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1. Introduction

The *Sorgenfrey line* is the set of real numbers topologized by a basis of half-open intervals each of which is closed on the right in the reals. The *Sorgenfrey plane* is the topological product of two copies of the Sorgenfrey line. A *neighbourhood assignment* on a topological space S is a function defined on the points of S such that for each point p in S the value of f at p denoted $f(p)$ is an open subset of S which contains p . We shall call a subset D of a topological space S *discrete* in S if but only if no point of S is a limit point of D . A topological space S is said to be a D -space provided that for each neighbourhood assignment f on S there is a subset D of S which is discrete in S such that the set of f -values of the points in D , denoted $f(D)$, covers S .

The notion of a D -space originated with van Douwen [4]. Let us place D -spaces in relation to other perhaps more familiar covering properties of topological spaces. Each semi-stratifiable space [3] is quickly seen to be a D -space. It follows that each semi-metrizable space, developable space and stratifiable or M_3 -space is a D -space. For definitions see [1]. Recall that a space is *irreducible* [2] provided that each open cover of the space can be refined to an open cover of the space no proper subset of

which covers the space. It is quickly seen that each D-space is irreducible as is each paracompact and subparacompact space. In a conversation R. W. Heath asked whether each subparacompact space is a D-space. Not only have we been unable, as yet, to settle this question, but we do not even know of a first countable paracompact space which is not a D-space. In a conversation E. K. van Douwen asked if each Lindelöf space is a D-space and in a joint paper with W. F. Pfeffer [5] several interesting questions are raised with regard to D-spaces. It is the purpose of this note to settle one of these questions posed to us originally by W. F. Pfeffer: Is each finite topological product of copies of the Sorgenfrey line hereditarily a D-space? It is known [5] that the Sorgenfrey line is hereditarily a D-space. We argue inductively:

Theorem. Each subspace of a finite topological product of copies of the Sorgenfrey line is a D-space.

2. Some Definitions and Three Lemmas

If \mathcal{U} is a collection of sets then \mathcal{U}^* will be used to denote the union of the elements of \mathcal{U} . For each positive integer n , for each subspace S of the topological product of n copies of the Sorgenfrey line, for each point p in S and for each positive integer i we define a subset $S(p,i)$ of S which we call a *region* of S as follows: Let e denote the point in Euclidean n -space, E^n , such that each of its coordinates is the largest integral multiple of 2^{-i} which is less than the corresponding coordinate of p . A point q

belongs to $S(p,i)$ provided q is a point of S each of whose coordinates is greater than the corresponding coordinate of e but no greater than the corresponding coordinate of p . Notice that the regions of S defined in this way form an open basis for S .

Further, a function f defined on S will be referred to as a *centered assignment* on S if but only if for each point p in S the f -value of p is $S(p,i)$ for some positive integer i and the least such positive integer will be called the f -index of p and will be denoted by $i_f(p)$. The *inside* of $f(p)$, denoted $f^O(p)$, is the possibly empty subset of $f(p)$ to which a point q belongs if but only if no coordinate of q is the corresponding coordinate of p . (Note that a point may consequently lie in the f -value of p without being in the inside of the f -value of p .)

Two observations, α) and β) below, follow quickly from the above definitions and will be needed in the proof of lemmas two and three:

α) If q is a point in $f^O(p)$ and $i_f(p) \leq i_f(q)$ then $f(q)$ is a subset of $f^O(p)$.

β) If q is a point in $f^O(p)$ and r is a point in $f(p)$ each of whose coordinates is greater than the corresponding coordinate of q and if $i_f(r) \leq i_f(p)$ then q is in $f^O(r)$.

Lemma 1. Suppose that n is a positive integer greater than one and that each subspace of the topological product of fewer than n copies of the Sorgenfrey line is a D -space. If f is a centered assignment on a subspace S of the topological product of n copies of the Sorgenfrey line and B is

a closed subset of S no point of which lies in the inside of the f -value of another point in B then there is a subset D of B which is discrete in S such that $f(D)$ covers B .

Proof. Suppose the hypothesis of the Lemma and suppose that some point in B doesn't lie in the f -value of any other point in B . Let D_0 denote the set of all such points in B . D_0 is discrete in S . Suppose that $f(D_0)$ doesn't cover B . Let F_1 denote the closed set $B - f(D_0)^*$. We make some definitions:

If σ is a set then $c(\sigma)$ will denote the cardinality of σ . If p is a point in E^n and k is a positive integer no greater than n then p_k will denote the k -th. coordinate of p . If p and q are two points in E^n then $p|q$ is that subset of the positive integers no greater than n to which an integer k belongs if but only if p_k is q_k . For each point p in F_1 let $T(p)$ denote the set of all points in B different from p which contain p in their f -value. $T(p)$ is not empty and for each point q in $T(p)$ $p|q$ is not empty since by the Lemma's hypothesis p is not in $f^0(q)$. Let $n(p)$ denote the largest integer no greater than $c(p|q)$ for any point q in $T(p)$. Let $T'(p)$ denote a maximal subset of $T(p)$ such that if q and r are two points in $T'(p)$ then $p|q$ isn't $p|r$ but $c(p|q)$ is $n(p)$. $T'(p)$ is a finite set. Let $O(p)$ denote the common part of $f(p)$ with the intersection of all the sets in $f(T'(p))$. $O(p)$ is an open set which contains p . Let $\Delta(p)$ denote the collection of all sets $p|q$ for q in $T'(p)$. $\Delta(p)$ is a finite collection of sets.

Remark γ) below follows quickly from these definitions and will be used more than once in this proof:

γ) If p and x are two points in F_1 and p is in $O(x)$ and x' is in $T'(x)$ then $p|x'$ is a subset of $x|x'$ and $n(p) \leq c(p|x') \leq c(x|x') = n(x)$. Consequently if $n(p)$ is $n(x)$ then $p|x'$ is $x|x'$.

Let j_1 denote the largest integer in $n(F_1)$ and let G_1 denote the set of all points p in F_1 such that $n(p)$ is j_1 . Suppose that x is a point in $F_1 - G_1$ and that p is a point of F_1 in $O(x)$. By γ) $n(p) \leq n(x)$ and p is not in G_1 . We have shown that G_1 is a closed subset of F_1 . We shall now show that there is a subset $D(G_1)$ of G_1 which is discrete in S such that $f(D(G_1))$ covers G_1 .

Let p_1 denote a point in G_1 such that if q is any other point in G_1 then $\Delta(q)$ is not a proper subset of $\Delta(p_1)$. Let H_1 denote the set of all points q in G_1 for which $\Delta(q)$ is $\Delta(p_1)$. Suppose that x is a point in $G_1 - H_1$ and that p is a point of G_1 in $O(x)$. Suppose x' is any point in $T'(x)$. Since $n(p)$ is $n(x)$ by γ) $p|x'$ is $x|x'$. x' is in $T(p)$ and $c(p|x')$ is $n(p)$. Consequently there is a point p' in $T'(p)$ such that $p|p'$ is $x|x'$. We have shown that $\Delta(x)$ is a subset of $\Delta(p)$. It follows that p cannot be a point of H_1 and that H_1 is a closed subset of G_1 .

Let us say that two points p and q in H_1 are related if but only if for each integer k in $\Delta(p_1)^*$, p_k is q_k . Let $P(H_1)$ denote the partition of H_1 by this relation. Each set in this partition is closed in S and as a subspace of S is homeomorphic to a subspace of the product of fewer

than n copies of the Sorgenfrey line and is consequently a D -space, by the Lemma's hypothesis. For each set A in $P(H_1)$ then let $D(A)$ denote a subset of A which is discrete in S and such that $f(D(A))$ covers A . Let $D(H_1)$ denote $D(P(H_1))^*$. Certainly $f(D(H_1))$ covers H_1 . Let us show that $D(H_1)$ is discrete in S . Suppose that x were a limit point of $D(H_1)$. Since H_1 is closed, x is in some element, A say, of $P(H_1)$. $D(A)$ is discrete in S so there is some point p of $D(H_1) - D(A)$ in $O(x)$. Since p isn't in $D(A)$ there is an integer k in $\Delta(p_1)^*$ such that x_k is not p_k . $\Delta(p_1)$ is $\Delta(x)$ so there is a point x' in $T'(x)$ such that k is in $x|x'$. Since $n(p)$ is $n(x')$ by γ $p|x'$ is $x|x'$ and consequently x_k is p_k . This contradiction shows $D(H_1)$ to be discrete in S .

Suppose that $f(D(H_1))$ fails to cover G_1 . Let K_2 denote the closed set $G_1 - f(D(H_1))^*$ and let p_2 denote a point in K_2 such that if q is any other point in K_2 then $\Delta(q)$ is not a proper subset of $\Delta(p_2)$. Let H_2 denote the set of all points q in K_2 for which $\Delta(q)$ is $\Delta(p_2)$. Arguing as above for H_1 there is a subset $D(H_2)$ of H_2 which is discrete in S and such that $f(D(H_2))$ covers H_2 .

Suppose that $f(D(H_2))$ fails to cover K_2 . Let K_3 denote the closed set $K_2 - f(D(H_2))^*$ and let p_3 denote a point in K_3 such that ...

...

Since $\Delta(G_1)$ is finite, for some positive integer m , this process must terminate after m steps. Let $D(G_1)$ denote the union of $D(H_1), \dots, D(H_m)$. $D(G_1)$ is a subset of

G_1 which is discrete in S such that $f(D(G_1))$ covers G_1 .

Suppose that $f(D(G_1))$ doesn't cover F_1 . Let F_2 denote the closed set $F_1 - f(D(G_1))^*$. Let j_2 denote the largest integer in $n(F_2)$ and let G_2 denote the set of all points p in F_2 such that $n(p)$ is j_2 . Notice that $j_2 < j_1$. Arguing as above for G_1 there is a subset $D(G_2)$ of G_2 which is discrete in S such that $f(D(G_2))$ covers G_2 .

Suppose that $f(D(G_2))$ doesn't cover F_2 . Let F_3 denote ...

...

For some positive integer i this process must terminate after i steps since $\dots < j_2 < j_1$. Let D denote the union of $D_0, D(G_1), \dots, D(G_i)$. D is a subset of B which is discrete in S such that $f(D)$ covers B . Lemma 1 is proved.

Lemma 2. Suppose that n is a positive integer and that f is a centered assignment on a subspace S of the topological product of n copies of the Sorgenfrey line. If T is a subset of S whose points have a common f -index and if the closure of T in S is covered by the insides of the f -values of the points in T then there is a countable subset D of T which is discrete in S such that the union of the insides of the f -values of the points in D is the union of the insides of the f -values of the points in T .

Proof. Assume the hypothesis of the Lemma and let \bar{T} denote the closure of T in S . In the topological space induced on the points of S by E^n , $f^0(T)$ is an open cover of \bar{T} . Since E^n is hereditarily Lindelof there is a countable subset C of T such that $f^0(C)$ covers \bar{T} . Construct a

well-ordered subset D of C each of whose proper initial segments is finite such that 1) if p is a point in D then $f^{\circ}(p)$ contains a point of \bar{T} which is not in $f^{\circ}(d)$ for any point d which precedes p in D and 2) $f^{\circ}(D)$ covers \bar{T} . We show that D satisfies the conclusion of the Lemma.

Suppose that t is a point in T . Then for some point d in D , t is in $f^{\circ}(d)$. Since $i_f(d)$ is $i_f(t)$, by α) $f(t)$ a fortiori $f^{\circ}(t)$ is a subset of $f^{\circ}(d)$. Thus $f^{\circ}(T)^*$ is a subset of $f^{\circ}(D)^*$. But D is a subset of T consequently $f^{\circ}(D)^*$ is $f^{\circ}(T)^*$.

To see that D is discrete suppose that x were a limit point of D . Since x is in \bar{T} , for some point d in D , x is in $f^{\circ}(d)$. Since x is a limit point of D there is a point p which follows d in D and lies in $f^{\circ}(d)$. Since $i_f(d)$ is $i_f(p)$ by α) $f^{\circ}(p)$ is a subset of $f^{\circ}(d)$. But $f^{\circ}(p)$ contains a point of \bar{T} which is not in $f^{\circ}(d)$. This contradiction completes the proof of Lemma 2.

Lemma 3. Suppose that n is a positive integer greater than one and that each subspace of the topological product of fewer than n copies of the Sorgenfrey line is a D -space. If f is a centered assignment on a subspace S of the topological product of n copies of the Sorgenfrey line and M is a closed subset of S which contains a set T whose points have a common f -index which is no greater than the f -index of any point of M which is not in T , then there is a subset D of M which is discrete in S such that $f(D)$ covers T .

Proof. Suppose the Lemma's hypothesis and suppose that \bar{T} is not covered by $f^{\circ}(T)$, for otherwise the conclusion

of this Lemma would follow immediately from Lemma 2. Let $B(T)$ denote the closed set $\bar{T} - f^{\circ}(T)^*$ which is a subset of M .

Suppose that p and q are two points in $B(T)$ and that q is in $f^{\circ}(p)$. Then p is in $\bar{T} - T$ and there is a point r of T in $f(p)$ each of whose coordinates is greater than the corresponding coordinate of q . Since p is in M $i_f(r) \leq i_f(p)$ by the Lemma's hypothesis and by β) q is in $f^{\circ}(r)$. But now q is in $f^{\circ}(T)^*$ hence q is not in $f^{\circ}(p)$. This contradiction proves that no point of $B(T)$ lies in the inside of the f -value of any other point in $B(T)$.

Using Lemma 1 there is a subset $D(T)$ of $B(T)$ which is discrete in S such that $f(D(T))$ covers $B(T)$. Notice that at least one point of T is in $f(D(T))^*$.

If $f(D(T))$ covers T then the conclusion of the Lemma follows.

Suppose that $f(D(T))$ doesn't cover T . Construct a well ordered collection V of subsets of T and define for each set X in V a subset $B(X)$ of M and a subset $D(X)$ of $B(X)$ as follows:

The first set in V is T and each of $B(T)$ and $D(T)$ is as already defined.

Suppose that some initial segment I of V has been constructed and that for each set X in I both $B(X)$ and $D(X)$ have been defined.

If $f(D(I)^*)$ covers T then the construction terminates and V is I .

If $f(D(I)^*)$ doesn't cover T then $T - f(D(I)^*)^*$ is the

first set in V to follow each set in I . Let X denote $T - f(D(I)^*)^*$.

If \bar{X} is a subset of $f^{\circ}(X)^*$ then use Lemma 2 to define $D(X)$ to be a subset of X which is discrete in S such that $f^{\circ}(D(X))$ covers \bar{X} . Define $B(X)$ to be $D(X)$ and the construction terminates. In this case V is the union of I and $\{X\}$.

If \bar{X} is not a subset of $f^{\circ}(X)^*$ then define $B(X)$ to be the closed set $\bar{X} - f^{\circ}(X)^*$ which is a subset of M . Since X is a subset of T , an argument entirely analogous to the one used earlier in the proof of this Lemma for $B(T)$ establishes that no point of $B(X)$ lies in the inside of the f -value of any other point of $B(X)$. Using Lemma 1 define a subset $D(X)$ of $B(X)$ which is discrete in S such that $f(D(X))$ covers $B(X)$. Again notice that at least one point of X is in $f(D(X))^*$ which ensures that the construction terminates.

Let D denote $D(V)^*$. D is a subset of M and $F(D)$ covers T . To complete the proof it remains only to show that D is discrete in S .

Suppose that W precedes X in V and let I denote the collection of all sets in V which precede X . $D(X)$ is a subset of \bar{X} which in turn is a subset of $M - f(D(I)^*)^*$. $D(W)$ is a subset of $f(D(I)^*)^*$. It follows that $D(W)$ doesn't intersect $D(X)$ and that no two sets in $D(V)$ intersect. For each set A in $D(V)$ let $E(A)$ denote the set in V such that $D(E(A))$ is A .

Suppose that P is any subset of D which has exactly one point in common with each set in $D(V)$. We show that P is discrete in S .

Well-order $D(V)$ so that the set A in $D(V)$ precedes the set C in $D(V)$ if but only if $E(A)$ precedes $E(C)$ in V . Well order P so that the point p in P precedes the point q in P if but only if the set in $D(V)$ which contains p precedes in $D(V)$ the set which contains q .

Claim. If the point p precedes the point q in P and A is the set in $D(V)$ which contains p then q but not p is in $f^0(E(A))^*$. Let us argue this claim. Let C denote the set in $D(V)$ which contains q and let I denote the collection of sets in V which precede $E(C)$. A is a subset of $B(E(A))$ which is $\overline{E(A)} - f^0(E(A))^*$ so p is not in $f^0(E(A))^*$ as claimed. Suppose q were not in $f^0(E(A))^*$. Since $E(C)$ is a subset of $E(A)$, $B(E(C))$ which is $\overline{E(C)} - f^0(E(C))^*$ is a subset of $\overline{E(A)}$. Since q is in $D(E(C))$ which is a subset of $B(E(C))$, q is in $\overline{E(A)}$. But since q is not in $f^0(E(A))^*$, q must be in $B(E(A))$. Since $f(A)$ covers $B(E(A))$, q is in $f(A)^*$ and hence in $f(D(I))^*$. But C is a subset of $\overline{E(C)}$ which is a subset of $M - f(D(I))^*$. Thus q is not in C . This contradiction establishes the claim.

Now suppose that x were a limit point of the point set P . Let m denote the f -index of each point in T . There are two points p and q of P in the region $S(x,m)$ such that p precedes q in P and such that no coordinate of p is greater than the corresponding coordinate of q . p is in $S(q,m)$. Let A be the set in $D(V)$ which contains p . Using the above claim, q but not p is in $f^0(E(A))^*$. Let t be a point of T in $E(A)$ such that q but not p is in $f^0(t)$. Since $i_f(t)$ is no greater than m by α) $S(q,m)$ is a subset of $f^0(t)$

and p is in $f^{\circ}(t)$. This contradiction shows that P is discrete in S . But then D must be discrete in S and the proof of Lemma 3 is complete.

3. Proof of the Theorem

Suppose that the Theorem is false. Let n be the least positive integer m for which there is a subspace S in the topological product of m copies of the Sorgenfrey line which is not a D -space. Since the Sorgenfrey line is hereditarily a D -space n must be greater than one. Let g be a neighbourhood assignment defined on S such that for no set D which is discrete in S does $g(D)$ cover S . Since the regions of S form an open basis for S we may choose for each point p in S a positive integer i such that $S(p,i)$ is a subset of $g(p)$ and define $f(p)$ to be $S(p,i)$. f is a centered assignment such that for no set D which is discrete in S does $f(D)$ cover S .

Define $M(1)$ to be S and let $m(1)$ be the least integer in $i_f(M(1))$. Define $T(1)$ to be the set of all points in $M(1)$ whose f -index is $m(1)$. By Lemma 3 there is a subset $D(1)$ of $M(1)$ which is discrete in S such that $f(D(1))$ covers $T(1)$. $f(D(1))$ doesn't cover S .

Define $M(2)$ to be the closed set $M(1) - f(D(1))^*$. Let $m(2)$ be the least integer in $i_f(M(2))$. Let $T(2)$ denote the set of all points in $M(2)$ whose f -index is $m(2)$. Again by Lemma 3 there is a subset $D(2)$ of $M(2)$ which is discrete in S such that $f(D(2))$ covers $T(2)$. $m(1) < m(2)$ and the union of $f(D(1))$ and $f(D(2))$ covers all those points of S whose f -index is no greater than $m(2)$. This union doesn't cover S .

Define $M(3)$ to be the closed set $M(2) - f(D(2))^*$.

Let $m(3)$ be ...

...

Continue in this fashion to define $D(1), D(2), \dots$. Let D denote the union of these sets. $f(D)$ covers S and D is discrete in S . This contradiction proves the Theorem.

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