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# Research Announcement: HOMOGENOUS HEREDITARILY INDECOMPOSABLE CONTINUA

by

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*Research Announcement*

# HOMOGENOUS HEREDITARILY INDECOMPOSABLE CONTINUA

**Wayne Lewis**

The point and the pseudo-arc are the only known examples of homogeneous hereditarily indecomposable continua. Both of these continua are planar.

It is known that any other example must be either one-dimensional (though not necessarily planar) or infinite-dimensional. In 1951, R. H. Bing [2] showed that there exist hereditarily indecomposable continua of every dimension. He observed that if M is an n-dimensional ($n \in Z^+$) hereditarily indecomposable continuum, then M contains a point p such that every non-degenerate subcontinuum containing p is n-dimensional. Since M also contains one-dimensional subcontinua, any non-degenerate, finite-dimensional, homogeneous, hereditarily indecomposable continuum must be one-dimensional.

By Bing's work, there exist strongly infinite-dimensional hereditarily indecomposable continua. Rubin, Schori, and Walsh [23] have shown that every strongly infinite-dimensional compactum contains an hereditarily strongly infinite-dimensional compactum. Thus there exist hereditarily strongly infinite-dimensional, hereditarily indecomposable continua. It is not known if such a continuum can be homogeneous.

*Question* 1. Does there exist an hereditarily indecomposable, homogeneous continuum which is infinite-dimensional?

To date, very little work has been done on higher dimensional, especially infinite-dimensional, hereditarily indecomposable continua. The only obvious restrictions on homogeneity are those imposed by composant considerations.

For one-dimensional, homogeneous, hereditarily indecomposable continua some more specific constraints are known. J. T. Rogers, Jr. [22] has recently shown that any separating, homogeneous plane continuum is decomposable. His argument does use separation properties of the plane, and it is not clear how or if it can in general be extended to show that homogeneous, hereditarily indecomposable curves are acyclic, or even tree-like.

*Question* 2. Is every homogeneous, hereditarily indecomposable curve acyclic? Tree-like?

F. B. Jones [12] has shown that any decomposable, homogeneous plane continuum has a continuous decomposition into mutually homeomorphic, homogeneous, non-separating plane continua. He has also shown [13] that any non-separating, homogeneous plane continuum is indecomposable, and he [14] and Charles Hagopian [10] have shwon that any indecomposable homogeneous plane continuum is hereditarily indecomposable. Thus, if there is another homogeneous plane continuum besides the four known examples, then there is one which is hereditarily indecomposable and tree-like. Since Bing [3] has shown that every nondegenerate chainable homogeneous continuum is a pseudo-arc, any other homogeneous plane continuum must be non-chainable.

*Question* 3. (Cook) Does there exist a (planar) non-chainable, homogeneous, (hereditarily indecomposable) tree-like continuum?

Many of the candidates [4], [7], [9] presented for other examples of homogeneous curves (both planar and non-planar) have the property that every non-degenerate proper subcontinuum is a pseudo-arc. This implies that the continuum is almost chainable. The author has recently shown [17] that almost chainable, homogeneous continua are chainable.

This means [6] that if M is a non-chainable, hereditarily indecomposable, homogeneous curve then there cannot be a bound on the number of junction links of one-dimensional covers of M of sufficiently small mesh (or, equivalently, if M is written as an inverse limit of finite graphs, there cannot be a bound on the number of vertices of the graphs of order greater than two).

Thus, any non-chainable, homogeneous, hereditarily indecomposable curve M must contain many non-chainable proper subcontinua. In fact, there must be a continuous decomposition of M into such subcontinua. Also, if M contains any non-degenerate chainable subcontinua then there is a continuous decomposition of M into maximal pseudo-arcs with the decomposition space $\tilde{M}$ also being hereditarily indecomposable and homogeneous. If the answer to the next question is no, then M would contain no chainable subcontinua.

*Question* 4. If M is an hereditarily indecomposable contiuum with a continuous decomposition into pseudo-arcs such that the decomposition space is a pseudo-arc, must M itself be a pseudo-arc?

If M is a planar, non-chainable, hereditarily indecomposable, homogeneous continuum, then it would contain $c = 2^{\omega_0}$ disjoint mutually homeomorphic non-chainable proper subcontinua. While Tom Ingram [11] has constructed $c = 2^{\omega_0}$ disjoint, non-chainable, hereditarily indecomposable tree-like continua in the plane, his continua are not mutually homeomorphic, and it is not known if there exist uncountably many disjoint copies of a single such continuum in the plane.

*Question* 5. (Bing [4]) If M is an atriodic tree-like continuum in the plane, does there exist an uncountable collection of mutually exclusive continua in the plane, each member of which is homeomorphic to M?

The following question is related and is suggested by known results [19] about the point and the pseudo-arc.

*Question* 6. If M is a homogeneous, non-separating plane continuum, does there exist a continuous collection of continua, each homeomorphic to M, filling up the plane?

The author earlier showed [18] that the pseudo-arc admits stable homeomorphisms other than the identity. While the question as asked by Bing [5] was initially motivated by consideration of hereditary indecomposability, the construction developed by the author used the fact that proper subcontinua of the pseudo-arc are homogeneous. The relation between homogeneity and the existence of stable homeomorphisms other than the identity has taken on renewed interest because Judy Phelps [21] has recently shown that every 2-homogeneous continuum is representable (implying n-homogeneity for every n, as well as other nice properties) if it admits a stable homeomorphism other than the identity. Neither the author nor Phelps knows of a homogeneous continuum which does not admit such a homeomorphism, and she has asked the following question.

*Question* 7. (Judy Phelps [21]) Does every non-degenerate homogeneous continuum admit a stable homeomorphism other than the identity?

For different reasons, an answer to the above question would be of interest in the restricted case of continua which are either 2-homogeneous or hereditarily indecomposable.

Of course, because of composant restrictions, an indecomposable continuum cannot be 2-homogeneous. However, taking composant restrictions into account, the pseudo-arc comes as close to being n-homogeneous for each n as one could reasonably expect. G. R. Lehner [16] has shown that if $\{x_1, x_2, \cdots, x_n\}$ are points of the pseudo-arc all in different composants then there exists a homeomorphism h of the pseudo-arc onto itself with $h(x_i) = y_i$ for each $1 \leq i \leq n$. One might call this composant-wise strong n-homogeneity. (Lehner's results are actually somewhat stronger than this.) It is natural to ask whether every homogeneous, hereditarily indecomposable continuum has this property.

*Question* 8. Is every homogeneous, hereditarily indecomposable continuum composant-wise strongly n-homogeneous (in the above sense)?

The question is also of interest if one allows some of the points to be in the same composant, though then one cannot always preserve indexing.

In 1955, F. B. Jones [15] asked what effect hereditary equivalence has on homogeneity. Though few specific results along this line are known, work in both areas suggests a relation between the two properties. Both the point and the pseudo-arc are hereditarily equivalent. The arc is the only other hereditarily equivalent continuum known, and Howard Cook [8] has shown that any additional such continuum must be hereditarily indecomposable and tree-like.

*Question* 9. (F. B. Jones [15]) What effect does hereditary equivalence have on homogeneity in continua?

**References**

1. R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. 15 (1948), 729-742.
2. ________, *Higher-dimensional hereditarily indecomposable continua*, Trans. Amer. Math. Soc. 71 (1951), 267-273.
3. ________, *Each homogeneous nondegenerate chainable continuum is a pseudo-arc*, Proc. Amer. Math. Soc. 10 (1959), 345-346.

4. ________, *Snake-like continua*, Duke Math. J. 18 (1951), 653-663.

5. ________, *The pseudo-arc*, Summary of lectures and seminars, Summer Institute on Set-Theoretic Topology, A.M.S., Madison (1955), 72-75.

6. C. E. Burgess, *Homogeneous continua which are almost chainable*, Canad. J. Math. 13 (1961), 519-528.

7. H. Cook, *Concerning three questions of Burgess about homogeneous continua*, Colloq. Math. 19 (1958), 241-244.

8. ________, *Tree-likeness of hereditarily equivalent continua*, Fund. Math. 68 (1978), 203-205.

9. L. Fearnley, *Hereditarily indecomposable circularly chainable continua*, Ph.D. Thesis, Univ. of London, 1970.

10. C. L. Hagopian, *Indecomposable homogeneous plane continua are hereditarily indecomposable*, Trans. Amer. Math. Soc. 224 (1976), 339-350.

11. W. T. Ingram, *Hereditarily indecomposable tree-like continua*, Fund. Math. 103 (1979), 61-64.

12. F. B. Jones, *On a certain type of homogeneous plane continuum*, Proc. Amer. Math. Soc. 6 (1955), 735-740.

13. ________, *Certain homogeneous unicoherent indecomposable continua*, Proc. Amer. Math. Soc. 2 (1951), 855-859.

14. ________, *Homogeneous plane continua*, Proc. Auburn Topology Conf., Auburn University, 1969.

15. ________, *On homogeneity*, Summary of lectures and seminars, Summer Institute on Set-Theoretic Topology, A.M.S., Madison (1955), 68-70.

16. G. R. Lehner, *Extending homeomorphisms on the pseudo-arc*, Trans. Amer. Math. Soc. 98 (1961), 369-394.

17. W. Lewis, *Almost chainable homogeneous continua are chainable* (to appear in Houston J. Math.).

18. ________, *Stable homeomorphisms of the pseudo-arc*, Canad. J. Math. 31 (1979), 363-374.

19. ________ and J. J. Walsh, *A continuous decomposition*

*of the plane into pseudo-arcs*, Houston J. Math. 4 (1978), 209-222.

20. E. E. Moise, *A note on the pseudo-arc*, Trans. Amer. Math. Soc. 64 (1949), 57-58.

21. J. K. Phelps, *A condition under which 2-homogeneity and representability are the same in continua* (to appear in Fund. Math.).

22. J. T. Rogers, Jr., *Homogeneous separating plane continua are decomposable* (preprint).

23. L. Rubin, R. M. Schori, and J. J. Walsh, *New dimension-theory techniques for constructing infinite-dimensional examples* (preprint).

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