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1. Introduction

There is a class of problems involving the juxtaposition of geometry and topology which may be viewed as balancing tables on hills or ridges. Intuitively, we shall think of a table as a horizontal polygon (or horizontal line segment) with vertical legs at each corner (or endpoint). The legs can be arbitrarily long but of equal length. A hill is the graph of a continuous function whose values are taken along a vertical axis. A ridge is a horizontal simple closed curve. These problems are elementary in statement and the known results are often non-trivial or striking. We proceed in reverse chronological order--starting with the most recent developments and then looking at possible roots in earlier work. Previously unpublished findings will be stated as numbered results.

2. Table Theorems

In 1970, Roger Fenn published the Table Theorem ([Fe]):

Let a "hill" be given, i.e., *a continuous non-negative function* f: $\mathbf{R}^2 \to \mathbf{R}$ *which is zero outside a compact convex disk,* D. *Also suppose the side length* d > 0 *of a square table is given. Then the table can be placed "on" the hill,* i.e., *there is a square in* $\mathbf{R}^2$ *of side* d *with center in* D *such that* f *takes a single value on the vertices.*

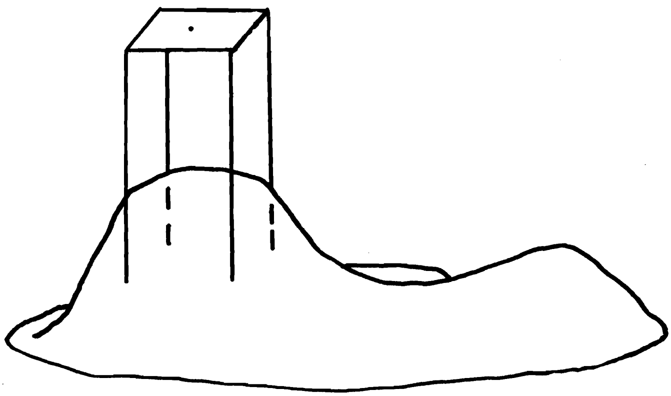


Figure 1. The feet of the square table (with arbitrarily long legs of equal length) can always be placed on the ground so that the table top is horizontal and the center is over the compact convex support of the hill.

Here is an intuitive justification for the Table Theorem (for a proof see [Fe]). The set of possible positions for a square with center in D can be viewed as a solid torus $D \times S^1$ where the center is at $d \in D$ and the rotation is determined by $s \in S^1$. For each such (d,s) the vertices of the square are at points a_1, a_2, a_3, and a_4; and we let $Q_i = (a_i, f(a_i))$ for each i. Now Q_1, Q_2, and Q_3 determine a plane in $\mathbf{R}^3$ and we let $\phi(d,s)$ be the (signed) height of Q_4 above the plane.

Then $A = \{(d,s) \mid \phi(d,s) = 0\}$ gives us those squares for which the table is balanced, but not necessarily horizontally (i.e., the Q_i's are coplanar). Note that A "looks like" four spanning surfaces in $D \times S^1$--any quarter-loop in $D \times S^1$ meets A. For ϕ has opposite signs at the ends of a quarter-loop, where these ends are a quarter turn from each other.

Now let $\psi(d,s)$ be the projection into $\mathbf{R}^2$ of the unit normal to the plane determined by Q_1, Q_2, and Q_3. Then $B = \{(d,s) | \Psi(d,s) = 0\}$ gives us those squares for which the first three feet can be placed horizontally. Note that B "looks like" a non-trivial loop in $D^2 \times S^1$--any spanning disk meets it. For as we let d move about ∂D, $s \in S^1$ fixed, $\psi(d,s)$ points out of D. We get a non-trivial map $\psi_s: \partial D \to E^2 - \{0\}$. So for any spanning disk of $D \times S^1$ with boundary equal to $\partial D \times s$, ψ must somewhere be zero.

Thus we expect "apparent spanning disk" A and "apparent loop" B to meet.

Although the truth of some such Theorem seems natural, the precise statement requires perhaps as much cleverness as the proof. Note the following non-trivial restrictions in the Theorem:

1. The center of the table is over D.
2. The support of the hill, D, is convex.
3. The table top is square.
4. The hill is non-negative.

Quite recently, variations in the Table Theorem changing each of these restrictions have been considered.

Last year E. H. Kronheimer and P. B. Kronheimer ([KK]) removed consideration of the center of the table by concluding that all four vertices of the square can be placed in D if ∂D does not contain the vertices of any smaller square.

To see this requires the use of Fenn's Theorem. If D contains the origin and is "nice" we may choose $\Lambda > 1$ large enough so that if the square's center is in ΛD then so is

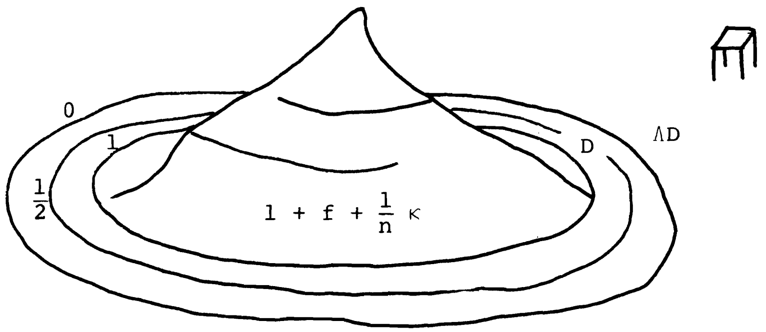


Figure 2. Under the appropriate conditions, the table will only balance with all four vertices over D.

at least one vertex. Construct a hill as indicated in Figure 2, using similar copies $r(\partial D)$ of ∂D for level curves at height $(\Lambda-r)/(\Lambda-1)$, $1 \leq r \leq \Lambda$, and inside D use function $1 + f + (1/n)\kappa$ where κ defines a cone. Then if ∂D does not contain the vertices of a smaller square the only possible balancing is with all vertices in D. Take the limit as n increases. If D was not "nice" to start with, it is a limit of "nice" disks.

Also, recently, the present author showed the necessity of the convexity assumption on D ([Mel]).

Suppose $d > 0$ is given. Construct ∂D as follows (see Figure 3). Take a large circle in the plane with center at the origin. Add the two segments from $(-\varepsilon,0)$ to the circle which make angles of $\pm\varepsilon$ with the positive x-axis ($\varepsilon > 0$ and small). Throw out the small arc of the circle these segments cut off and round off the two corners there. Note that the only squares with vertices on ∂D have center $(0,0)$. Define f by taking the cone on ∂D with apex over $(-\gamma,0)$, slightly to the left of $(-\varepsilon,0)$. The level curves are

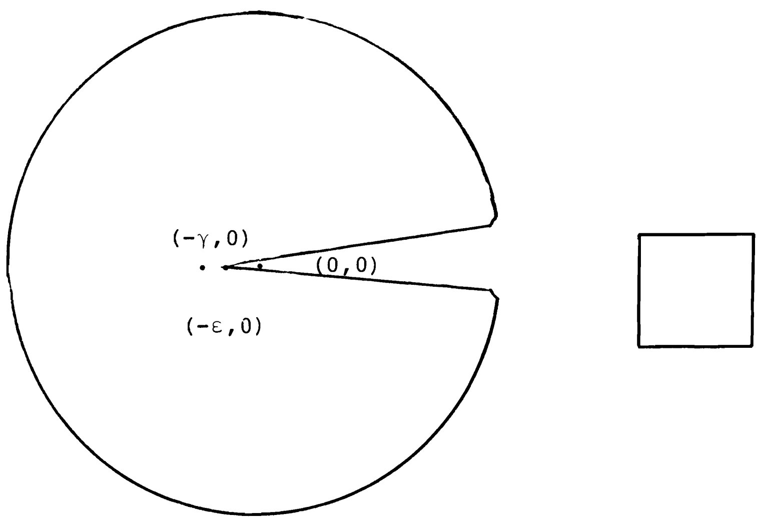


Figure 3. The cone on this curve with apex over $(-\gamma,0)$ gives a hill on which we cannot balance the square table.

similar to ∂D. Now if the original circle is large, the table won't balance at zero height. There is only one non-zero height at which it could balance, but for γ close to ε, the center will not be over D. Note that this example also shows the necessity of a convexity assumption in Kronheimer and Kronheimer's result.

As a final type of generalization to Fenn's Table Theorem, we consider non-square tables. Results in this area have had further implications related to non-negative hills and the concept of center of the table.

If our table has more than four vertices, a hill which is a cone on an appropriate ellipse shows it may fail to balance. For this we just need the fact that five points determine a conic. We then use an ellipse such that no ellipse of that eccentricity passes through the vertices (see [Me3]). To get a single hill on which no *regular* n-gon, $n \geq 5$, will balance, Fenn ([Fe]) refers to Eggleston ([Eg]). Similarly, with four vertices, a cone on a circle shows we need only consider cyclic quadrilaterals. The question of whether a table in the shape of a cyclic quadrilateral can balance on any hill remains open.

With triangular tables there are several results due to Zaks ([Za]), Kronheimer and Kronheimer ([KK]), and the author ([Me3]). For example, given any triangular table with any point chosen as a "center," the table can be translated (no rotation) until it balances on a hill ([Me3]). In a rare result for hills with possibly negative heights, Kronheimer and Kronheimer have shown ([KK]) that an equilateral triangular table will balance with all feet on such a hill if and only if ∂D contains the vertices of an equal or larger equilateral triangle.

Kronheimer and Kronheimer used Fenn's result requiring the center of the square table to be over the hill to get a result where all of the table was over the hill. We can reverse this process to convert Kronheimer and Kronheimer's result for equilateral triangles on hills with perhaps negative height, to get the corresponding result for tables with a "center" point over the hill. To do this we need:

Lemma1. Fix any point called the "center" of an equilateral triangle of side s. *If a closed convex disk,* D, *doesn't contain the vertices of the triangle under any translation, then some translation of the triangle has* *"center" in the disk and all three vertices on or outside the disk.*

Proof. We may assume the "center" is in the interior of the triangle. Let S be a segment from the apex of the triangle, through the "center," ending at E in the base of the triangle. Place the base tangent to ∂D at E, with the "center" and D on the same side of the base. Then if the "center" is not in D, move it (and the triangle) toward the point of tangency until we have the desired result. If the "center" is in D, translate the triangle continuously so that the base meets D in a chord broken into the same ratio by E as the base is. Translate until the apex of the triangle touches ∂D and we are done.

Theorem 1. *We can balance an equilateral triangular table on a hill, perhaps with negative heights, with arbitrary prechosen "center" over convex compact support disk,* D.

Proof. We may assume the "center" is in the interior of the triangle. If we can't balance the equilateral triangular table with all three vertices over D, then by Kronheimer and Kronheimer ∂D does not contain the vertices of an equal or larger equilateral triangle. It then follows that D does not contain the vertices of a congruent triangle (see [Me2] for a proof which doesn't use the convexity of D). So by Lemmal, we can balance the table at zero height.

The next stage would be tables with just two legs, which is what we'll consider in the next Section.

3. A Hill in Flatland

Since a table with only two legs and a line segment for a top is of one dimension less than our earlier tables, we are naturally lead to consider hills which are the graphs of one variable.

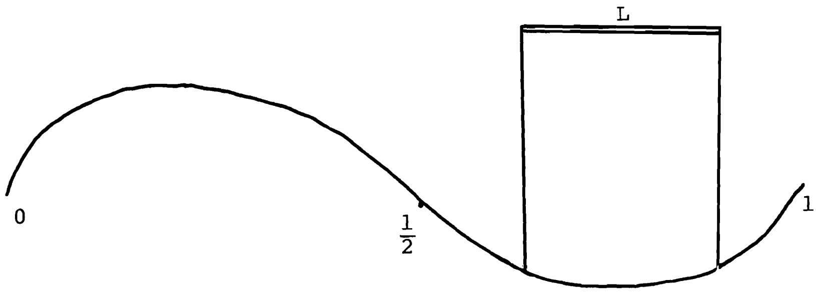


Figure 4. The Table Theorem in Flatland. The table will balance on any hill if and only if L = 1/n.

Let's consider the hill to be the graph of a function f: [0,1] → ℝ with f(0) = f(1). Then the Theorem proved by Paul Levy ([Le]) in 1934 was that a given segment can be placed horizontally with both endpoints on any such graph if and only if its length is 1/n for some positive integer n. For example, in Figure 4 we can see that lengths 1 and 1/2 balance but nothing in between. Similar examples show that a length not equal to 1/n may fail to balance. To see that 1/n will always balance, Heinz Hopf gave the following proof ([Ho]).

Note that a table of length L will balance if and only if the graph G shifted by L (G_L) meets G. Add to G an upward vertical ray from a maximal point and a downward vertical ray from a minimal point to get G'. Let M be the set of lengths which don't balance on G. Then if a, b $\in$ M so is a + b since G'_{-a} and G'_b are separated by G'--thus G'_{-a} and $(G'_{-a})_{a+b}$ don't meet. Since $1 \notin M$, $1/n$ or $(n-1)/n \notin M$. So $1/n$ or $(n-2)/n \notin M, \cdots$, etc. Hence $1/n \notin M$.

The restriction on the length of the table to 1/n is in stark contrast to the case with a 2-dimensional table. One reason is the fact that we are insisting on both legs inside the interval, as Kronheimer and Kronheimer do in 2-dimensions. If instead we only require the "center" in the interval (by analogy to Fenn) any length table will balance.

Theorem 2. *If* $f: \mathbf{R} \to \mathbf{R}$ *is continuous and zero off* (0,L) *and* $\ell \in \mathbf{R}$ *is arbitrary, then there is a translation,* τ, *of* $\mathbf{R}$, *such that* $\tau(\ell) \in [0,L]$ *and* $f(\tau(0)) = f(\tau(1))$.

Proof. We think of [0,1] as a table. We may assume the "center," ℓ, is in the interior of the table (otherwise place the "center" at 0 or L) and that $1 < L$ (otherwise place the endpoints appropriately on opposite sides of (0,L)).

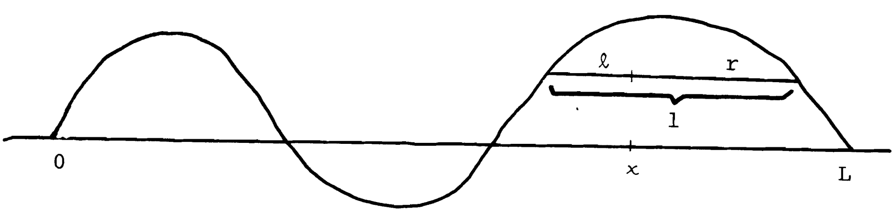


Figure 5. Balancing a table with center over [0,L].

The "center" is ℓ units from the left endpoint and $r = 1 - \ell$ from the right endpoint. Now we wish to show that for some $x \in [0,L]$, $f(x - \ell) = f(x + r)$.

Suppose false. Then by continuity we may assume $f(x - \ell) < f(x + r)$ for all $x \in [0,L]$ (the $>$ case is similar). Since $f(x - \ell) = 0$ for $x \leq \ell$, $f(x + r) > 0$ for $x \in [0,\ell]$. But then $f(x + 1 - \ell) = f(x + r) > 0$ for $x \in [0,\ell]$ and $f(x + 1 + r) > 0$. By induction $f(x + n + r) > 0$ if $x \in [0,\ell]$ and $x + n + r \leq L$. In particular, in any unit length subinterval of $[0,L]$ the set of points with $f(x) > 0$ has measure greater than ℓ. Similarly, using $f(x - \ell) < f(x + r) = 0$ for $x \in [L - r, L]$, we have that in any unit subinterval as above, the set of points with $f(x) < 0$ has measure greater than r. But $r + \ell = 1$, and we have a contradiction.

A variation of this problem is to change the domain of the function from $[0,L]$ (or $\mathbf{R}^1$) to the unit circle in $\mathbf{R}^2$. Then to balance a table, it must have length at most 2 and this condition suffices. For if we rotate a chord equal to the length of the table around the circle, at some time one endpoint is at a minimum point and at some time at a maximum point. So at some time f takes the same value at both endpoints.

A less immediate fact which seems related to the open question from the previous section about cyclic quadrilaterals is the following.

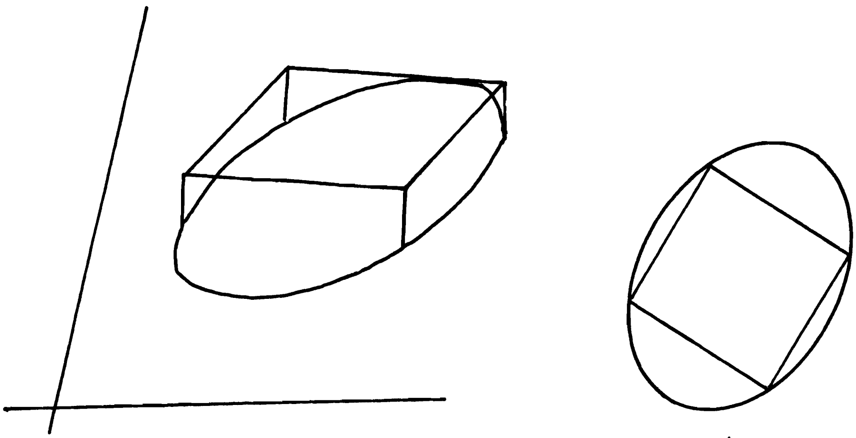
Theorem 3. *Let* f *be a continuous real-valued function on the unit circle,* C, *with graph in* $\mathbf{R}^3$. *Let* A_1, A_2, A_3, *and* A_4 *be any points of* C. *Then there are (at least two) rotations* r *of* C *so that* $\{rA_i, f(rA_i)\}_{i=1}^{4}$ *is a coplanar set in* $\mathbf{R}^3$.

Proof. We may assume that the A_i's are distinct and cyclically ordered, with $D = A_1A_3 \cap A_2A_4$. Let $\alpha = A_1D/A_1A_3$, $0 < \alpha < 1$. For $0 \leq \theta \leq 2\pi$, let r_θ be the rotation of $\mathbf{R}^2$ about the origin through an angle of θ. Let $h(\theta) = \alpha f(r_\theta A_3) + (1-\alpha)f(r_\theta A_1)$. Note that there is a point of the segment between $(r_\theta A_i, f(r_\theta A_i))$, $i = 1$ or 3, directly over $r_\theta D$ at a height of $h(\theta)$. The average value of h with respect to θ, $(h_{av})_\theta$, is $(1/2\pi) \int h(\theta)d\theta = (f_{av})_\theta$. Similarly, if $g(\theta)$ is the height above $r_\theta D$ of the segment between $(r_\theta A_i, f(r_\theta A_i))$ $i = 2$ or 4, then $(g_{av})_\theta = (f_{av})_\theta$. So for (at least 2) values of θ, $h(\theta) = g(\theta)$, and so for these θ the diagonals of $(r_\theta A_i, f(r_\theta A_i))$ $i = 1,2,3,4$ meet at $(r_\theta D, h(\theta))$, and so the four points are coplanar.

4. Planar Closed Curves

The preceding result is similar to the final situation we will consider. Instead of placing a table on a closed curve in space which lies over a circle, we will consider closed planar curves. This brings us to the oldest results.

In 1913 Emch ([Em]) showed that any oval (a "smooth" convex curve) contains the vertices of a square. (The problem appears to have originated with Toeplitz ([To]) in 1911.)



Top View

Figure 6. Given oval $\mathcal{O}$ there is some horizontal square table with all four feet on $\mathcal{O}$.

Emch's method was first to show that in each direction there is exactly one rhombus with a diagonal in that direction and at most one rhombus with side in that direction. To see the existence of the former rhombus, he considered the centers of chords in the given direction and in the perpendicular direction, to get two interesecting curves. The intersection point is the center for the rhombus. Such rhombi vary continuously as we continuously change direction, and so by turning 90° we get back to the same rhombus. At some intermediate point we have equal diagonals, and so a square.

Further work by Schnirelman ([Sc]), Guggenheimer ([Gu]), and Jerrard ([Je]) has reduced the hypotheses on the curve. Most recently, Jerrard has shown that the result holds for any piecewise C^1 curve in the plane. But whether every planar simple closed curve contains the vertices of a square is still open.

Recently, Vaughan ([Va]) has presented a beautifully simple proof that any planar simple closed curve S contains the vertices of a rectangle. Let $M = \{\{x,y\} | x, y \in S\}$ and define $f: M \to \mathbf{R}^3$ by letting $(f_1(\{x,y\}), f_2(\{x,y\}))$ be the midpoint of segment xy and $f_3(\{x,y\})$ equal the length of xy. Now M is a Möbius strip and if S doesn't contain the vertices of a rectangle f is one-to-one. So then f(M) is a Möbius strip which lies in the closed half-plane $z \geq 0$ and meets $z = 0$ in its boundary S. Adding the interior of S we get a projective plane embedded in $\mathbf{R}^3$, a contradiction.

In contrast to this, we have:

Example 1. A simple closed curve in space may fail to contain the vertices of a rectangle.

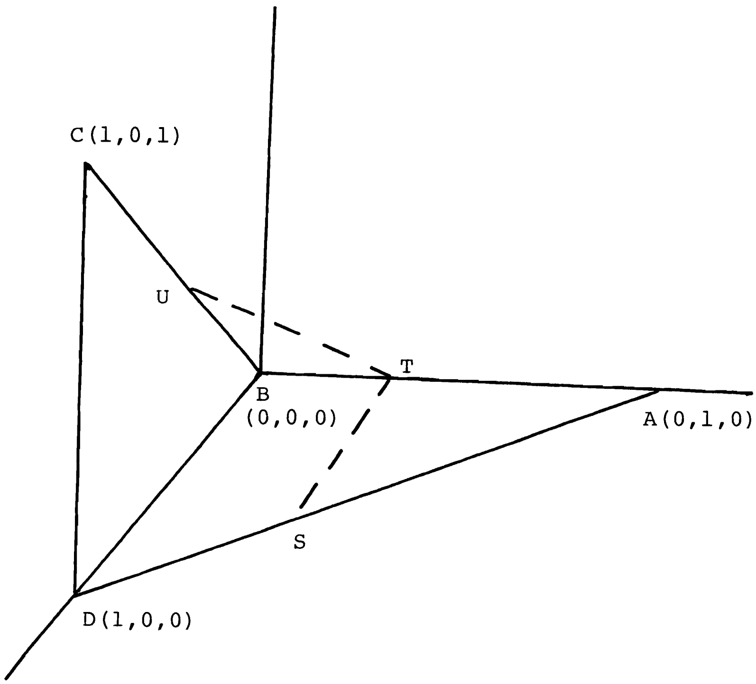


Figure 7. No rectangle has vertices on closed curve ABCD.

Construction. Let Q = ABCD be the quadrilateral in $\mathbf{R}^3$ with A, B, C, and D equal to (0,1,0), (0,0,0), (1,0,1), and (1,0,0) respectively. Suppose a rectangle, R, has vertices on Q. Consider the planes Σ_A and Σ_C determined by ABD and CBD respectively. All four vertices of R can't lie in one of these planes; hence two vertices must lie in each plane. Considering planes containing a side of Q we see that no side can contain opposite corners of R. Since no side of Q is parallel to (and not in) either Σ_A or Σ_C, vertices of R must lie cyclically, one to a side of Q. Now the plane determined by R meets Σ_A and Σ_C in parallel lines, so R has 2 sides parallel to the line BD. Let R = STUV, $S \in DA$ and $T \in AB$, where $ST || DB$. Then $\measuredangle STU < 90°$, a contradiction.

We close with consideration of triangular tables on ridges. (For details and further results see [Me2].) Given a planar set, X, call a point P a *vertex point* if some equilateral triangle has all vertices on X with one vertex at P. Now, before looking at ridges, we explore another continuous curve, the triod, an embedding of the letter "T."

A somewhat technical approximation and intersection argument shows that if T is a planar triod then one of its endpoints is a vertex point.

We can then conclude that for any planar triod, every point of some leg, except perhaps the junction point where the legs meet, is a vertex point. For otherwise every leg would have a non-vertex point. These three non-vertex points would determine a subtriod with no endpoint a vertex point, violating the previous result.

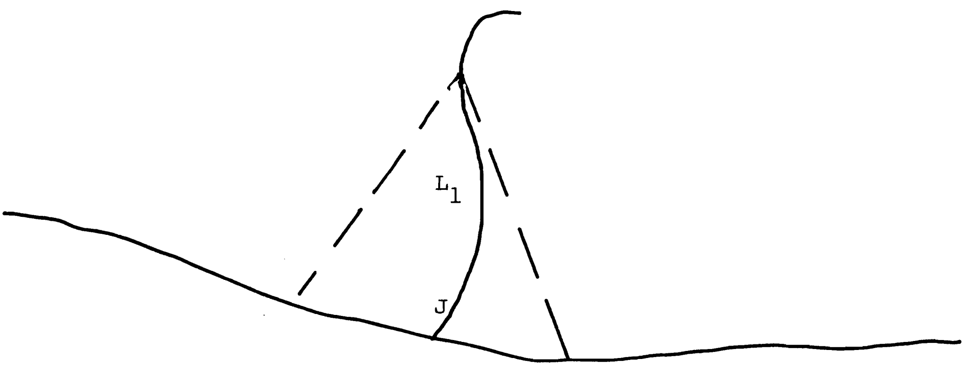


Figure 8. Every point of leg L_1, except the junction point J, is the vertex for an equilateral triangle with all vertices on the triod. There is always such a leg.

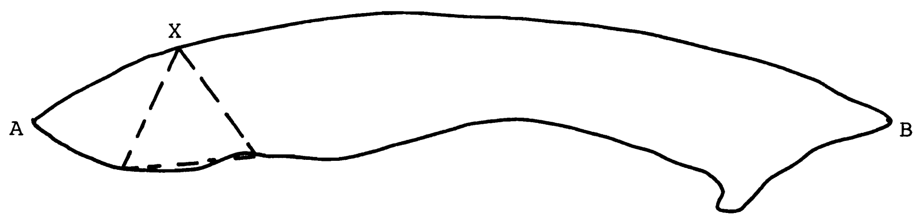


Figure 9. At most 2 points of a simple closed planar curve can fail to be vertex points. If $X \neq A,B$ is on the curve, then there is an equilateral triangle as above.

Further, we see that for a simple closed planar curve, S, at most 2 points can fail to be vertex points. For suppose we had three non-vertex points, A, B, and C. Then there is a planar triod in the closure, D, of the interior of S with endpoints at A, B, and C. So there is an equilateral triangle with one vertex at say C and the other two vertices in D. Expand the triangle radially from C until a second vertex meets S. Then, keeping the vertex at C fixed, move this second vertex to a point of S maximally distant from C, moving the third vertex so we always have an equilateral triangle. At some point we must have all 3 vertices on S, so C is a vertex point.

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