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PROBLEM SECTION

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PROBLEM SECTION

CONTRIBUTED PROBLEMS

Problems followed by the notation [SETOP] were presented at the problem sessions at the SETOP symposium which was held at Erindale College, University of Toronto, in July and August of 1980. The other problems, some of which were also posed at SETOP, are related to articles in this volume.

A. Cardinal Invariants

9. (*van Douwen*) If G is an infinite countably compact group, is $|G|^\omega = |G|$?

10. (*van Douwen*) Is the character, or hereditary Lindelöf degree, or spread, equal to the weight for a compact F-space? for a compact basically disconnected space? (Yes for compact extremally disconnected spaces by a result of Balcar and Franek.)

B. Generalized Metric Spaces and Metrization

19. (*Zhou*) A space X is said to have a small diagonal if any uncountable subset of $X^2-\Delta$ has an uncountable subset with closure disjoint from the diagonal. Must a compact T_2-space with a small diagonal be metrizable? [SETOP]

20. (*Zhou*) Is a strongly ω_1-compact, locally compact space with a G_δ-diagonal metrizable? [SETOP]

C. Compactness and Generalizations

30. (*Watson*) Is there a pseudocompact, meta-Lindelöf space which is not compact? [Yes if CH] [SETOP]

31. (*Watson*) Is there a pseudocompact, para-Lindelöf space which is not compact? [SETOP]

32. (*Nyikos*) Is every separable, first countable, normal, countably compact space compact? [No if $p = \omega_1$]

33. (*Vaughan*) Is there a separable, first countable, countably compact, non-normal space? [Yes if $p = \omega_1$ or $b = c$, hence yes if $c \leq \omega_2$] [SETOP]

34. (*Przymusinski*) Can every first countable compact space be embedded in a sequentially separable space? (A space is *sequentially separable* if it has a countable subset D such that every point is the limit of a sequence from D.) [Yes if CH] [SETOP]

35. (*Nyikos*) Is CH *alone* enough to imply the existence of a locally compact, countably compact, hereditarily separable space which is not compact? a perfectly normal, countably compact space which is not compact? [Under "CH + there exists a Souslin tree" there is a single example with all these properties, and various non-Lindelöf spaces have been constructed under CH that are countably compact and hereditarily separable, or perfectly normal, locally compact and hereditarily separable.] [SETOP]

36. (*van Douwen*) Does there exist in ZFC a separable normal countably compact noncompact space? (Examples exist if MA or if $p = \omega_1$).

D. Paracompactness and Generalizations

23. (*Swardson, attributed to Blair*) Does MA + $\neg$CH imply that every perfectly normal space of nonmeasurable cardinal is realcompact?

24. (*Watson, attributed to Arhangel'skiĭ and Tall*) Is every normal, locally compact, metacompact space paracompact? [Yes if V = L; yes for perfectly normal spaces.]

25. (*Watson*) Is there a locally compact, perfectly normal space which is not paracompact? [Yes if MA or if there exists a Souslin tree. A "real" example must be collectionwise Hausdorff under V = L but must not be under MA + $\neg$CH; if one adds $\aleph_2$ random reals to a model of V = L the example must be collectionwise normal in the model.]

26. (*Navy*) Is every collectionwise normal para-Lindelöf space paracompact?

27. (*Wicke*) Is every collectionwise normal meta-Lindelöf space paracompact? what if it is first countable? [SETOP]

28. (*Wicke*) Is there a meta-Lindelöf space which is not weakly θ-refinable? [Yes if CH] [SETOP]

E. Separation and Disconnectedness

8. (*Watson*) Is there a locally compact, normal, non-collectionwise normal space? [Yes if MA + $\neg$CH or "Shelah's Principle"]

9. (*Watson*) Is there a perfectly normal, collectionwise Hausdorff space which is not collectionwise normal? [Yes if $E(\omega_2)$ or MA + $\neg$CH or V = L]

G. Mappings of Continua and Euclidean Spaces

11. (*E. Lane*) What is a necessary and sufficient condition in order for a space to satisfy the C insertion property for (n usc, n ℓsc)?

12. (*van Douwen*) Let **H** denote the half-line, [0,+∞]. Is every continuum of weight $\leq \omega_1$ a continuous image of $\mathbf{H}^* = \beta H - H$? [Yes for metrizable continua, i.e., weight $\leq \omega_1$]

H. Homogeneity and Mappings of Other Spaces

7. (*van Douwen*) Does there exist a rigid zero-dimensional separable metrizable space which is absolutely Borel, or at least analytic?

M. Manifolds

4. (*Nyikos*) Is every normal manifold collectionwise normal? [Yes if PMEA] [SETOP]

N. Measure and Topology

3. (*Steprans*) Is there a measure zero subset X of **R** such that any measure zero subset of **R** is contained in some translate of X? in the union of countably many translates of X? [SETOP]

P. Products, Hyperspaces, and Similar Constructions

11. (*van Douwen*) Let **H** be the half-line (0,∞). Does there exist a characterization of $\mathbf{H}^*$ under CH? For example:

a. Under CH, if L is a σ-compact connected LOTS with exactly one endpoint, with $\omega \leq w(L) \leq c$, is L* homeomorphic to $\mathbf{H}^*$?

b. More generally, does CH imply that $\mathbf{H}^*$ is (up to homeomorphism) the only continuum of weight c that is an F-space, has the property that nonempty G_δ's have nonempty interior, and is one-dimensional, indecomposable, hereditarily uni-coherent, and atriodic?

12. (*van Douwen*) Does there exist in ZFC a apsce that is homeomorphic to $\mathbf{N}^*$, but not trivially so? Is it at least consistent with $\neg$CH that such a space exists? (For example, under CH $(\mathbf{N} \times \mathbf{N}^*)^* \approx \mathbf{N}^*$.) [More generally, if CH then X^* is homeomorphic to $\mathbf{N}^*$ whenever X is locally compact, Lindelöf, (strongly) zero-dimensional, and noncompact. This follows from results of Comfort and Negrepontis, Math. Zeitschr. 107 (1968), 53-58.]

13. (*van Douwen*) Write $X_0 \simeq X_1$ if there are open $U_i \subseteq X_i$ with compact closure in X_i for $i = 0,1$ such that $X_0 - U_0$ and $X_1 - U_1$ are homeomorphic. Then X^* and Y^* are homeomorphic if $X \simeq Y$ but not conversely. Does there exist in ZFC a pair of locally compact realcompact (preferably separable metrizable) spaces X,Y such that X^* is homeomorphic to Y^*, but $X \not\simeq Y$?

S. Problems Closely Related to Set Theory

5. (*Steprans*) If there is a non-meager subset of $\mathbf{R}$ of cardinality $\aleph_1$, is there a Luzin set? [SETOP]

6. (*Steprans*) If there is a measure zero subset of $\mathbf{R}$ of cardinaltiy $\aleph_1$, is there a Sierpinski set? [SETOP]

7. (*Przymusinski*) Does there exist a σ-set of cardinality $\aleph_1$? (A *σ-set* is a separable metric space in which every F_σ-set is a G_δ-set.) [Yes if MA; under MA there even exists a σ-set of cardinality C.] [SETOP]

8. (*Przymusinski*) A q-*set* is a metrizable space in which every subset is a G_δ. Is every q-set strongly zero-dimensioned? linearly orderable? [SETOP]

INFORMATIN ON EARLIER PROBLEMS

Three results of the last year and a half have had a considerable effect upon the problems that have been posed in the past issues (not to mention on the face of general topology itself!). In chronological order, they are: Caryn Navy's construction of assorted para-Lindelöf, non-paracompact spaces, both in ZFC and under the axiom that there is a Q-set (implied by MA + ¬CH); Bill Fleissner's construction of a normal metacompact nonmetrizable Moore space under CH and under an axiom so weak that its negation implies the consistency of a proper class of measurable cardinals; and the solution, using PFA, of the S-space problem by Todorčević and Baumgartner (each coming up with his own proof). First we list the effects of these results, then move on to other problems which have been solved in whole or in part.

A. Problems in the Three Categories

E2 (vol 1): (*Wage*) Is there a strong S-space that is extremally disconnected? *Partial answer.* No if PFA.

Classic Problem II (vol 1) related problem: Is every perfectly normal space with a point countable base collectionwise normal? *Partial answer.* Yes if PMEA, but no unless it is consistent that there is a proper class of measurable cardinals.

A6 (in Problems from Other Sources, vol 2): If X is a regular space of countable spread, does X = Y ∪ Z where Y is hereditarily Lindelöf and Z is hereditarily separable? *Solution.* Yes if PFA, no if CH or there exists a Souslin line (Roitman).

D20 (in Problems from Other Sources, vol 2): (*Hodel*) Does a regular, hereditarily ccc p-space or wΔ-space with a G_δ-diagonal have a countable base? *Solution.* No if CH (*Kunen*), yes if PFA.

Classic Problem VII (vol 2), related problem B: Does there exist a Dowker space which is hereditarily separable? *Solution.* Yes if CH or if there exists a Souslin line (*Rudin*), no if PFA.

From "A Survey of two problems" by P. Nyikos.

1. Is there an S-space? *Solution.* Yes under numerous conditions (see article), no if PFA. For the following related problems, we list only those results that are most important, or that have been obtained since the article was written, referring the reader to the article for a more complete listing of consistency results.

A. Does there exist a countably compact S-space? Yes if CH (*Hajnal and Juhász*), no if PFA.

B. Is there an S-space of cardinality > c? Yes if $\diamond$ (*Fedorchuk*), no if PFA.

C. Does there exist a perfectly normal or hereditarily normal S-space? Yes if CH (*Kunen*) or there exists a Souslin tree (*Nyikos*); no if PFA.

D. Does there exist a first countable S-space? Yes if CH (*Juhász and Hajnal*) or there exists s Souslin line (*Dahroug*); yes is also consistent with MA + $\neg$CH (*Avraham and Todorčević*), no is also (PFA + $c = \aleph_2$ implies MA + $\neg$CH).

E. Does there exist a locally connected S-space? Yes if CH (*Rudin and Zenor*), no if PFA.

F. Same as D20 above.

2. Is every regular para-Lindelöf space paracompact? *Solution.* No (*Navy*). *Remarks.* While a full listing of the problems solved by Caryn Navy's examples is not practical here, it might be noted that the problem of obtaining first countable counterexamples has not been fully solved yet, although there are normal Moore examples in many models where there are normal Moore spaces at all. Also unsolved (even consistency results are lacking) is the problem of whether there is an example which is collectionwise normal.

D22 (vol 3): (*Reed*) Does there exist a strongly collectionwise Hausdorff Moore space which is not normal? *Partial answer.* Yes if there is a Q-set: every para-Lindelöf Moore space is strongly collectionwise Hausdorff.

B18 (vol 4): (*Burke*) Does every regular space with a σ-locally countable base have a σ-disjoint base? *Partial answer.* No if there is a Q-set, because then there is a para-Lindelöf nonmetrizable normal Moore space (*Navy*) and no nonmetrizable normal Moore space can have a σ-disjoint base or even be screenable.

B. Other Problems

Classic Problem III (vol 1): Is every screenable normal space paracompact? *Partial answer.* No if $\diamond^{++}$ (*M. E. Rudin*). The counterexample she constructs does not have a σ-disjoint base, and it is not known whether it is collectionwise normal or realcompact.

Classic Problem VIII (vol 2): Is every γ-space quasi-metrizable? *Partial answer.* (*R. Fox*) There is a

Hausdorff γ-space that is not quasi-metrizable. It is not known whether it is regular.

Related problems: Is every γ-space with an ortho-base quasi-metrizable? Is every linearly orderable γ-space quasi-metrizable? *Solution.* Yes to both (*Kofner*).

A3 (vol 2): (*van Douwen*) Is every point-finite open family in a ccc space σ-centered? *Solution.* No (*Ortwin Förster*).

C5 (vol 2): (*Comfort*) Is it a theorem in ZFC that $\omega_1^* \neq \omega_0^*$? ($\kappa^*$ denotes the Stone-Čech remainder of a discrete space of cardinal κ). *Solution.* Yes (*Glazer*, AMS Abstracts 2, p. 180).

A5 (vol 3): (*Arhangel'skiĭ*) Let c(X) denote the cellularity of X. Does there exist a space X such that $c(X)^2 > c(X)$? *Partial answer.* (*Galvin*) If $2^\kappa = \kappa^+$ then there is an X with c(X) = and $c(X^2) = \kappa^+$. (Fund. Math. 108 (1980), 33-48.) It is unknown if ZFC implies the existence of κ such that $2^\kappa = \kappa^+$. It is known that to deny it for the first uncountable strong limit cardinal is to assume the consistency of there being a proper class of measurable cardinals.

A6 (vol 3): (*Arhangel'skiĭ*) Does there exist a compact space X such that $c(X) = t(X) < d(X)$? *Partial answer.* Yes if CH or there exists a Souslin line.

C15 (vol 3): (*Nyikos*) Does there exist a first countable compact T_1 space of cardinality > c? A compact T_1 space with points G_δ and cardinality > c? *Solution.* No (*Gryzlov*, Soviet Math Doklady 21 (1980), 506-509).

Shelah's announcement reported in vol 3, that "yes" is consistent, was in error.

H5 a (vol 3): (*van Douwen*) Does there exist a homogeneous zero-dimensional separable metrizable space which cannot be given the structure of a topological group? *Solution.* Yes (*van Douwen*).

P9 (vol 3): (*Nyikos*) If a product of two spaces is homeomorphic to 2^κ, must one of the factors be homeomorphic to 2^κ? *Solution.* Yes (*Shchepin*).

Errata

The Bockstein separation property was incorrectly stated in vol 4 (p. 642) and should read: disjoint open sets are contained in disjoint cozero sets. (cf. M. Bockstein, Fund. Math. 35 (1948), 242-246.)

In problem C28 (vol 4, p. 639) "the ω_1-tunnel axiom holds" should read "$p = \omega_1$."