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1. Introduction

This paper concerns the following question raised by Markov [21] in 1945:

Does every infinite group admit a non-discrete Hausdorff topology?

Since the appearance of Markov's paper, there has been substantial research related to this question. For example, Kertész and Szele [19] proved in 1953 that every infinite abelian group admits a non-discrete metrizable topology. About a decade later, Comfort and Ross [4] discovered a natural one-to-one correspondence between the collection of admissible totally bounded Hausdorff topologies on an abelian group G and the collection of point-separating groups of characters on G. The same authors also found an elegant generalization of the Kertész-Szele theorem. More recently, Fletcher and Liu [10] have shown that the full homeomorphism groups of a large class of topological spaces do admit non-discrete Hausdorff topologies and Taimanov [30] has proved that all *'big'* subgroups of a permutation group are, indeed, non-trivially topologizable. It is also known now that, assuming CH, there exist infinite groups which admit no topology except the two trivial ones.

Although our main concern here is to topologize groups, not all sections of this paper are directly related to Markov's question. In §2, we give a simple and yet important extension of the Kertész-Szele theorem. In §3, we extend a result of Prodanov [23] by showing that any admissible Hausdorff topology on a countable group can be condensed into an admissible metrizable topology. In §4, we show that under some compatibility conditions, an ideal on a given set induces admissible non-discrete Hausdorff topologies on certain groups of permutations of the set. This leads to an extension of a result of Taimanov [30]. In §5, the *method of ideals* is applied to define several admissible topologies on the full homeomorphism groups of certain classes of spaces. For example, we show that if a topological space X has the property that each of its finite subsets is periodic, then the full homeomorphism group of X admits a non-trivial topology; this extends a result of Fletcher and Liu [10]. In §6, some algebraic properties of the full homeomorphism groups of some spaces are studied and a question of Fletcher and Liu [10] is answered.

A word of caution is in order. Our terminology often differs from that of Fletcher and Liu [10]. For example, the *weak Galois spaces* of [10] are, in our terminology, those *spaces in which each proper closed subset is periodic.*

2. Topologies on Groups with Infinite Centers

Let $(G,\cdot)$ be a group and let τ be a topology on G, such that $(G,\cdot,\tau)$ is a topological group. Then $(G,\cdot)$ is said to *admit* the topology τ and τ is said to be an *admissible* topology on $(G,\cdot)$. An admissible topology τ on $(G,\cdot)$ is said to be *subgroup generated* provided the neighbourhood filter of the identity element e of $(G,\cdot)$ has a filter base each element of which is a subgroup of $(G,\cdot)$.

2.1. *Remark.* It is well known [3] that if G is a group and if β is a filter base on G satisfying

(B1) For each $V \in \beta$, there exists a $W \in \beta$ such that $W \cdot W \subset V$

(B2) For each $V \in \beta$, there is a $W \in \beta$ such that $W^{-1} \subset V$; and

(B3) For each $g \in G$ and each $V \in \beta$, there exists a $W \in \beta$ such that $g^{-1} \cdot W \cdot g \subset V$

then there is a unique admissible topology on G for which β is a neighbourhood filter base of the identity element e of G. The following simple extension of the Kertész-Szele theorem is worth recording.

2.2. *Theorem. Every group with infinite center admits a non-discrete metrizable topology.*

Proof. Let G be a group such that the center Z(G) of G is infinite. In view of the Kertész-Szele theorem, there exists an admissible non-discrete metrizable topology τ on Z(G). Let β be the collection of all τ-neighbourhoods of e in Z(G). Since Z(G) is the center, it is clear that $g^{-1} \cdot A \cdot g = A$ for any $g \in G$ and any $A \subset Z(G)$. It is now easy to verify that β is a filter base at e for an admissible topology τ^* on G; and that the topology τ^*, like τ, is non-discrete and metrizable.

2.3. *Remark.* Shelah [28] has shown that, assuming CH, there is a group G of cardinality $\aleph_1$, such that G admits only the two trivial (the discrete and the indiscrete) topologies. Let n be any positive integer and let C_n be the cyclic group of order n. Note that the group $G \times C_n$ has exactly n elements in its center. It is easy to show that $G \times C_n$ admits no non-discrete Hausdorff topology. Hence, given CH and any positive integer n, there is group having exactly n elements in its center and admitting no non-discrete Hausdorff topology.

2.4. *Remark.* An interesting generalization of the Kertész-Szele theorem discovered by Comfort and Ross [4] states that every infinite abelian group is a dense non-discrete subgroup of a locally compact Hausdorff abelian group. Whether a similar statement holds for all groups with infinite centers is not known.

2.5. *Remark.* Another result of Comfort and Ross [4] states that, given any abelian group G, there is a natural one-to-one correspondence between the collection of admissible totally bounded Hausdorff topologies on G and the collection of point-separating groups of characters on G; each admissible totally bounded topology on G is the weak topology of the group of its continuous characters. Combining this fact with the observation that for each countable abelian group G there exists a countable group of characters which separates the points of G, one can easily show that every countable infinite abelian group has $\exp(\aleph_0)$ admissible totally bounded metrizable topologies. Consequently any group with infinite center admits at least $\exp(\aleph_0)$ metrizable topologies.

3. Topologies on Countable Groups

In this section we extend a result of Prodanov [23], as promised in the introduction.

3.1. *Theorem. For any admissible Hausdorff topology* τ *on a countable group* G, *there exists an admissible metrizable topology* τ^* *on* G *such that* $\tau^* \subset \tau$.

Proof. If τ itself is metrizable (and, in particular if G is finite) then by letting $\tau^* = \tau$, we are done. So assume that G is infinite and let $G = \{x_n: n < \omega\}$ with $x_o = e$, the identity element of G. Fix a sequence $\{V_n: n < \omega\}$ of τ-neighbourhoods of e such that $\cap V_n = \{e\}$. Partition the set N of positive integers into a collection $\{A_n: n \in N\}$ of sets such that each A_n is infinite. Let $f: N \to G$ be defined by the rule: $f(n) = x_j$ if and only if $n \in A_j$. Now choose a sequence $\{W_n: n < \omega\}$ of τ neighbourhoods of e inductively as follows: Let $W_0 = V_0 \cap V_0^{-1}$ and suppose W_j has already been defined for each $j < n$. Then, we let W_n to be any τ-neighbourhood of e satisfying the following three conditions.

(i) $W_n = W_n^{-1}$

(ii) $W_n^2 \subset V_{n-1} \cap W_{n-1}$, and

(iii) $[f(n)]^{-1} \cdot W_n \cdot f(n) \subset W_{n-1}$

Such a choice of W_n is clearly possible. The decreasing sequence $\{W_n: n < \omega\}$ is a filter base at e which induces an admissible topology τ^* on G satisfying all of the required conditions.

Let us note some easy consequences of this theorem.

3.2. *Corollary.* (Prodanov [23]). *Every countable minimal abelian Hausdorff group is metrizable.*

In view of our theorem and the well-known fact that any countable dense-in-itself metrizable space is homeomorphic to the space Q of rational numbers, we have the following result:

3.3. *Corollary. If a countable group admits a non-discrete Hausdorff topology then it also admits a topology under which it is homeomorphic to the space of rational numbers.*

3.4. *Corollary. If a countable group admits a totally bounded Hausdorff topology then it also admits one which is totally bounded and metrizable.*

4. Method of Ideals

An *ideal* $\mathcal{I}$ on a non-empty set X is a non-empty collection of subsets of X such that

(i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$

(ii) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$; and

(iii) $X \notin \mathcal{I}$.

By an *ideal on a topological space* we shall mean an ideal on the underlying set of the space. The group of all permutations on a set X is denoted by $\Pi(X)$. Note that the group operation of $\Pi(X)$ is composition of maps, and that the identity element e of $\Pi(X)$ is the function for which $e(x) = x$ for all $x \in X$. For a subset G of $\Pi(X)$ and a subset A of X we define $A^*(G) = \{g \in G: g(a) = a \text{ for all } a \in A\}$.

4.1. *Definition.* Let $\mathcal{I}$ be an ideal on a non-empty set X and let G be a subgroup of $\Pi(X)$. The ideal $\mathcal{I}$ is said to be *compatible* with the group G provided the following conditions are satisfied

(i) $A \in \mathcal{I}$ and $g \in G$ implies $g(A) \in \mathcal{I}$

(ii) For each $A \in \mathcal{I}$, $A^*(G) \neq \{e\}$; and

(iii) For each $g \in G$, $g \neq e$, there is some $A \in \mathcal{I}$ such that $g \notin A^*(G)$.

Suppose an ideal $\mathcal{I}$ on a set X and a subgroup G of $\Pi(X)$ are given. Let $\mathcal{I}$ be compatible with G and let $\beta = \{A^*(G): A \in \mathcal{I}\}$. It is easily seen that β satisfies conditions (B1) and (B2) of 2.1. Furthermore, from the compatibility conditions (ii) and (iii) it follows that $\{e\} \notin \beta$ and $\cap\{A^*(G): A \in \mathcal{I}\} = \{e\}$. Also note that each member of β is a subgroup of G. We claim that β also satisfies condition (B3) of 2.1. To verify this claim, take any $g \in G$ and any $A \in \mathcal{I}$ and let $C = A \cup g(A)$. Then $C \in \mathcal{I}$ and $g^{-1} \cdot C^*(G) \cdot g \subset A^*(G)$. These observations lead us to the following result.

4.2. *Theorem. Let $\mathcal{I}$ be an ideal on a non-empty set* X *and let* G *be a subgroup of* $\Pi(X)$. *If $\mathcal{I}$ is compatible with* G, *then it induces an admissible, subgroup generated, non-discrete, Hausdorff topology on* G.

Now, we will examine one particular ideal which we shall later find to be very useful. Before defining the ideal and examining its compatibility, we need to introduce some terminology. In the remainder of this section, X will denote an arbitrary fixed infinite set, Y is a fixed infinite subset of X and G is an arbitrarily fixed subgroup of $\Pi(X)$. Our goal is to find conditions under which the ideal $\mathcal{I}_0(Y)$, of all finite subsets of Y is compatible with G.

4.3. *Definition.* The set Y is said to be G-*invariant* provided $y \in Y$ and $g \in G$ implies $g(y) \in Y$.

4.4. *Definition.* The identity element e of G is said to be *pure with regard to* the set Y provided $g \in G$ and $g(y) = y$ for all $y \in Y$ implies $g = e$ (i.e. $g(x) = x$ for all $x \in X$).

4.5. *Remark.* If the identity element of G is pure with regard to Y, then it is clear that for $g_1, g_2 \in G$, the condition $g_1(y) = g_2(y)$ for all $y \in Y$ implies $g_1 = g_2$.

The following theorem is merely a reformulation of the compatibility conditions of 4.1.

4.6. *Theorem. The ideal* $\mathcal{I}_0(Y)$ *of all finite subsets of* Y *is compatible with* G *if and only if the following conditions are satisfied*

(i) Y *is* G-*invariant*

(ii) $A \in \mathcal{I}_0(Y)$ *implies* $A^*(G) \neq e$, *and*

(iii) *The identity element of* G *is pure with regard to* Y.

4.7. *Corollary. The ideal* $\mathcal{I}_0$ *of all finite subsets of* X *is compatible with* G *if and only if* $A \in \mathcal{I}_0$ *implies* $A^*(G) \neq e$.

4.8. *Lemma. Let* Y *be* G-*invariant and let* $|G| > |Y|$. *Then for any* $y \in Y$, $|\{y\}^*(G)| > |Y|$.

Proof. Let y be an arbitrary (but fixed) element of Y. For any $t \in Y$; let $P(t) = \{g \in G: g(y) = t\}$. As Y is G-invariant, $\cup\{P(t): t \in Y\} = G$; and therefore, in view of $|G| > |Y|$, there is some $t_1 \in Y$ such that $|P(t_1)| > |Y|$. Let us fix some $h \in G$ such that $h(t_1) = y$. Then the set $H = \{hg: g \in P(t_1)\}$ is such that $|H| > |Y|$. Since H is a subset of the group $\{y\}^*(G)$, the lemma is proved.

4.9. *Theorem. Let* Y *be* G-*invariant and let the identity element of* G *be pure with regard to* Y. *Then* $|G| > |Y|$ *implies that the ideal* $\mathcal{I}_0(Y)$ *is compatible with* G.

Proof. Let $A = \{y_1, y_2, \cdots, y_n\}$ be an arbitrary finite subset of Y. Let $G_1 = \{g \in G: g(y_1) = y_1\}$ and for any integer k, $1 < k \leq n$, define $G_k = \{g \in G_{k-1}: g(y_k) = y_k\}$. Clearly $G_n = A^*(G)$. A repeated use of Lemma 4.8 shows that $|G_n| > |Y|$.

The following result is obtained from Theorem 4.9 by letting $Y = X$.

4.10. *Corollary.* (Taimanov [30]). *Let* X *be an infinite set and let* G *be a subgroup of* $\Pi(X)$ *such that* $|G| > |X|$. *Then* G *admits a non-discrete, Hausdorff topology.*

The importance of Theorems 4.6 and 4.9 is brought out by the following examples.

4.11. *Example.* Let R be the real line with the usual topology and let G be the group of all surjective homeomorphisms on R for which the set Q of rationals is invariant. Since Q is dense in R, the identity element of G is pure with regard to Q. Applying Theorem 4.6 (or Theorem 4.9) we conclude that G admits a non-discrete Hausdorff topology.

4.12. *Example.* Let τ denote the usual topology on the real line R. Let τ_1 be the topology on R generated by $\tau \cup \{Q\}$; and let τ_2 be the topology on R obtained by expanding τ by isolating each element of Q. Let G_i, i = 1,2; be the group of all homeomorphisms of (R,τ_i) onto itself. Note that Q is dense in (R,τ_i) and therefore the identity element of G_i is pure with regard to Q. Applying either Theorem 4.6 or Theorem 4.9 we conclude that each G_i admits a non-discrete Hausdorff topology.

4.13. *Example.* Let X be the set of all non-zero integers. For each positive integer n, define f_n as follows: $f_n(n) = -n$, $f_n(-n) = n$ and $f_n(j) = j$ for all $j \in X$ such that $|j| \neq n$. Let G be the subgroup of $\Pi(X)$ generated by the set $\{f_n$: n is a positive integer$\}$. It is easy to see that the ideal $\mathcal{I}_0$ of all finite subsets of X is compatible with G (apply 4.7) and therefore G admits a non-discrete Hausdorff topology. Note that G is countable and thus Theorem 4.9 does not apply here.

5. Topologies on Homeomorphism Groups

The group of all homeomorphisms of a topological space X onto itself is denoted by H(X) and is known as the *full homeomorphism group* of X. Since every group is isomorphic to the full homeomorphism group of some topological space [13], topologizing the full homeomorphism groups of large classes of topological spaces could be a fruitful approach to Markov's question. Although various kinds of topologies on homeomorphism groups have been studied for a long time, it is only recently that Fletcher and Liu [10] started studying topological properties which guarantee the existence of non-discrete Hausdorff topologies on the full homeomorphism groups. In this section we use the method of ideals to find more general properties than those given in [10], which also insure the existence of suitable topologies on the homeomorphism groups.

5.1. *Definition.* A subset A of a topological space X is said to be *periodic* if there exists $h \in H(X)$, $h \neq e$, such that $h(a) = a$ for all $a \in A$. A subset A of X is said to be *strongly periodic* if for each $x \notin A$, we can find some $h \in H(X)$ such that $h(a) = a$ for all $a \in A$ and $h(x) \neq x$.

5.2. *Theorem. If each finite subset of a topological space* X *is periodic then the ideal* $\mathcal{I}_0$ *of all finite subsets of* X *is compatible with* H(X)*; and consequently* H(X) *admits a subgroup generated non-discrete Hausdorff topology.*

Proof. The conclusion of this theorem easily follows from Corollary 4.7 by taking $G = H(X)$.

5.3. *Lemma. The intersection of all the finite dense subsets of a* T_0 *space* X *is dense in* X.

Proof. If the collection of finite dense subsets of X is empty then the intersection of the collection is X, and so we are done. If there is a finite dense subset F in X, then F clearly contains a subset A such that A is dense and is minimal among the dense sets. We claim that every finite dense subset of X contains A. If possible suppose there is a finite dense subset B of X which does not contain A, let $a \in A-B$. Since B is finite and dense in X, there is some $b \in B$ such that $a \in \{b\}^-$. As no proper subset of A is dense, $b \notin A$. Since X is T_0 and since $a \in \{b\}^-$, so $b \notin \{a\}^-$. Therefore, there is some $a_1 \in A$, $a_1 \neq a$, such that $b \in \{a_1\}^-$. But then $a \in \{a_1\}^-$; a contradiction to the fact that no proper subset of A is dense in X.

5.4. *Theorem. If each proper closed subset of a* T_0 *space* X *is periodic, then so is every finite subset of* X.

Proof. Let F be an arbitrary finite subset of X. If F is not dense in X, then $\overline{F}$ is periodic and therefore so is F. Let us suppose that F is dense in X. First, note that the hypothesis that each proper closed set of X is periodic implies that X has no isolated points. This fact and the T_0 axiom together imply that X is infinite. Let A be the intersection of all the finite dense subsets of X. Then F contains A, and, by Lemma 5.3, A is dense in X. Clearly F-A is not dense in X and so there exists $a \in A$ such that $a \notin \overline{F-A}$. As no proper subset of A is dense in X, $a \notin \overline{A-\{a\}}$. Consequently there is an open set G containing a such that $G \cap F = \{a\}$. Now X-G is a proper closed subset of X and so there is some $f \in H(X)$, $f \neq e$, such that $f(x) = x$ for all $x \in X-G$. If we show that $f(a) = a$, then the periodicity of F would be established. We note that the fact that A is the smallest among the finite dense subsets of X implies that h(A) = A for all h $\in$ H(X); and therefore, in particular, f(A) = A. Since F - $\{a\} \subset$ X-G and f(x) = x for all x $\in$ X-G, it follows that f(a) = a. This completes the proof of the theorem.

5.5. *Corollary.* (Fletcher and Liu [10]). *If each proper closed subset of a* T_0 *space* X *is periodic, then* H(X) *admits a subgroup generated, non-discrete, Hausdorff topology.*

5.6. *Remark.* Let X be a T_0 space such that each proper closed subset of X is periodic. Then it follows from Theorems 5.2 and 5.4 that the ideal $\mathcal{I}_0$ of all finite subsets of X is compatible with H(X). It is also easy to check that if X has a finite dense set and if A is the smallest finite dense subset of X, then the ideal of all finite subsets of X-A is also compatible with H(X). It should be noted that the two ideals are equivalent in the sense that both induce the same topology on H(X).

5.7. *Remark.* A topological space may be such that there may be several different ideals on X compatible with H(X). In each of the following examples, we have, besides the ideal of finite subsets, some other ideal compatible with H(X).

(i) X is non-compact and each compact subset of X is periodic; the ideal generated by compact subsets of X.

(ii) X is uncountable and each countable subset of X is periodic; the ideal of all countable subsets of X.

(iii) X is a dense-in-itself (= no isolated points) T_1 space and each nowhere dense subset of X is periodic; the ideal of all nowhere dense subsets of X.

5.8. *Example.* Let R be the real line with the usual topology and let G be the group of all homeomorphisms of R onto R. We will examine the following topologies on G.

(i) τ_c; the compact open topology.

(ii) τ_0; the topology induced by the ideal $\mathcal{I}_0$ of all finite subsets of R.

(iii) τ_n; the topology induced by the ideal $\mathcal{I}_n$ of all bounded nowhere dense subsets of R.

(iv) τ_N; the topology induced by the ideal $\mathcal{I}_N$ of all nowhere dense subsets of R.

(v) τ_k; the topology induced by the ideal $\mathcal{I}_k$ of all subsets of R which have compact closures.

It is well-known [1] that τ_c admissible; the fact that the other four topologies are also admissible can be easily verified by applying Theorem 4.2. It is also straightforward to show that these five topologies are all distinct. Note that if a and b are any two real numbers such that $a < b$ and if we are given some $g \in G$ such that $g(a) = a$ and $g(b) = b$ then $a < h(x) < b$ for any x in the interval (a,b). This observation leads one to prove that $\tau_c \subset \tau_0$. It is easy to check that the following inclusion relationship holds:

(a) $\tau_c \subset \tau_0 \subset \tau_n \subset \tau_k \cap \tau_N$; and

(b) Neither of τ_k and τ_N contains the other.

Note that (G,τ_c) and (G,τ_k) are both metrizable whereas the local weight of (G,τ_0) equals the cardinality of the continuum. It is also easy to show that the local weight of (G,τ_n) equals that of (G,τ_N). Since R is of second category in itself, the local weight of (G,τ_n) must be uncountable. In Cohen's original models ZFC + $\neg$CH, the local weight of these two groups equals the cardinality of the continuum. However, we do not know whether ZFC alone can imply this result. In this context, it is known that if κ denotes the ubiquitous cardinal defined by Hechler [16], then there is a collection of cardinality at most κ, of nowhere dense subsets of R which covers R.

6. Algebraic Properties of H (X)

Fletcher and Liu [10] have proved that if X is a Hausdorff space such that each proper closed subset of X is periodic, then H(X) is infinite and nonabelian. This result lead them to raise the following questions:

Is every infinite nonabelian group isomorphic to the full homeomorphism group of some T_0 space X satisfying the condition that each proper closed subset of X is periodic? If so, can X also be taken to be completely regular?

In this section we will answer these questions in the negative. We will also construct an example of T_0 space X such that each proper closed subset of X is periodic and yet the full homeomorphism group of X is abelian.

6.1. *Theorem. Let a topological space* X *be such that each finite subset of* X *is periodic. Then* H(X) *does not satisfy the minimum condition for subgroups.*

Proof. The conclusion follows easily from the fact that H(X) admits a subgroup-generated, non-discrete Hausdorff topology.

6.2. *Theorem. If each proper closed subset of* X *is periodic and if* X *contains an infinite collection of pairwise disjoint non-empty open sets, then* H(X) *does not satisfy the maximum condition for subgroups.*

Proof. Let $\{A_n: n < \omega\}$ be a countably infinite collection of pairwise disjoint non-empty open subsets of X. Let $G_n = \{h \in H(X): h(x) = x \text{ if } x \notin A_j \text{ for } 0 \leq j \leq n\}$. We can easily show that $\{G_n: n < \omega\}$ is an increasing sequence of subgroups of H(X). Using the hypothesis that each proper closed subset of X is periodic and the imposed conditions on $\{A_n: n < \omega\}$, it can be easily seen that for each n, G_n is properly contained in G_{n+1}.

6.3. *Corollary. Let* X *be a topological space such that each proper closed subset of* X *is periodic. Then the following statements hold.*

(i) *If* X *is* T_0, *then* H(X) *does not satisfy the minimum condition for subgroups.*

(ii) *If* X *is Hausdorff then* H(X) *does not satisfy the maximum condition for subgroups.*

6.4. *Theorem. If a topological space* X *satisfies any one of the following conditions, then the center of the group* H(X) *is trivial.*

(i) *For any* x,y ∈ X, *there is some* h ∈ H(X) *such that* h *fixes one of* x,y *and moves the other.*

(ii) *Each point of* X *is strongly periodic.*

(iii) X *is* T_0 *and each proper closed subset of* X *is strongly periodic.*

(iv) X *is Hausdorff and each proper closed subset of* X *is periodic.*

Proof. We will only show that if condition (iv) is satisfied, then H(X) has a trivial center. The proof of the remaining cases is similar. Take some $h \in H(X)$, $h \neq e$. Then, there is some $x \in X$ such that $h(x) \neq x$. Since X is Hausdorff and $h \in H(X)$, we can find disjoint open neighbourhoods L and M of x and h(x) such that $h(L) = M$. Since X-L is periodic, there is some $f \in H(X)$ such that $f(z) = z$ for all $z \notin L$ and $f(p) \neq p$ for some $p \in L$. Clearly $fh(p) = h(p) \neq hf(p)$; and therefore $fh \neq hf$.

6.5. *Example.* This is an example of a T_0 space X such that each proper closed subset of X is periodic and yet H(X) is abelian. Let < be the lexicographic order on the Euclidean plane E^2 and let E^2 be given the topology generated by the collection $\{R(a,b): (a,b) \in E^2\}$, where $R(a,b) = \{(x,y) \in E^2: (a,b) < (x,y)\}$. Let X be the subspace of E^2 defined by:

$$X = \{(m,n): m \text{ and } n \text{ are integers and } m \geq 1\}.$$

The m-th column of X is the set V_m of all points of X with first coordinate m. Note that X is T_0 and each element of X has a smallest neighbourhood. It is easy to verify that $h \in H(X)$ if and only if it satisfies the following three conditions: (i) $h \in \Pi(X)$, (ii) $h(V_m) = V_m$ for all $m \geq 1$, and (iii) the restriction of h to V_m is a homeomorphism on V_m for all $m \geq 1$. Note that each proper closed subset of X is periodic. Let H(X) be given the admissible topology induced by the ideal $\mathcal{J}_0$ of all finite subsets of X. Let Z be the discrete group of integers and let Z^ω be given the product topology. Let $\lambda: Z^\omega \to H(X)$ be defined as follows: For a sequence $s = (m_j)$, $1 \leq j < \omega$ of integer, if $f_s: X \to X$ is the map $f_s(p,q) = (p,q+m_p)$ then $\lambda(s) = f_s$. It is easy to see that λ is a topological isomorphism between Z^ω and H(X). In particular, H(X) is abelian.

6.6. *Example*. Let 2 denote the discrete group {0,1} and let 2^ω be the cartesian product of 2 taken ω-times. Here, we will construct a topological space X with every finite set periodic, such that H(X) is isomorphic to 2^ω. For each positive integer n, let $A_n = \{(n,k)$: k is an integer and $1 \leq k \leq n\}$; $B_n = \{(n,k): (n,-k) \in A_n\}$; $A_{nj} = \{(n,k) \in A_n: k \leq j\}$; $B_{nj} = \{(n,k): (n,-k) \in A_{nj}\}$; and $X = \{(x,y): (x,y) \in A_n \cup B_n$ for some $n \geq 1\}$. Let X be given the topology generated by all sets A_{nj} and B_{nj}. It is clear that for any $h \in H(X)$, the restriction of h to $A_n \cup B_n$ is either the identity map or it maps A_n onto B_n and B_n onto A_n in a unique way. Let H(X) be given the admissible topology induced by the ideal $\mathcal{J}_0$ of all finite subsets of X. Then we can easily show that there is a topological isomorphism between H(X) and 2^ω.

7. Related Results and Problems

Given a property (e.g. compactness, precompactness etc.) of topologies on groups one would like to know which algebriac groups admit a topology having the given property. There are numerous results in literature addressing this problem. Here, we will list a few results related to Markov's question.

(i) Marshall Hall, Jr. [14] has shown that every free group admits a totally bounded Hausdorff topology. Since a free group is infinite, such a topology is necessarily non-discrete. Also, the fact that free groups are residually nilpotent can be used to show that every free group admits a non-discrete metrizable topology.

(ii) If a group G admits a totally bounded Hausdorff topology then it admits a finest such topology [8]. Each abelian group G has enough characters to separate points of G and therefore by the correspondence theorem of Comfort and Ross [4], G admits a totally bounded Hausdorff topology. If G is infinite such a topology will necessarily be non-discrete.

(iii) Not every group admits a totally bounded Hausdorff topology (see [6] for references).

(iv) The class of abelian groups admitting a compact Hausdorff topology is easy to describe (see 25.25 of [18]) and so is the class of abelian groups admitting at least one non-discrete, locally compact, Hausdorff topology.

Finally, we list some questions, a few of these have already been mentioned.

(1) *Does every countably infinite group admit a non-discrete Hausdorff topology? What about infinite solvable groups?*

(2) *Which groups admit non-discrete metrizable topologies?*

(3) *Can it be shown in* ZFC *alone that there are infinite groups which admit no non-discrete Hausdorff topologies?*

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