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Research Announcement:

ON A CLASSIFICATION OF WEAKLY INFINITE-DIMENSIONAL COMPACTA

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Research Announcement

ON A CLASSIFICATION OF WEAKLY INFINITE-DIMENSIONAL COMPACTA

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1. Introduction

In this paper, the spaces we consider will be metrizable, and by a compactum we shall mean a compact space.

After the classical dimension theory of subsets of Euclidean spaces was set up, and such excellent expositions of the theory as Hurewicz's and Wallmans's "Dimension Theory" and K. Kuratowski's "Topology" appeared, interest arose in the classification of the spaces to which the dimension function assigned the value ∞.

We shall discuss here some results and problems in this topic, restricting ourselves to the class of compacta; the details will appear in [P1]. As one can expect when dealing with transfinite classification of compacta, analytic sets theory provides an appropriate tool for investigations. It seems to us, that what makes the subject interesting is a certain interplay between the classical concepts of the theory of analytic sets and the concepts which are based upon dimension theory.

To begin, let us recall the definitions and some basic facts about countable-dimensional and weakly infinite-dimensional compacta--two important classes of infinite-dimensional compacta defined explicitly for the first time

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[1]This paper was written while the author was a Visiting Assistant Professor at the University of Washington.

by P. S. Aleksandrov in an expository paper [A] in 1951.

2. Countable-Dimensional Spaces and Transfinite Dimensions

A space is *countable-dimensional* if it is a union of countably many zero-dimensional sets; this class includes all finite-dimensional spaces, since each such space is a finite union of zero-dimensional sets.

The transfinite dimensions ind and Ind are the ordinal valued functions obtained through the extension by transfinite induction of the classical notions of small or large transfinite dimension respectively (i.e. for example, Ind $X \leq \alpha$ if for each pair of closed subsets A, B of X there is a partition L in X between A and B with Ind $X < \alpha$, α being an ordinal); the values of the transfinite dimensions considered in the class of separable metrizable spaces are always countable ordinals. For a comprehensive survey of the topic we refer the reader to [E].

The transfinite dimensions were first considered by W. Hurewicz who proved that for a complete space X the transfinite dimension ind is defined if and only if X is countable-dimensional. There exists a function ϕ which maps the set of countable ordinals into itself such that for each countable-dimensional compactum X ind $X \leq$ Ind $X \leq$ ϕ(ind X). Thus, although L. A. Luxemburg [L] has shown that the exact relations between transfinite dimensions ind and Ind are very interesting, globally both of the functions provide essentially the same classification of countable-dimensional compacta.

3. Weakly Infinite-Dimensional Compacta

A continuous map $f: X \to I^{\omega}$ of a compactum X onto the Hilbert cube I^{ω} is *essential* if for each n the composition $p_n \circ f: X \to I^n$ of the map and the projection $p_n: I^{\omega} \to I^n$ onto the n-dimensional cube is essential (recall, that $g: X \to I^n$ is essential if each continuous extension of $g|g^{-1}(\partial I^n)$ over X maps X onto I^n).

A compactum X is *strongly infinite-dimensional* if it has an essential map onto the Hilbert cube, and it is *weakly infinite-dimensional* if no such a map exists.

Hurewicz proved that countable-dimensional compacta are weakly infinite-dimensional, and, on the other hand, there are quite natural examples of weakly infinite-dimensional compacta which are not countable-dimensional [P2].

4. The Lusin-Sierpinski Index of a Weakly Infinite-Dimensional Compactum

In this section we shall describe a certain natural classification of weakly infinite-dimensional compacta which turns out to be closely related to a classical notion of a Lusin-Sierpinski index.

Let Fin ω be the set of all finite non-empty subsets of the set ω of natural numbers endowed with the Brouwer-Kleene order $\prec$, i.e. $\sigma \prec \tau$ means that there is an $n \in \omega$ such that $\sigma \cap \{1,\cdots,n-1\} = \tau \cap \{1,\cdots,n-1\}$ and $n \in \sigma\backslash\tau$.

Given a continuous map $f: X \to I^{\omega}$ of the compactum X into the Hilbert cube let us put

$$(*)\ M(f) = \{\sigma \in \text{Fin}\ \omega: p_{\sigma} \circ f: X \to I^{\sigma} \text{ is essential}\},$$

where p_σ: $I^\omega \to I^\sigma$ is the projection. By type M(f) we shall denote the order type of the set M(f) ordered by $\prec$.

Lemma 4.1. *A compactum* X *is weakly infinite-dimensional if and only if for each continuous map* f: X $\to I^\omega$ *the set* M(f) *is well-ordered, further if* X *is weakly infinite-dimensional then the set of countable ordinals* $\mathcal{M}$(X) = {type M(f): f: X $\to I^\omega$} *has a largest element.*

Given a weakly infinite-dimensional compactum X let us denote by index (X) the largest ordinal in the set $\mathcal{M}$(X) defined in the above lemma; evidently, index (X) is a topological invariant.

To interpret the index from the point of view of the theory of analytic sets let us denote by $\underline{H}$ the hyperspace of the Hilbert cube and let $\underline{S}$ be the set of all strongly infinite-dimensional compacta in I^ω. Let us choose further an arbitrary family $\{(A_i,B_i): i \in \omega\}$ of pairs of closed disjoint subsets of I^ω such that for each pair (A,B) of closed disjoint sets in I^ω there are infinitely many indices i for which $A \subset A_i$ and $B \subset B_i$, and finally, let us put, for each $\sigma \in$ Fin ω, $\underline{W}_\sigma = \{X \in \underline{H}$: if L_i is a partition in I^ω between A_i and B_i then $\bigcap\{L_i: i \in \sigma\} \cap X \neq \emptyset\}$. In this way we have defined a closed Lusin sieve $\underline{W} = \{\underline{W}_\sigma: \sigma \in$ Fin $\omega\}$ in the hyperspace $\underline{H}$. Let us recall that the set L($\underline{W}$) sifted by the sieve $\underline{W}$ consists of the points X such that the set $M(X) = \{\sigma: X \in \underline{W}_\sigma\}$ is not well-ordered, and that for each $X \notin L(\underline{W})$ the Lusin-Sierpinski index of X is the order type of the set M(X). One can easily check that $L(\underline{W}) = \underline{S}$ and that the Lusin-Sierpinski index of a compactum $X \notin L(\underline{W})$ coincides with the topological invariant index (X).

In particular, from the general properties of Lusin-Sierpinski indices, it follows that the index is bounded over each analytic set $\underline{A}$ in $\underline{H}$ disjoint from $\underline{S}$ which yields easily the following fact.

Proposition 4.2.

(A) $\sup\{\text{index }(X)\colon \text{ind } X \leq \alpha\} < \omega_1$.

(B) If $\underline{D}$ *is an upper semi-continuous decomposition of a compactum* X *into weakly infinite-dimensional compacta, then* $\sup\{\text{index }(S)\colon S \in \underline{D}\} < \omega_1$.

We shall see later in this paper that ind can be unbounded over a set of countable-dimensional compacta with bounded index.

We do not know how regular the transfinite dimensions are from the point of view of the descriptive set theory: is the set $\underline{C}$ of all countable-dimensional compacta in $\underline{H}$ a $\underline{CA}$ (i.e. coanalytic) set (it is a $\underline{PCA}$-set), or are the transfinite dimensions bounded over each analytic set $\underline{A} \subset \underline{C}$? This question seems to be of interest also because, as R. D. Mauldin kindly pointed out to the author, the transfinite dimensions provide quite natural examples of so-called monotone inductive operators investigated in descriptive set theory [C-M].

The second part of the above question can be equivalently stated in the following way:

Question 4.3. Let $\underline{D}$ be an upper semi-continuous decomposition of a compactum X into countable-dimensional compacta. Is it true that

$$\sup\{\text{ind } S: S \in \underline{D}\} < \omega_1?$$

Remark 4.4. It was observed in [P2] that there exists a non-countable-dimensional compact space X and a continuous map $f: X \to C$ onto the Cantor set such that all fibers $f^{-1}(t)$ are countable-dimensional. One can prove that in this example $\sup\{\text{ind} f^{-1}(t): t \in C\} < \omega_1$; this and some other results related to Question 4.3 will be published elsewhere.

5. On Two Questions of D. W. Henderson

Finite-dimensional spaces can be characterized by means of the essential maps onto finite-dimensional cubes. It seems to us rather unlikely that there exists an adequate notion of essential maps for transfinite dimensions, but we don't know any argument which would allow us to replace the phrase "rather unlikely that there exists" by the phrase "there is no." But whatever the situation is, we think that it is an interesting and stimulating idea to extend the notion of essential maps beyond the class of finitely-dimensional spaces and to use such a notion for classification of infinitely-dimensional compacta (we actually considered the essential maps onto the Hilbert cube). In this section we discuss a natural concept of essential maps onto "transfinite cubes," due to D. W. Henderson [H].

Henderson defined AR-compacta $H_1, H_2, \cdots, H_\alpha, \cdots, \alpha < \omega_1$ and their "boundaries" ∂H_α by transfinite induction as follows: let H_1 be the unit interval I, $\partial H_1 = \partial I = \{0,1\}$, $p_1 = \{0\}$ and assume that for all $\beta < \alpha$ the compacta H_β, their "boundaries" ∂H_β, and the points $p_\beta \in \partial H_\beta$ are defined. If $\alpha = \beta + 1$ then we let $H_{\beta+1} = H_\beta \times I$, $\partial H_{\beta+1} = (\partial H_\beta \times I) \cup (H_\beta \times \partial I)$, and $p_{\beta+1} = (p_\beta, p_1)$. If α is limit, let K_β be the union of H_β and a half-open arc A_β such that $A_\beta \cap H_\beta = \{p_\beta\} = \{$the end point of $A_\beta\}$, and define H_α to be the one-point compactification of the union $\bigoplus_{\beta<\alpha} K_\beta$, $\partial H_\alpha = H_\alpha \setminus \bigcup_{\beta<\alpha} (H_\beta \setminus \partial H_\beta)$ and let p_α be the compactifying point.

Henderson called a continuous map $f\colon X \to H_\alpha$ essential if each continuous extension of $f|f^{-1}(\partial H_\alpha)$ over X maps X onto H_α (notice that for $\alpha < \omega$, H_α is the α-dimensional cube I^α, ∂H_α is the boundary of I^α, and the notion of the essential maps coincides with the classical one).

Henderson proved that if a countable-dimensional compactum X has an essential map onto H_α then Ind $X \geq \alpha$.

Theorem 5.1. *If a weakly infinite-dimensional compactum* X *admits an essential map onto* H_α *then index* (X) $\geq \alpha$. *In particular, a compactum which has an essential map onto each* H_α *is strongly infinite-dimensional.*

The second part of Theorem 5.1 answers affirmatively a question raised by Henderson in [H]; the next theorem answers negatively another question from [H].

Theorem 5.2. *There exists a compactum* S *which is a* *countable disjoint union of finite polytopes such that* Ind S = α *but* S *does not have any essential map onto* H_α.

The proof of this theorem, which we sketch below, essentially uses a theorem of Hurewicz which is based upon an existence of an analytic set which is not Borel. As a result, the image of the compactum S whose existence we prove is quite vague and a more explicit construction would be of interest. In particular we don't know what the smallest α in the theorem is, or whether it is possible to construct a compactum X with Ind X = α + 1 such that X has an essential map onto H_α but doesn't have any essential map onto $H_{\alpha+1}$ (cf. also Remark 4.4).

The reasoning we are going to present shows that in fact there exists a countable ordinal λ such that for each $\alpha > \lambda$ there exists a compactum S (which is a countable disjoint union of finitely-dimensional compacta) such that Ind S = α but S doesn't have any essential map onto H_λ.

Let X be a weakly infinite-dimensional compactum which is not countable-dimensional (see sec. 3) and let $f_i\colon X \to W_i$ be a 1/i-map (i.e. the fibers $f_i^{-1}(y)$ have diameter less than 1/i) onto a polyhedron W_i. Let I be the unit interval and let $q_1, q_2, \cdots$ be the enumeration of the set Q of rational numbers from I. Let Z be the compactum obtained from the product I × X by attaching to each set $\{q_i\} \times X$ the polyhedron W_i by the map f_i and let p: Z → I be the natural "projection." Then we have

(1) $$p^{-1}(q_i) \underset{\text{top}}{=} W_i \qquad \text{for} \qquad i = 1,2,\cdots,$$

(2) $p^{-1}(t) \underset{\text{top}}{=} X$ for $t \in I \backslash Q$.

Let us denote by $\underline{I}$ the hyperspace of the interval, let $\underline{Q} = \{T \in \underline{I}: T \subset Q\}$, and let $\underline{Z}$ be the hyperspace of Z. For each $T \in \underline{I}$ let us put

(3) $X(T) = p^{-1}(T)$

Since, as one can easily check, Z is weakly infinite-dimensional, for each $T \in \underline{I}$ we have

(4) $\text{index}\ (X(T)) \leq \text{index}\ (Z) < \omega_1$

Let us prove that

(5) $\sup\{\text{Ind}\ X(T): T \in \underline{Q}\} = \omega_1$

(notice that if $T \in \underline{Q}$ then X(T) is a disjoint union of countably many polytopes $p^{-1}(q_i)$, cf. (1)). One can easily verify that the sets $\underline{Z}_\alpha = \{S \in \underline{Z}: \text{Ind}\ S \leq \alpha\}$ are analytic and thus for each α the set $\underline{A}_\alpha = \{T \in \underline{I}: X(T) \in \underline{Z}_\alpha\}$ is analytic, because $\underline{A}_\alpha$ is the inverse image of the set $\underline{Z}_\alpha$ under the Borel function $T \to X(T)$. Since X is not countable-dimensional, it follows from (2) that $\underline{A}_\alpha \subset \underline{Q}$ for each $\alpha < \omega_1$, and hence by (1) we have $\bigcup_{\alpha<\omega_1} \underline{A}_\alpha = \underline{Q}$. To prove (5) it is now enough to recall a classical Hurewicz's theorem which says that $\underline{Q}$ is not an analytic set, and thus $\underline{A}_\alpha \neq \underline{Q}$ for each $\alpha < \omega_1$.

Having proven (5) let us choose $T \in \underline{Q}$ such that $\text{Ind}\ X(T) > \text{index}(Z)$ and let us put $S = X(T)$. By (4) we see that index S < Ind S and we are done by Theorem 5.1.

6. Universal Functions for the Families of Compacta With Transfinite Dimension $\leq \alpha$

A classical idea of universal functions for a given family of sets can be applied to prove the following result.

Theorem 6.1. *For each* $\alpha < \omega$, *there exists a continuous function* $\Phi_\alpha\colon \omega^\omega \to \underline{H}$ *of the irrationals into the hyperspace of the Hilbert cube such that:*

(i) if $X \in \underline{H}$ *and* $\operatorname{ind} X \leq \alpha$ *then* $X = \Phi_\alpha(t)$ *for some* $t \in \omega^\omega$,

(ii) $\operatorname{ind} G_\alpha = \alpha$, *where* $G_\alpha = \{(t,x)\colon x \in \Phi_\alpha(t)\}$.

Speaking briefly, the idea of the proof is as follows: we construct the functions Φ_α by transfinite induction in such a way that given the functions Φ_β for $\beta < \alpha$ we define Φ_α with the property that there exists a collection of partitions in the space G_α which determines the dimension of G_α such that each of the partitions is parametrized by a function Φ_β.

One can prove also a counterpart to the theorem for the transfinite dimension Ind. Since each G_α is a complete space there exists a compactum X_α such that the remainder $X_\alpha \setminus G_\alpha$ is countable-dimensional [E; 4.15]. Thus we obtain the following corollary (analogous fact is also true if ind is replaced by Ind).

Corollary 6.2. *For each* $\alpha < \omega_1$ *there exists a countable-dimensional compactum* X_α *which contains topologically all compacta* S *with* $\operatorname{ind} S \leq \alpha$.

We don't know what the smallest possible transfinite dimension ind of such a compactum X_α is.

Remark 6.3. There is a continuous map $\phi\colon C \to \underline{H}$ of the Cantor set into the hyperspace $\underline{H}$ such that each $\Phi(t)$

is countable-dimensional but the space $G(\Phi) = \{(t,x): x \in \Phi(t)\}$ is not countable-dimensional.

To define such a map Φ let us modify the map $f: X \to C$ described in Remark 4.4. Using an idea of E. Michael and A. H. Stone [M-S; proof of Theorem 1.1] one can construct a compactum $Z \supset X$ with $\dim(Z\backslash X) = 0$ and an open extension $g: Z \to C$ of the map f, and then it suffices to let $\Phi(t) = g^{-1}(t)$, since $G(\Phi)$ is homeomorphic to Z. Moreover, there exists $\alpha < \omega_1$ such that Φ satisfies the condition $\operatorname{ind} \Phi(t) \leq \alpha$ for all t; see Remark 4.4.

Let us mention also that for each $\alpha < \omega_1$ one can define a continuous map $\Phi: C \to \underline{H}$ such that each $\Phi(t)$ is finitely-dimensional but $\operatorname{ind} G(\Phi) \geq \alpha$; this follows immediately from a construction of Yu. M. Smirnov [S].

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