if t is a member of the α-th level of T, i.e., if $\{s \in T: s < t\}$ is well-ordered in type α by $\leq$, and α is a limit ordinal, then $V(t) = \bigcup\{V(s): s < t\}$; and

(c) there is a branch, β, through T such that $X = \bigcup\{V(t): t \in \beta\}$. (By a "branch through T" I mean a $\beta \subseteq T$ which is linearly ordered by $\leq$ and has the property that if $s < t \in \beta$, then $s \in \beta$, and there is no $t \in T$ such that $s < t$ for all $s \in \beta$.)

$\mathcal{V}$ is *shrinkable* iff there is a precisely-indexed, closed refinement, $\mathcal{F}$, of satisfying (a) and (b) above (with V replaced by F) and such that $\cup\{F(t): t \in \beta\} = X$ whenever β is a branch through T for which $\cup\{V(t): t \in \beta\} = X$. Finally, X is a *P(T)-space* iff every T-cover of X is shrinkable.

Note that every P(λ)-space (see [S1] for definition and references) is a P(T)-space for some tree T of height ω.

4.13. *Definition.* A space X is *non-Archimedean* iff it has a rank 1 base, i.e., a base, $\mathcal{B}$, such that if $B_0, B_1 \in \mathcal{B}$ and $B_0 \cap B_1 \neq \emptyset$, then either $B_0 \subseteq B_1$, or $B_1 \subseteq B_0$. It is easy to see that such a base can be indexed by the nodes of a tree, $\langle T, \leq \rangle$, in such a way that $B(t) \supseteq B(s)$ iff $t < s$. For any tree $\langle T, \leq \rangle$, $n\mathcal{A}(T)$ denotes the class of all non-Archimedean spaces having bases so indexed by T. (The class $\mathcal{Z}(\lambda)$ of [S1] is $n\mathcal{A}(T_\lambda)$, where T_λ has height ω and branches λ times at each node.)

4.14. *Theorem.* [S3]. *Let* $\langle T, \leq \rangle$ *be a tree, and let* X *be an orthocompact* P(T)*-space. Let* $Y \in n\mathcal{A}(T)$*; then* $X \times Y$ *is orthocompact.*

The proof of 4.14 is long, involved, tedious, and similar in principle to the proof of [S1, Thm. 2.3].

In the notation of [S1], it was there conjectured that in case $T = T_\lambda$ for some $\lambda \geq \omega$, the converse to 4.14 is also true. This conjecture is false, as may be seen from the following example.

4.15. *Example.* The Michael line, X is hereditarily paracompact [M]. Let Y be the irrationals; X and Y are both non-Archimedean, so, by a result of Nyikos [Ny], X×Y is hereditarily metacompact. Moreover, if $Z \in n\mathcal{A}(T_\omega)$, then $Z \subseteq Y$, so in fact X is $n\mathcal{A}(T_\omega)$-metacompact. But X×Y is not normal, so X is not a $P(T_\omega)$-space.

5. Mapping Theorems

It is not difficult to see that orthocompactness is not inversely preserved by perfect open maps: consider the projection to ω_1 in the product $\omega_1 \times D_2^{\omega_1}$. For some time, the question of "forward" preservation under a perfect map was unsettled. Recently, D. Burke [Bu] has given an example which shows that orthocompactness is not preserved by perfect maps. Here we prove a couple results which give some conditions under which orthocompactness is preserved.

5.1. *Theorem. Let* $f: X \to Y$ *be an open, finite-to-one map. If* X *is orthocompact, so is* Y.

Proof. Let $\mathcal{V} = \{V_\alpha : \alpha \in \kappa\}$ be an open cover of Y. For $\alpha \in \kappa$, let $W_\alpha = f^{-1}[V_\alpha]$, and let $\mathcal{W} = \{W_\alpha : \alpha \in \kappa\}$. Let $\mathcal{R} = \{R_\alpha : \alpha \in \kappa\}$ be a precisely-indexed Q-refinement of $\mathcal{W}$ covering X. For $y \in Y$, let $H(y) = \bigcap\{f[c(x,\mathcal{R})] : f(x) = y\}$, and let $\mathcal{H} = \{H(y) : y \in Y\}$; clearly $\mathcal{H}$ is an open cover of Y refining $\mathcal{V}$. Finally, if $z \in H(y) \in \mathcal{H}$, then, for each $x \in f^{-1}[\{y\}]$, there is an $x(z) \in c(x,\mathcal{R})$ such that $f(x(z)) = z$; but then $H(z) \subseteq \bigcap\{f[c(x(z),\mathcal{R})] : f(x) = y\}$ $\subseteq H(y)$, since $c(x(z),\mathcal{R}) \subseteq c(x,\mathcal{R})$. Thus for any $y \in Y$, $c(y,\mathcal{H}) = H(y)$, which is open, and $\mathcal{H}$ is a Q-cover.

5.2. *Theorem.*(*) *Let* $f: X \to Y$ *be a closed may with the property that* $D = \{y \in Y: |f^{-1}[\{y\}]| > 1\}$ *has ultraparacompact closure in* Y. *If* X *is orthocompact, then so also is* Y. (Recall that a space is ultraparacompact iff each open cover of it can be refined to a clopen partition of the space.)

Proof. Let $\mathcal{U}$ be an open cover of Y. By the ultraparacompactness of $cl_Y D$, $\mathcal{U}$ has an open refinement, $\mathcal{V}$, such that $ord(y,\mathcal{V}) = 1$ for each $y \in D$. Let $\mathcal{W} = \{f^{-1}[V]: V \in \mathcal{V}\}$, and let $\mathcal{R} = \{R(V): V \in \mathcal{V}\}$ be a Q-refinement of $\mathcal{W}$ covering X and so indexed that $R(V) \subseteq f^{-1}[V]$ for each $V \in \mathcal{V}$. For each $V \in \mathcal{V}$, let $H(V) = Y\setminus f[X\setminus R(V)]$, and let $\mathcal{H} = \{H(V): V \in \mathcal{V}\}$. Clearly $\mathcal{H} \subseteq \tau Y$, and $\mathcal{H}$ covers $Y\setminus D$.

For any $y \in D$, there is a unique $V_y \in \mathcal{V}$ containing y; thus, $R(V_y)$ is the only member of $\mathcal{R}$ to meet the fibre of y, so $y \in H(V_y)$, and $\mathcal{H}$ therefore covers all of Y.

Finally, fix $y \in Y$, and let $\mathcal{R}(y) = \{R \in \mathcal{R}: f^{-1}[\{y\}] \subseteq R\}$. Let $G = Y\setminus f[X\setminus \cap\mathcal{R}(y)]$; then G is open, and $y \in G$. But $G = Y\setminus f[\cup\{X\setminus R: R \in \mathcal{R}(y)\}] = Y\setminus \cup\{f[X\setminus R]: R \in \mathcal{R}\} = \cap\{Y\setminus f[X\setminus R]: R \in \mathcal{R}(y)\} = c(y,\mathcal{H})$, so $\mathcal{H}$ is a Q-collection.

Referneces

[B] C. J. R. Borges, *On a counterexample of A. H. Stone*, Quart. J. Math., Oxford (2) 20 (1969), 91-95.

[Bu] D. Burke, *Orthocompactness and perfect mappings*, Proc. Amer. Math. Soc. 79 (1980), 484-486.

[C] R. A. Conover, *Normality and products of linearly ordered spaces*, Gen. Top. Appl. 2 (1972), 215-225.

(*) Recently, Gary Gruenhage [G] has proven a strengthened version of Theorem 5.2 in which ultraparacompactness of $cl_Y D$ is replaced by paracompactness.

[vD] E. K. van Douwen, *A technique for constructing honest locally compact submetrizable examples* (preprint).

[D] C. H. Dowker, *On countably paracompact spaces*, Can. J. Math. 3 (1951), 219-224.

[FL] P. Fletcher and W. F. Lindgren, *Orthocompactness and strong Cech completeness in Moore spaces*, Duke Math. J. 39 (1972), 753-766.

[G] G. Gruenhage, *On closed images of orthocompact spaces*, Proc. Amer. Math. Soc. 77 (1979), 389-394.

[J] H. Junnila, *Covering properties and quasi-uniformities of topological spaces*, Doctoral Dissertation, VPI & SU, 1978.

[L] D. Lutzer, *On generalized ordered spaces*, Dissertationes Math. (= Rozprawy Mat.) 89 (1971), 1-39.

[M] E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. 69 (1963), 375-376.

[Mo1] K. Morita, *Paracompactness and product spaces*, Fund. Math. 50 (1961), 223-236.

[Mo2] ________, *Products of normal spaces with metric spaces II*, Sci. Rep. Tokyo Kyoiku Daigaku (A) 8 (1963), 87-92.

[N] K. Nagami, *Hereditary properties of product spaces*, Tôhoku Math. J. 21 (1969), 39-46.

[No] N. Noble, *Products with closed projections. II*, Trans. Amer. Math. Soc. 160 (1971), 169-183.

[Ny] P. Nyikos, *Notes on countable rank* (preprint).

[P] T. Przymusiński, *Normality and paracompactness in subsets of product spaces*, Fund. Math. 91 (1976), 161-165.

[RS] M. E. Rudin and M. Starbird, *Products with a metric factor*, Gen. Top. Appl. 5 (1975), 235-238.

[S1] B. M. Scott, *Toward a product theory for orthocompactness*, Studies in Topology (N. M. Stavrakas and K. R. Allen, ed.), Academic Press, New York, 1975, 517-537.

[S2] ________, *Orthocompactness is normality in finite products of locally compact* LOTS*'s*, Set-Theoretic Topology (G. M. Reed, ed.), Academic Press, New York, 1977, 339-348.

[S3] ________, *The behavior of orthocompactness in products, with emphasis on certain analogies with that of normality under similar circumstances*, Doctoral Dissertation, University of Wisconsin-Madison, 1975.

[S4] ________, *Some 'almost-Dowker' spaces*, Proc. Amer. Math. Soc. 68 (1978), 359-364.

[St] M. Starbird, *The normality of products with a compact or a metric factor*, Doctoral Dissertation, University of Wisconsin-Madison, 1974.

[Sn] A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. 54 (1948), 977-982.

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