TOPOLOGY PROCEEDINGS

Volume 6, 1981

Pages 99–113

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN:	0146-4124

THE DEVELOPMENT OF AND GAPS IN THE THEORY OF PRODUCTS OF INITIALLY M–COMPACT SPACES

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According to A. Tychonoff's theorem, obtained in 1930, every product of compact spaces is compact. Since that time, topologists and other matematicians have been considering properties similar to, but weaker than, compactness, and they have been attempting to determine the extent to which analogues of Tychonoff's theorem hold for these properties.

One such property in which there has been considerable interest is initial m-compactness. Beginning in 1938 when E. Čech asked if countable compactness is productive, or perhaps earlier when P. Alexandroff and P. Urysohn published [AU], it became apparent that some questions concerning products of "nearly compact" spaces might turn out to be both very interesting and difficult to answer. In this talk we shall focus our attention on initially m-compact product spaces for various infinite cardinal numbers m.

The cardinality of a set X will be denoted by $|X|$, and for a filter base $\mathcal{F}$ on a topological space X, the set of adherent points of $\mathcal{F}$, $\cap\{\overline{F}: F \in \mathcal{F}\}$, will be denoted by ad $\mathcal{F}$. The successor of a cardinal number m will be denoted by m+. We shall write c for $2^{\aleph_0}$, and MA (CH, GCH) will denote Martin's Axiom (the Continuum or Generalized Continuum Hypothesis).

A topological space is called *initially* m-*compact* (where m is an infinite cardinal number) if any of the following equivalent conditions holds:

(i) for every filter base $\mathcal{F}$ on X, if $|\mathcal{F}| \leq m$ then ad $\mathcal{F} \neq \emptyset$;

(ii) for every open cover $\mathcal{V}$ of X, if $|\mathcal{V}| \leq m$ then $\mathcal{V}$ has a finite subcover; or

(iii) for every subset A of X, if $|A| \leq m$ then A has a complete accumulation point, i.e., a point every neighborhood of which contains $|A|$ points of A.

Initially $\aleph_0$-compact spaces are called *countably compact*. These ideas are due to Alexandroff and Urysohn.

That initial m-compactness could be quite different from compactness was discovered in the early 50's. Examples published in 1952 by H. Terasaka [T2] and in 1953 by J. Novák [N3] showed that, in general, countable compactness was not productive and, in fact, that one could construct two countably compact completely regular Hausdorff spaces whose product is not even pseudocompact (by *pseudocompact* one means a space on which every continuous real valued function is bounded).

During the last 20 years the theory of countably compact spaces and product spaces has been extensively developed. While there has also been much activity in the area of initially m-compact spaces, for m $\geq \aleph_0$, the theory of the latter is, at this time, much less complete than the theory of countably compact spaces. In both areas a common trend, since the discoveries of Novák and Terasaka, has been the search for well behaved properties, only slightly stronger than initial m-compactness, which force certain product spaces to be initially m-compact. We shall outline much of the progress that has occurred since 1953 and indicate some of the gaps in the theory and open problems on which further work is needed.

In the late 50's Z. Frolík [F1], [F2] and I. Glicksberg [G] independently considered conditions somewhat stronger than initial m-compactness which could be used to produce initially m-compact product spaces. Several very useful theorems were obtained.

Theorem 1. Glicksberg [G]. *Let* X *be a product of no more than* m *completely regular Hausdorff spaces each of which is initially* m*-compact. Then* X *is initially* m*-compact if either* (i) *all but one of the factors are locally compact or* (ii) *all but one of the factors have character* $\leq$ m.

Frolík's results concerned the case $m = \aleph_0$ and the *family* F of completely regular Hausdorff spaces X such that every infinite subset of X has an infinite subset whose closure in X is compact.

Theorem 2. Frolík [F2]. *Every product of countably many members of* F *is a member of* F.

Another nice theorem concerning a property which forces certain product spaces to be countably compact was obtained by A. H. Stone in 1966.

Theorem 3. Stone [SS2]. *Every product of at most* $\aleph_1$ *sequentially compact spaces is countably compact.*

Next, N. Noble [N1] studied the *family* F' of T_1-spaces X such that every infinite subset of X has an infinite subset whose closure in X is compact. He succeeded in improving Frolík's and Glicksberg's theorems by obtaining, among other results, the following.

Theorem 4. Noble [N1]. *For each space* $X \in F'$ *and countably compact* T_1*-space* Y, *the space* $X \times Y$ *is countably compact.*

For completely regular Hausdorff spaces, Theorem 4 was also used by T. Isiwata [I].

Theorem 5. Noble [N2]. *Every product of at most* m+ *spaces, each of which is initially* m*-compact and of character* $\leq$ m, *is initially* m*-compact.*

About the same time three other authors, S. L. Gulden, W. M. Fleishman, and J. H. Weston, defined a topological space X to be m-*bounded* provided that for every subset A of X, if $|A| \leq m$ then there is a compact subset K of X with $A \subset K$. Although this concept appeared to be much stronger than initial m-compactness, it was different from those used by Glicksberg, and one could easily show that it was productive and, therefore, would produce initially m-compact product spaces.

In 1969 I defined a T_1-space X to be *strongly initially* m*-compact* provided that for every filter base $\mathcal{F}$ on X, if

$|\mathcal{F}| \leq m$ then there exists a compact subset K of X such that $\mathcal{F}|K$ is a filter base. For T_1-spaces, this concept generalized m-boundedness, and for $m = \aleph_0$, it generalized sequential compactness and was equivalent with membership in F'. One could prove that strong initial m-compactness is finitely productive, and the product of a strongly initially m-compact space and an initially m-compact space is initially m-compact. In addition, I was able to use the concept, for the case $m = \aleph_0$, to strengthen Theorem 3.

Theorem 6. Stephenson [SS1]. *Every product of at most* $\aleph_1$ *spaces in* F' *is countably compact.*

The best property of this type was discovered and later refined by J. E. Vaughan in the early and mid 70's. He defined a space X to be $\underline{(1)_m}$ provided that for every filter base $\mathcal{F}$ on X, if $|\mathcal{F}| \leq m$, then there exist a compact set $K \subset X$ and a filter base $\mathcal{G}$ on X such that $|\mathcal{G}| \leq m$, and $\mathcal{G}$ is finer than both $\mathcal{F}$ and the filter base of all open sets containing K [V1]. Later [V4] he defined a space X to be TI m-*compact* provided that for every filter base $\mathcal{F}$ on X, if $|\mathcal{F}| \leq m$ then there exists a finer total filter base $\mathcal{G}$ on X with $|\mathcal{G}| \leq m$. (A filter base $\mathcal{G}$ on X is called *total* if every finer filter base has an adherent point--this concept is due to B. J. Pettis--see [P], [V2].) A regular, TI m-compact space is $(1)_m$.

Vaughan's concept might be considered "best" in that it made possible a simultaneous generalization of most of the theorems above, namely, by Theorems 7 and 8.

Theorem 7. Vaughan [V1]. *Let* $X = \Pi\{X_a : a \in A\}$, *where each* X_a *is* $(1)_m$.

(i) *If* $|A| \leq m$ *then* X *is* $(1)_m$.

(ii) *If* $|A| \leq m^+$ *then* X *is initially* m-*compact.*

Theorem 8. Stephenson-Vaughan [SV]. *If a space* X *is* $(1)_m$ *and* Y *is an initially* m-*compact space, then* $X \times Y$ *is initially* m-*compact.*

Analogous theorems were obtained for his second definition.

Theorem 9. Vaughan [V4]. *Let* $X = \Pi\{X_a : a \in A\}$ *where each* X_a *is* TI m-*compact.*

(i) *If* $|A| \leq m$ *then* X *is* TI m-*compact.*

(ii) *If* $|A| \leq m^+$ *then* X *is initially* m-*compact.*

Theorem 10. Vaughan [V4]. *If* X *is* TI m-*compact and* Y *is an initially* m-*compact space, then* $X \times Y$ *is initially* m-*compact.*

To verify that Theorems 7 and 8 (or 9 and 10) include most of the previous ones, it suffices to note that one has (in some cases, require T_1) the following implications.

$$
\begin{array}{lcl}
\left.\begin{array}{l}\text{locally compact}\\ \quad\quad + \\ \text{initially m-compact}\end{array}\right\} & \searrow & \begin{array}{l}\text{m-bounded}\\ \Downarrow\\ \text{strongly initially m-compact}\end{array}\\
 & & \Downarrow \\
\left.\begin{array}{l}\text{of character} \leq m\\ \quad\quad + \\ \text{initially m-compact}\end{array}\right\} & \searrow & \begin{array}{l}(1)_m\\ \Downarrow\\ \text{TI m-compact}\\ \Downarrow\\ \text{initially m-compact}\end{array}
\end{array}
$$

A number of interesting examples, some quite simple, show that most of these concepts are distinct. We briefly describe them.

Example 11. Let X be the set of ordinal numbers $< m+$, with the order topology, and let $P = X^n$, where n is a cardinal number. Then P is m-bounded but not compact. P is locally compact if and only if $n < \aleph_0$, and P is of character $\leq m$ if and only if $n \leq m$.

Example 12. The following space, C, similar to ones due to H. H. Corson, I. Glicksberg, L. S. Pontryagin, and J. Kister, is m-bounded but not initially $m+$-compact. For each ordinal number $a < m+$ choose a compact Hausdorff space X_a with $|X_a| \geq 2$ and fix a point $p_a \in X_a$. Let C be the set of all points x in the product space $\Pi\{X_a\}$ such that $|\{a: x_a \neq p_a\}| \leq m$. In case each X_a is a topological group with identity p_a, then C is a topological group.

Example 13. Frolík [F1], [F2]. If $x \in \beta N \backslash N$ let $K_x = \beta N \backslash \{x\}$. Then each $K_x \in F$ but is not $\aleph_0$-bounded or sequentially compact. Moreover, there exist sets A and B such that $|A| = c$ and $|B| = 2^c$, but $\Pi\{K_x: x \in A\} \notin F'$ and $\Pi\{K_x: x \in B\}$ fails to be countably compact. In 1975 S. H. Hechler proved [H1], under MA, the space $\Pi\{K_x: x \in C\}$ is in F if $|C| < c$ and is countably compact if $|C| \leq c$.

Example 14. M. E. Rudin [R]. If CH holds, then there exists a separable, noncompact, sequentially compact space. (In Trans. Amer. Math. Soc. 155 (1971), 305-314, Franklin and Rajagopalan obtain such a space within ZFC.)

The next, very nice construction, as well as a topological group version of it [SS1], was discovered by Victor Saks while he was a graduate student working under W. W. Comfort. It shows that for every $m \geq \aleph_0$, if m has the discrete topology, then there exists an initially m-compact space $m \subset P \subset \beta m$ such that $|P| \leq 2^m$. For the case $m = \aleph_0$, the result that such a space exists is due to Frolík [Fl].

Example 15. Saks [SS1]. Let m be an infinite cardinal number. For a subset A of a compact space, denote by A' a set obtained as follows: for each filter base $\mathcal{F}$ on A such that $|\mathcal{F}| \leq m$ and $|F| \leq m$ for every $F \in \mathcal{F}$, choose one adherent point $p_{\mathcal{F}}$ of $\mathcal{F}$, and let A' be the set of all such adherent points. Now in any Hausdorff compactification of the discrete space m, define $P_0 = m$, $P_b = (\cup\{P_a: a < b\})'$ for each ordinal number $0 < b < m+$, and, finally, $P = \cup\{P_b: b < m+\}$. Then, as pointed out by Saks, (i) P is initially m-compact, (ii) P is not m-bounded if the compactification has cardinality $> 2^m$, and (iii) P can be obtained as a topological group by a similar construction. Saks and I also noted in [SS1] that for m regular and for a compactification of cardinality $> 2^m$, P fails to be strongly initially m-compact. In 1974 [SV], Vaughan and I proved, moreover, that if the compactification is βm, then P fails to be $(1)_m$, whether m is regular or not.

Example 16. Eric van Douwen [vD3]. Let m be an uncountable regular cardinal number and $\leq$ its usual well order. Call a subset C of m a *cub set* if sup C = m and if $\sup B \in C \cup \{m\}$ for every subset B of C. Next, viewing m as a discrete space, define V to be the following subspace of βm:

$$V = \cup\{\overline{I}: I \subset m \text{ and } I \cap C = \emptyset \text{ for some cub set } C\}.$$

The space V is then locally compact and initially m-compact (hence strongly initially m-compact) but not m-bounded or of character $\leq$ m.

Example 17. Vaughan [V5]. If V is as in Example 16 and n is a cardinal number, then the space $X = V^n$ is (i) not strongly initially m-compact if $n \geq \aleph_0$, (ii) not TI m-compact if $n \geq 2^m$ and (iii) initially m-compact if GCH.

Before discussing some open questions concerning the productivity of initial m-compactness, let us consider two positive theorems of a different sort than those discussed earlier.

The first is a general reduction theorem which, for the case $m = \aleph_0$, strengthened A. H. Stone's reduction theorem in [SS2].

Theorem 18. Saks [S1]. *Let* $X = \Pi\{X_a: a \in A\}$ *and* $m \geq \aleph_0$. *Then* X *is initially* m-*compact if and only if* $\Pi\{X_b: b \in B\}$ *is initially* m-*compact for each* $B \subset A$ *such that* $|B| \leq 2^{2^m}$.

The second is a surprising result I obtained and later improved which concerns only singular cardinals m, i.e., m such that m can be expressed as a sum of fewer, smaller cardinals. In set theory some of the most unexpected (and interesting) results have turned out to be those concerning singular cardinals--the same has been the case in this area.

Theorem 19. Stephenson [SV]. *Let* m *be a singular cardinal number and suppose that* $2^n \leq m$ *for every* $n < m$. *Then initial* m-*compactness is productive.*

Thus, in contrast with the situation for $m = \aleph_0$, it follows from GCH that initial m-compactness is productive for every singular cardinal number m. Even without the assumption of GCH, a standard technique shows that Theorem 19 applies to cofinally many cardinals (given a, define $m_0 = a$, $m_{i+1} = 2^{m_i}$, and $m = \sup\{m_i : i \in \omega\}$).

Obtaining answers to some of the problems below would significantly help the development of the theory of initially m-compact spaces and product spaces.

Problem 20. Let m be a singular cardinal number. Is every TI m-compact space strongly initially m-compact (m-bounded)? Is every strongly initially m-compact space m-bounded?

Problem 21. Let m be a regular uncountable cardinal number. Is every product of strongly initially m-compact (TI m-compact) spaces initially m-compact?

Problem 22. Using Example 13, Vaughan [SV] proved that, for $m = \aleph_0$, it cannot be determined within ZFC if the bounds in the conclusion of Theorem 7 can be improved. What can be said for $m > \aleph_0$?

Problem 23. Can the bound in Saks' Theorem 18 be improved within ZFC? Saks [S1] showed that for $m = \aleph_0$, if MA then it is the best possible.

Several, but not all, of Problems 20-23 have been raised in the literature--see [SS1], [SV], [S1], and [V5]. Before stating some additional ones, I would like to state four important results obtained by Eric van Douwen during the last 2 or 3 years. These results have filled major gaps in the theory of initially m-compact product spaces. Several of them are based on a very nice technique van Douwen has devised for constructing spaces X_0, X_1 such that each X_i is a union of (i) the nonuniform ultrafilters in βm and (ii) a space similar to the type described in Example 12.

Theorem 24. van Douwen [vD2]. *If* $c > \aleph_\omega$ *and* MA, *then there are two initially* $\aleph_\omega$*-compact normal spaces whose product is not initially* $\aleph_\omega$*-compact.*

Theorem 25. van Douwen [vD2]. *If* MA, *then there exist two normal spaces, each initially* m*-compact for every* $m < c$, *whose product is not even countably compact.*

Theorem 26. van Douwen [vD2]. *Let* m *be an uncountable regular cardinal number and assume* GCH. *There exist two normal initially* m*-compact spaces whose product is not initially* m*-compact.*

Theorem 27. van Douwen [vD1]. *Assume* MA. (i) *There exist two countably compact topological groups whose product* *is not countably compact.* (ii) *Assume* MA *and the negation of the Continuum Hypothesis. There are two initially* $\aleph_1$*-compact topological groups whose product is not countably compact.*

Problem 28. In 1966, W. W. Comfort and K. Ross published a proof that every product of pseudocompact groups is pseudocompact [CR], and about that time Comfort raised the related question of whether or not every product of countably compact groups is countably compact. What other results besides Theorem 27 can be obtained concerning initially m-compact groups and their products?

Problem 29. While Theorem 19 implies that initial m-compactness is productive for cofinally many singular cardinals m, it follows from Theorems 19 and 24 that it cannot be determined within ZFC if initial $\aleph_\omega$-compactness is productive. For which singular cardinals $m > c$ is an analogous result true? See 7.2 of [vD2].

Problem 30. Can the assumption of GCH in Theorem 26 be deleted?

In 1966 A. H. Stone asked [SS2] if every product of sequentially compact spaces is countably compact. M. Rajagopalan and R. Grant Woods proved in 1977 that under CH, Stone's question has a negative answer, and J. E. Vaughan proved in 1978 that under an additional axiom, there exist sequentially compact, perfectly normal spaces whose product is not countably compact.

Problem 31. Within ZFC, is every product of sequentially compact spaces countably compact?

Problem 32. Within ZFC, does there exist a first countable, countably compact space that is not $\aleph_0$-bounded?

Like Čech's question in 1938, the problems listed above (some of which also have been raised elsewhere) are likely to turn out to be difficult but very worthy of efforts to solve them.

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