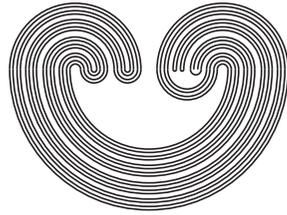

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WELL PARTIALLY ORDERED SETS AND LOCAL BASES

Gary Grabner

A collection of subsets of a set X is said to be *Noetherian (of subinfinite rank)* provided that every sub-collection well ordered by \subseteq (with incomparable members) is finite. The concepts of a base of subinfinite rank and a Noetherian base of subinfinite rank were introduced in [G,N] and studied further in [F,G], [G], [L,N], $[N_1]$, $[N_2]$, $[N_3]$ among others. An important result from [F,G] proved by Ortwin Förster is that every T_1 space having a base of subinfinite rank is hereditarily metacompact.

A collection of subsets of a set X is said to be *well ranked* provided that it is the countable union of Noetherian collections of subinfinite rank. The concept of a well ranked base was introduced in [G,N] where it is shown, for example, that a compact T_2 -space having a well ranked base is metrizable. In [G,N] it is also shown that the property that every point of a topological space has a well ranked base is preserved by countable products. A topological space in which every point has a well ranked local base is called a *wrb-space*.

A topological space in which every point has a local base linearly ordered by set inclusion is called a *lob-space*. The class of lob-spaces, introduced by S. Davis in [Da], contains the class of first countable spaces. The class of globular spaces (defined in section 2), introduced

by B. Scott in [Sc] contains the class of lob-spaces which, unlike the class of lob-spaces, is closed under finite products. Many of the interesting properties of lob-spaces and globular-spaces are given in [N₃].

In section 1, we discuss the properties of partially ordered sets satisfying the finite antichain condition (the natural partially ordered set analog of subinfinite rank). Using the results of this section we characterize globular spaces as those in which every point has a local base of subinfinite rank. Thus the class of wrb-spaces contains the class of globular spaces. In section 2, we study the properties of wrb-spaces and, in particular, show that various properties of globular spaces and lob-spaces are possessed by wrb-spaces.

Suppose X is a topological space. Let $\tau(X) = \{V \subseteq X; V \text{ is open}\}$. If \mathcal{M} is a collection of subsets of X and if $x \in X$ then let $(\mathcal{M})_x = \{H \in \mathcal{M}; x \in H\}$. If $Y \subseteq X$ then the topology of Y will be the usual subspace topology.

We will use Greek letters to denote ordinals and for convenience we will not distinguish between the cardinal κ and the first ordinal having cardinality κ . The first infinite cardinal will be denoted by ω . For any set A the cardinality of A will be denoted by $|A|$. Our set theoretic usage will be, in general, that of [M]. We also use some results of [J].

We now state several definitions which will be used in section 2.

Let X be a topological space. The space X is *weakly first countable* provided that at each point $x \in X$ there is

a decreasing sequence $\langle B(n,x) : n \in \omega \rangle$ of subsets of X such that $U \in \tau(X)$ if and only if for each $x \in U$ there is an $n(x,U) \in \omega$ such that $B(n(x,U),x) \subseteq U$, [A]. The space X is *sequential* provided that each of its sequentially closed subsets is closed, [F]. The *tightness* of X , denoted $t(X)$, is the smallest infinite cardinal κ such that if $A \subseteq X$ and $x \in \text{cl}(A)$ then there is a set $C \subseteq A$ with $|C| \leq \kappa$ and $x \in \text{cl}(C)$, [J]. If $x \in X$ then the *character of x in X* , denoted $\chi(x,X)$, is the smallest infinite cardinal κ such that x has a local base with cardinality less than or equal to κ , [J]. The *character of X* , denoted $\chi(X)$, is $\sup \{ \chi(x,X) : x \in X \}$, [J].

1. Partially Ordered Sets Satisfying the Finite Antichain Condition and Well Partially Ordered Sets

Let (P, \leq) be a partially ordered set and n an infinite cardinal. If $a, b \in P$ such that $a \not\leq b$ and $b \not\leq a$ then a and b are said to be *incomparable*, written as $a \text{ inc } b$. A subset Q of P is called *incomparable* provided that every two elements of Q are incomparable. A subset Q of P is said to be *cofinal* in P (with respect to \leq) provided that for every $a \in P$ there is a $b \in Q$ such that $a \leq b$. Let $\text{cf}(P, \leq) = \min\{|Q| : Q \text{ is cofinal in } P\}$. The poset (P, \leq) is said to

(1) satisfy the *finite antichain condition* (abbreviated f.a.c.) provided that every incomparable subset of P is finite.

(2) be *Noetherian* (*n-Noetherian*) provided that every subset of P well ordered by \leq is finite (has cardinality at most n).

(3) be *well partially ordered* (abbreviated w.p.o.) provided that every nonempty subset has at least one, but no more than a finite number of minimal elements [K].

(4) be *directed* provided that for every $p, q \in P$ there is an $x \in P$ such that $p \leq x$ and $q \leq x$.

The finite antichain condition is the natural partially ordered set analog of subinfinite rank. Indeed, a collection of sets with nonempty intersection and having subinfinite rank when partially ordered by \subseteq satisfies the finite antichain condition. By this observation the following characterization of base of subinfinite rank is easily established.

Theorem 1.1. A base β for a topological space X is a base of subinfinite rank if and only if for each $x \in X$ the partially ordered set $((\beta)_x, \subseteq)$ satisfies the finite antichain condition.

The concept of a well partial ordering was introduced around 1950 (see [K]) and is a generalization of a well ordering. The finite antichain condition is the corresponding generalization of a linear (total) ordering. The following is easily proved.

Theorem 1.2. (1) A poset (P, \leq) is well partially ordered if and only if it satisfies the f.a.c. and $(P, >)$ is Noetherian. (2) Every poset (P, \leq) satisfying the f.a.c. has a cofinal subset well partially ordered by \leq .

Corollary 1.3. A base β for a topological space X is a Noetherian base of subinfinite rank if and only if for

every $x \in X$ the partially ordered set $((\beta)_{x, \supset})$ is well partially ordered.

As in [D,M] the dimension of the partially ordered set (P, \leq) is the smallest cardinal n such that \leq is the intersection of n linear orders on P . Equivalently, the dimension of (P, \leq) is the smallest cardinal n such that (P, \leq) can be isomorphically embedded in the product of n linearly ordered sets. (See [O].) The following theorem is from [W₂].

Theorem 1.4. Let (P, \leq) be a partially ordered set of finite dimension. Then (P, \leq) is well partially ordered if and only if (P, \leq) is isomorphic to a subset of the product of a finite number of well ordered sets.

The following theorem is found in [D].

Theorem 1.5. If (P, \leq) is a partially ordered set and $k \in \omega$ is such that every incomparable subset of P has at most k elements then P can be expressed as the union of at most k subsets of P totally ordered by \leq .

In the case of partially ordered sets satisfying the finite antichain condition there is no such characterization. In fact, for any infinite cardinal n there exists a well partially ordered set which cannot be expressed as the union of less than n totally ordered subsets [W₁]. The next theorem (proved in [P]) is, however, a natural analog to Theorem 1.4 for partially ordered sets satisfying the f.a.c.

Theorem 1.6. Suppose (P, \leq) is a partially ordered set satisfying the finite antichain condition

(1) P is the union of a finite number of directed subsets.

(2) If P is directed, then it contains a cofinal subset which is order isomorphic to the product of a finite number of distinct regular cardinals.

Corollary 1.7. If (P, \leq) is a poset satisfying the finite antichain condition then $\text{cf}(P, \leq)$ is regular.

Corollary 1.8. Suppose (P, \leq) is a poset satisfying the finite antichain condition. If $|P| = n \geq \omega$ then there is a cofinal subset of P which is the union of less than n sets well ordered by \leq .

Corollary 1.7 and Corollary 1.8 are from $[M, P]$. The following result is from $[W_2]$.

Lemma 1.9. Any infinite well partially ordered set (P, \leq) contains a well ordered subset C with $|C| = |P|$. Hence, for every infinite cardinal n , every n -Noetherian well partially ordered set has cardinality at most n .

2. Spaces Having Well Partially Ordered Local Bases

Let Ω be a finite set of infinite regular cardinals, X a topological space and $x \in X$. A local base β at x is called an Ω -generalized linearly ordered base (Ω -glob) at x provided that (β, \supseteq) is order isomorphic to $(\prod \Omega, \leq)$ where \leq is the usual product partial ordering. If $\Omega = \emptyset$ then let $\prod \Omega = \emptyset$. A topological space in which each point x ,

for some finite set of infinite regular cardinals Ω_x , has an Ω_x -generalized linear ordered base is said to be *globular*. The class of globular spaces was introduced by Brian Scott as a generalization of lob-spaces which is closed under finite products (see [Sc]).

As a direct consequence of Theorem 1.4 and Theorem 1.6 we have the following characterization of globular spaces.

Theorem 2.1. A topological space is globular if and only if every point has a local base with subinfinite rank.

Corollary 2.2. Every topological space with a base of subinfinite rank is globular.

The following result also follows from Theorem 1.4 and Theorem 1.6.

Theorem 2.3. A topological space X is a wrb-space if and only if for each $x \in X$ and for each $n \in \omega$ there is a collection $\beta(n, x)$ of neighborhoods of x and a finite set of infinite regular cardinals $\Omega(x, n)$ such that $(\beta(n, x), \supseteq)$ is order isomorphic to $(\prod \Omega(n, x), \leq)$ where \leq is the usual product partial ordering and such that $\cup\{\beta(n, x) : n \in \omega\}$ is a local base at x .

The following theorem is from [Da].

Theorem 2.4. If X is a T_1 lob-space, then the following are equivalent:

- (a) X is first countable.
- (b) X is weakly first countable.

- (c) X is sequential.
- (d) X has countable tightness.
- (e) If $x \in X$ then $\{x\}$ is a G_δ -set.
- (f) If $x \in X$ and $\{x\}$ is not open, then there is a countable set $C \subseteq X - \{x\}$ with $x \in \text{cl}(C)$.

Example 2.5. Let $X = [(\omega_1+1) \times \omega] \cup \{(\omega_1, \omega)\}$. We topologize X by letting points of $(\omega_1+1) \times \omega$ be isolated and giving (ω_1, ω) a local base as in the product topology of $(\omega_1+1) \times (\omega+1)$. The space X is easily seen to be globular. However, $\{(\omega_1, \omega)\}$ is a G_δ -set and $(\omega_1, \omega) \in \text{cl}\{(\omega_1, n) : n < \omega\}$. Thus X meets conditions (f) and (e) of Theorem 2.4 but is clearly not first countable. This example is due to Brian Scott.

Theorem 2.6. If X is a wrb-space and for each $x \in X$, $\beta(x)$ is a well-ranked local base at x then the following are equivalent:

- (a) X is first countable.
- (b) X is weakly first countable.
- (c) X is sequential.
- (d) X has countable tightness.
- (e) The union of any collection of closed subsets of X totally ordered by \subseteq with uncountable cofinality is closed.
- (f) For all $x \in X$ and $\mathcal{G} \subseteq \beta(x)$ such that \mathcal{G} is totally ordered by \subseteq and has uncountable cofinality with respect to \supseteq , $x \in \text{int}(\bigcap \mathcal{G})$.

Proof. Clearly for every topological space (a) \Rightarrow (b) \Rightarrow

(c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f). We will prove that for wrb-spaces (f) \Rightarrow (a).

Let $x \in X$ and for every infinite cardinal κ let $[\kappa]$ be the statement

For every $\mathcal{G} \subseteq \beta(x)$ with $|\mathcal{G}| \leq \kappa$ there is a countable set $\mathcal{M} \subseteq \beta(x)$ such that for each $G \in \mathcal{G}$ there is an $H \in \mathcal{M}$ with $H \subseteq G$.

Clearly $[\omega]$ holds. Suppose λ is an uncountable cardinal and for every infinite cardinal $\kappa < \lambda$ $[\kappa]$ holds. Also suppose $\mathcal{G} \subseteq \beta(x)$ with $|\mathcal{G}| = \lambda$. Since \mathcal{G} is well ranked, let $\mathcal{G} = \cup\{K(n) : n < \omega\}$ where for each $n < \omega$ the collection $K(n)$ has subinfinite rank.

Suppose that λ is a singular cardinal. Since, by Corollary 1.7, for all $n < \omega$, $\text{cf}(K(n), \supseteq)$ is regular, for each $n < \omega$, there is a regular cardinal $\gamma(n) < \lambda$ and a cofinal (w.r.t. \supseteq) subset $H(n)$ of $K(n)$ with $|H(n)| = \gamma(n)$. For each $n < \omega$, by $[\gamma(n)]$, there is a countable set $\mathcal{M}(n) \subseteq \beta(x)$ such that for each $H \in H(n)$ there exists an $M \in \mathcal{M}(n)$ with $M \subseteq H$. Since $\cup\{\mathcal{M}(n) : n < \omega\}$ is cofinal in \mathcal{G} , the countable set $\cup\{\mathcal{M}(n) : n < \omega\}$ has the desired property.

Suppose that λ is an uncountable regular cardinal. By Corollary 1.8, for each $n < \omega$ there is a cardinal $\gamma(n) < \lambda$ such that for each $\alpha < \gamma(n)$ there is an $\mathcal{M}(\alpha, n) \subseteq K(n)$ totally ordered by \supseteq and such that $\cup\{\mathcal{M}(\alpha, n) : \alpha < \gamma(n)\}$ is a cofinal (w.r.t. \supseteq) subset of $K(n)$. For each $n < \omega$, let $S(n) = \{\alpha < \gamma(n) : \text{cf}(\mathcal{M}(\alpha, n), \supseteq) = \omega\}$ and for each $\alpha \in S(n)$ let $\mathcal{J}(\alpha, n)$ be countable cofinal (w.r.t. \supseteq) subset of $\mathcal{M}(\alpha, n)$. For each $n < \omega$ and each $\alpha \in \gamma(n)/S(n)$, since $x \in \text{int} \cap \mathcal{M}(\alpha, n)$, let $H(\alpha, n) \in \beta(x)$ such that $H(\alpha, n) \subseteq \text{int} \cap \mathcal{M}(\alpha, n)$

and let $\mathcal{S}(\alpha, n) = \{H(\alpha, n)\}$. Let $\mathcal{N} = \cup\{\mathcal{S}(\alpha, n) : n < \omega \text{ and } \alpha < \gamma(n)\}$ and notice that for each $G \in \mathcal{G}$ there is an $N \in \mathcal{N}$ with $N \subseteq G$. Since λ is an uncountable regular cardinal, $|\mathcal{N}| \leq \sup(\{\gamma(n) : n < \omega\} \cup \{\omega\}) < \lambda$. By $[|\mathcal{N}|]$ there is a countable set $\mathcal{H} \subseteq \beta(x)$ such that for each $N \in \mathcal{N}$ there is an $M \in \mathcal{H}$ with $M \subseteq N$. Thus \mathcal{H} is a countable subset of $\beta(x)$ having the desired property.

Hence $[\lambda]$ holds and so $[\kappa]$ holds for every infinite cardinal κ . By $[|\beta(x)|]$, let \mathcal{M} be a countable cofinal (w.r.t. \subseteq) subset of $\beta(x)$. Clearly \mathcal{M} is a countable local base at x . Therefore we have established that X is first countable.

The following result due to M. Ismail follows from Theorem 2.5.

Corollary 2.7. Every compact T_2 space having a base of subinfinite rank is first countable.

Proof. In a compact space the union of a collection of closed (compact) sets totally ordered by \subseteq with uncountable cofinality is countably compact. Since a T_1 space having a base of subinfinite rank is hereditarily metacompact, a countably compact subset of a compact T_2 space having a base of subinfinite rank is compact and hence closed. The corollary follows from Theorem 2.6(f).

Theorem 2.8. If X is a T_3 wrb-space then $\chi(X) \leq c(X)$.

Proof. Let $x \in X$ and β a well ranked base at x . Since X is T_3 , the collection $\beta^* = \{\text{int}(\text{cl}(B)) : B \in \beta\}$ is a base at x . It is straightforward to verify that β^* is well

ranked and that $(\beta^*, \underline{\supseteq})$ is $c(X)$ -Noetherian. Hence, by Lemma 1.9, $|\beta^*| \leq c(X)$ and so $\chi(x, X) \leq |\beta^*| \leq c(X)$.

Corollary 2.9. *Every T_3 wrb-space satisfying the countable chain condition ($c(X) = \omega$) is first countable.*

Both the class of first countable spaces and the class of wrb-spaces are closed under countable products. However no point in the product of uncountably many non-trivial T_1 -spaces has a countable local base. In the next theorem we strengthen this well known result.

Theorem 2.10. *Suppose X is the produce of uncountably many T_1 spaces having at least 2 points. Then no point of X has a well ranked local base.*

Proof. If $x \in X$ has a well ranked base then x has a well ranked base in every subspace of X in which it is contained. Thus without loss of generality we may assume that X is the product of uncountably many two point discrete spaces. If so, then X satisfies the countable chain condition but no point X has a countable local base. Thus by the proof of Theorem 2.7 no point of X has a well ranked local base.

The following theorem was proved for globular spaces in [Sc].

Theorem 2.11. *Suppose X is a T_1 wpob-space. Then X is countably compact if and only if it is sequentially compact.*

Proof. Assume X is countably compact and show that it

is sequentially compact. (The other implication is obvious.)

Let $\langle x(n) : n < \omega \rangle$ be a sequence in X , y a cluster point of a set $A = \{x(n) : n \in \omega\}$, and β a well ranked base at y . Since A is countable $(\{B \cap A : B \in \beta\}, \supseteq)$ is ω -Noetherian. Hence, by Lemma 1.9, $|\{B \cap A : B \in \beta\}| \leq \omega$. Let \mathcal{G} be a countable subset of β such that $\{G \cap A : G \in \mathcal{G}\} = \{B \cap A : B \in \beta\}$ and let $\mathcal{G} = \{G(n) : n < \omega\}$.

Let $n(0) = 0$. Suppose $k < \omega$ and for every $m \leq k$, $n(m) < \omega$ has been chosen such that if $j < m \leq k$ then $n(j) < n(m)$ and such that for all $m \leq k$, $x(n(m)) \in \bigcap \{G(j) : j < m\}$. Since y is a cluster point of A there is an $m < \omega$ such that $n(k) < m$ and $x(m)$ is in the open set $\bigcap \{G(j) : j < m + 1\}$. Let $n(k+1) = m$.

Let V be an open neighborhood of y . There is a $B \in \beta$ such that $B \subseteq V$ and a $k \in \omega$ such that $G(k) \cap A \subseteq B \cap A$. Since for all $m < \omega$, $x(n(m)) \in \bigcap \{G(j) : j < m\}$, $\{x(n(m)) : k < m < \omega\} \subseteq G(k) \cap A \subseteq B \cap A \subseteq B \subseteq V$. Thus the subsequence $\langle x(n(j)) : j < \omega \rangle$ converges to y .

If X is a globular space then, for every $x \in X$, $\chi(x, X)$ is a regular cardinal and $\chi(X) = t(X)$, [Sc]. If X is a wrb-space and $x \in X$ then, although $\chi(x, X)$ need not be regular, by Corollary 1.7, we can conclude that $\chi(x, X)$ is either a regular cardinal or has cofinality ω .

Question 2.12. Suppose X is a wrb-space. Does $\chi(X) = t(X)$?

The following characterizations of paracompactness in lob-spaces are from [Da].

Theorem 2.13. Suppose X is a regular lob-space. Then the following are equivalent

- (1) X is paracompact.
- (2) X is irreducible and \aleph -preparacompact.
- (3) X is θ -refinable and \aleph -preparacompact.

Question 2.14. Does Theorem 2.13 hold for regular wrb-spaces?

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