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by

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Introduction

As is well known, the product X^2 of a space X of countable tightness need not have countable tightness. Also if X is a CW-complex, X^2 is not always a CW-complex. In this paper, in the first section, we consider the products of spaces of countable tightness and k-spaces. In the second section, we consider the products and the metrizability of CW-complexes.

1. Products of k-Spaces and Spaces of Countable Tightness

All spaces are assumed to be regular and T_1. We consider cardinals to be initial ordinals, and let c denote the cardinality of the continuum. Let N be the set of natural numbers.

We need the following well known example. This example will play an important role in the products.

Let α be an infinite cardinal number. Let S_α be the space obtained from the disjoint union of α convergent sequences by identifying all the limit points. S_ω is especially called the *sequential fan*.

We now recall some basic definitions.

Let X be a space, and let $\mathcal{F} = \{F_\gamma : \gamma \in \Gamma\}$ be a closed covering of X. Then X has *the weak topology* with respect to $\mathcal{F}$, if $F \subseteq X$ is closed whenever $F \cap F_\gamma$ is closed in X for each $\gamma \in \Gamma$.

A space X is a k-*space* (resp. *sequential space*), if X has the weak topology with respect to the collection of all compact subsets (resp. compact metric subsets) of X.

A space X is a k_ω-*space* [11], if it has the weak topology with respect to a countable covering of compact subsets of X.

A space X has *countable tightness*, $t(X) \leqq \omega$, if $x \in \overline{A}$ in X, then $x \in \overline{C}$ for some countable $C \subseteq A$. It is known that every sequential space has countable tightness.

Proposition 1.1. (1) *If* $X \times S_c$ *is a* k-*space, then each closed, separable subset of* X *is locally countably compact.*

(2) *If* $X \times S_c$ *has countable tightness, then each* k_ω-*subspace of* X *is locally compact.*

Proof. (1) Suppose that there exists a closed, separable subset S of X which is not locally countably compact. Since S is regular and T_1, as is well known, the weight of S is equal or less than c. Hence some $x_0 \in S$ has a local base $\{U_\alpha: \alpha < m\}$ in S, $\omega \leq m \leq c$, such that each $\overline{U}_\alpha$ is not countably compact.

We now use the idea of E. Michael [10; Theorem 2.1]. For $\alpha < m$, since $\overline{U}_\alpha$ is not countably compact, there is a decreasing sequence $\{F_{\alpha n}; n \in N\}$ of non-empty closed subsets of $\overline{U}_\alpha$ with $\cap F_{\alpha n} = \emptyset$. Let $T_\alpha = \cup\{F_{\alpha n} \times n_\alpha; n \in N\}$, where n_α denotes the n-th term of the α-th sequence in S_m, and let $T = \bigcup_{\alpha<m} T_\alpha$. Then for each compact subset K of $S \times S_m$, $T \cap K$ is closed in $S \times S_m$, because K meets only finitely many T_α's and each $K \cap T_\alpha$ is a finite union of closed subsets of $S \times S_m$. But T is not closed in $S \times S_m$. This

implies that $S \times S_m$ is not a k-space. Since $S \times S_m$ is a closed subset of $X \times S_c$, $X \times S_c$ is not a k-space. This is a contradiction.

(2) If a space has countable tightness, so does every subspace. Thus we may assume that X is a k_ω-space. Since $t(X \times S_c) \leq \omega$, $X \times S_c$ has the weak topology with respect to the covering of all closed separable subsets of $X \times S_c$. Since each subset S of $X \times S_c$ is contained in $X \times \overline{\pi(S)}$, where $\pi\colon X \times S_c \to S_c$ is the projection, $X \times S_c$ has the weak topology with respect to a closed covering $\{X \times F;\ F$ is a closed separable subset of $S_c\}$. Since we can assume that each F is contained in some S_α, $\alpha < \omega_1$, F is a k_ω-space. By [11; (7.5)], each $X \times F$ is a k-space. Thus $X \times S_c$ is a k-space. Hence, by (1) each closed, separable subset of X is locally countably compact.

We now show that X is locally compact. Let X have the weak topology with respect to a countable covering of compact subsets X_i with $X_i \subseteq X_{i+1}$. For some $x_o \in X$, suppose $x_o \in \overline{X - X_i}$ for each i. Since $t(x) \leqq \omega$, there are countable subsets $C_i \subseteq X - X_i$ with $x_o \in \overline{C}_i$. Let $C = \overline{\cup_{i=1}^{\infty} C_i}$. Then $x_o \in \overline{C \cap (X - X_i)}$ for each i. Since the closed separable subset C of X is locally countably compact, there exists a countably compact subset K of C such that $x_o \in \overline{K \cap (X - X_i)}$ for each i. Since K is countably compact in X, it is easy to see that K is contained in some X_{i_o}. But $x_o \in \overline{K \cap (X - X_{i_o})} = \emptyset$. This is a contradiction. Thus each point of X is contained in some int X_i. Hence X is locally compact.

A space X is *strongly Fréchet* [14], i.e. countably bi-sequential due to E. Michael [12], if $x \in \overline{A_n}$ with $A_{n+1} \subseteq A_n$ then there exist $x_n \in A_n$ such that $x_n \to x$. If the A_n are all the same set, then such a space X is *Fréchet*.

Lemma 1.2. (cf. [15; 16(b) and p. 35]). *Every Fréchet space which is not strongly Fréchet contains a copy of* S_ω.

Recall that a space X is *symmetric* if there is a real valued, non-negative function d defined on $X \times X$ satisfying the conditions:

(1) $d(x,y) = 0$ whenever $x = y$; (2) $d(x,y) = d(y,x)$; and (3) $A \subseteq X$ is closed in X whenever $d(x,A) > 0$ for any $x \in X - A$. If we replace the condition (3) by the following: For $A \subseteq X$, $x \in \overline{A}$ if and only if $d(x,A) = 0$, then such a space is called *semi-metric*.

Corollary 1.3. *Suppose* $X \times S_c$ *has countable tightness.*

(1) *If* X *is Fréchet, then* X *is strongly Fréchet.*

(2) [CH]. *If* X *is symmetric, then* X *is semi-metric. When* X *is paracompact,* [CH] *can be omitted.*

Proof. (1) This follows from Proposition 1.1(2) and Lemma 1.2.

(2) Let X be a symmetric space. Every Fréchet and symmetric space is first-countable ([1; p. 129]), hence is semi-metric. So, we prove that X is Fréchet. To prove this, since $t(X) \leqq \omega$, it is sufficient to show that every closed, separable subset S of X is first countable. Since

S is regular and T_1, each point of S has a local base of cardinality $\leqq c$ in S. Then, under CH each point of S is a G_δ-set in S by [16; Theorem 10]. When X is a paracompact space, without [CH], the separable space S is Lindelöf. Thus, by [13; Theorem 2] S is hereditarily Lindelöf. Then each point of S is a G_δ-set in S. Hence, then in any case each point of S is a G_δ-set in S. Thus, by Proposition 1.1(2) and [8; Lemma 6.11], S is first countable.

A *bi*-k-*space*, according to E. Michael [12], is characterized as a bi-quotient image of a paracompact M-space. For the intrinsic definition of a bi-k-space, see [12; Definition 3.E.1].

Corollary 1.4. *Suppose* f: X → Y *is a closed map with* $t(Y) \leqq \omega$. *Let* X *be a paracompact bi*-k-*space* (resp. paracompact locally compact space). *Then* $Y \times S_c$ *is a* k-*space* (resp. $t(Y \times S_c) \leqq \omega$) *if and only if* Y *is locally compact.*

Proof. Let Y be locally compact. Then $Y \times S_c$ is a k-space (resp. $t(Y \times S_c) \leqq \omega$) by [3; 3.2] (resp. [9; Theorem 4]. So we prove the "only if" part. Suppose $Y \times S_c$ is a k-space. Then, by Proposition 1.1(1), Y has property (P): Every closed separable subset is locally countably compact. Then, since $t(Y) \leqq \omega$, it is easy to see that Y satisfies Lemma 9.1(b) in [12]. Indeed, if $\{F_n: n \in N\}$ is a decreasing sequence with $y \in \cap(\overline{F_n - \{y\}})$, then there exist $y_n \in F_n$ such that $\{y_n: n \in N\}$ is not closed in Y. Then, by [12; Theorem 9.9], each $\partial f^{-1}(y)$ is compact. Thus, by [12; Proposition 3.E.4], Y is a bi-k-space.

Next, we prove that Y is locally compact. Suppose not. Then there is a point $y_o \in Y$ such that $y_o \in \overline{Y - K}$ for every compact subset K of Y. Let $\mathcal{J} = \{X - K; K \text{ is compact in } Y\}$. Then $\mathcal{J}$ is a filter base accumulating at the point y_o. Since Y is bi-k, by [12; Lemma 3.E.2] there is a decreasing closed sequence $\{A_n: n \in N\}$ satisfying the following:

(a) $C = \cap A_n$ is compact;

(b) If V is an open subset of Y with $C \subseteq V$, then $C \subseteq A_n \subseteq V$ for some n; and

(c) $y_o \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathcal{J}$.

To prove some A_n is compact, suppose not. Since Y is paracompact, each A_n is not countably compact. Then there are closed discrete subsets D_n of A_n with $|D_n| = \omega$.

Let $Y_o = C \cup \bigcup_{n=1}^{\infty} D_n$ be a subspace of Y. Then Y_o is closed in Y. Let Z be a quotient space obtained from Y_o by identifying the compact subset C. Then, by (a) and (b), Z is not locally countably compact. Since Y_o satisfies (P) and the countable space Z is the perfect image of a closed separable subset of Y_o, so then Z is locally countably compact. This is a contradiction. Hence some A_{n_o} is compact. But, by (c), $y_o \in \overline{F \cap N_{n_o}} = \emptyset$. This is a contradiction. Hence Y is locally compact.

Finally we prove the parenthetical part. Let $t(Y \times S_c) \leq \omega$ and let T be any closed separable subset of Y. Then T is a closed image of a closed separable subset S of X. Since X is paracompact, S is Lindelöf. Since X is locally compact, it is easy to see that S is a k_ω-space.

Thus, since T is a quotient image of S, T is also a k_ω-space. Then, by Proposition 1.1(2), T is locally compact. Hence Y has Property (P). Thus, since $t(Y) \leqq \omega$, Y satisfies Lemma 9.1(b) in [12]. So, by [12; Theorem 9.9] each $\partial f^{-1}(y)$ is compact. Thus Y is locally compact.

Let α be an infinite cardinal. Recall that a space X is α-*compact* if every subset of X of cardinality α has an accumulation point in X.

Lemma 1.5. *Let* $f: X \to Y$ *be a closed map with* X *collectionwise normal and* Y *sequential. If* Y *contains no closed copy of* S_α, *then each* $\partial f^{-1}(y)$ *is* α-*compact.*

Proof. Suppose some $\partial f^{-1}(y_o)$ is not α-compact. Then there exists a closed discrete subset D of $\partial f^{-1}(y_o)$ with $|D| = \alpha$. Hence there is a discrete open collection $\{V_d; d \in D\}$ of X with $V_d \ni d$. For each $d \in D$, since $y_o \in \overline{f(V_\alpha) - \{y_o\}}$, y_o is not isolated in a sequential space $\overline{f(V_d)}$. So then there is a sequence $C_d = \{y_{dn}; n \in N\}$ such that $y_{\alpha n} \to y_o$ and $C_d \subseteq \overline{f(V_d)} - \{y_o\}$. Since $\{\overline{f(V_d)}; d \in D\}$ is hereditarily closure preserving, so is the collection $\mathcal{C} = \{C_d \cup \{y_o\}; d \in D\}$. Let Y_o be the union of $\mathcal{C}$. Then Y_o is closed in Y. Let Z be the disjoint union of $\mathcal{C}$, and let $g: Z \to Y_o$ be the obvious map. Then Z is metric and g is closed with $\partial g^{-1}(y_o)$ not α-compact. Hence, by [7; Lemma 2], Y_o contains a closed copy of S_α. Thus Y contains a closed copy of S_α. This is a contradiction.

From Proposition 1.1(2) and Lemma 1.5, we have

Corollary 1.6. *Let* $f: X \to Y$ *be a closed map with* X *paracompact sequential. If* $t(Y \times S_C) \leqq \omega$, *then each* $\partial f^{-1}(y)$ *is compact.*

By Lemma 1.5, we can generalize all results in this section as follows.

Generalization. Let S be a sequential space which is a closed image of a collectionwise normal space under f such that some $\partial f^{-1}(s)$ is not c-compact. Then, *for all results in this section we can replace* "S_C" *by* "S."

By this generalization, for example we have the following:

Let Y be a Fréchet space. Let X be a collectionwise normal sequential space, and let F be a closed subset of X. Suppose Z is a quotient space obtained from X identifying F. Then Y is strongly Fréchet or ∂F is c-compact, if $t(Y \times Z) \leqq \omega$.

2. CW-Complexes

The concept of CW-complexes due to J. H. C. Whitehead [17] is well-known. We recall some basic properties of CW-complexes. Let X be a CW-complex; that is, X is a complex which is closure finite (i.e. each cell of X is contained in a finite subcomplex), and which has the weak topology with respect to the closed covering $\{L_\gamma;\ \gamma \in \Gamma\}$ of all finite subcomplexes L_γ of X. Then for any subset Γ' of Γ, $L' = \bigcup_{\gamma \in \Gamma'} L_\gamma$ is closed in X and L' has the weak topology with respect to a closed covering $\{L_\gamma;\ \gamma \in \Gamma'\}$.

As a topological complex, C. H. Dowker [4] introduced the concept of the Whitehead complex. A space X is a *Whitehead complex*, if it is an affine complex (see [4; §1]) having the weak topology with respect to $\{\overline{e_\lambda}; \lambda\}$. Here $\{e_\lambda; \lambda\}$ is the cells of X. Recall that the *closure* $\overline{e_\lambda}$ of e_λ coincides with the topological closure in X of e_λ [4; p. 560], and this also holds in CW-complexes. Every Whitehead complex with the cells $\{e_\lambda; \lambda\}$ is a CW-complex with each $\overline{e_\lambda}$ a subcomplex [4; p. 558].

We need the canonical example S_2 due to S. P. Franklin [5; Example 5.1]. That is, $S_2 = (N \times N) \cup N \cup \{0\}$ with each point of $N \times N$ is an isolated point. A basis of neighborhoods of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m,n); m \geqq m_0\}$. And U is a neighborhood of 0 if and only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in N$.

Lemma 2.1. *Suppose that* X *has the weak topology with respect to a point-countable closed covering* $\{C_\alpha; \alpha\}$ *of* X.

(1) *Let each* C_α *be Fréchet. Then* X *is Fréchet if and only if* X *contains no copy of* S_2.

(2) *Let each* C_α *be metric. Then* X *is metric if and only if* X *is a paracompact, strongly Fréchet space.*

Proof. (1) Since S_2 is not Fréchet, the "only if" part follows from that every subset of a Fréchet space is Fréchet.

We prove the "if" part. Suppose X is not Fréchet. Since X is sequential, by [5; Proposition 7.3] X contains a subspace $M = (N \times N) \cup N \cup \{0\}$ which, with the sequential closure topology, is a copy of S_2. The countable space M intersects at most countably many C_α's, say C_{α_1}, C_{α_2} .... Let $X_n = \bigcup_{i=1}^{n} C_{\alpha_i}$ and let C be a compact subset of M. Then C has the weak topology with respect to a countable closed covering $\{X_n \cap C;\ n \in N\}$ of C. Hence C is contained in some $X_n \cap C$. Thus each convergent sequence in M is contained in some X_n. We also remark that each X_n is Fréchet, hence contains no copy of M.

We now use the method of proof of S. P. Franklin and B. V. Smith Thomas [6; Proposition 1]. Since $N \cup \{0\}$ is a convergent sequence in M, there is X_{n_o} with $N \cup \{0\} \subseteq X_{n_o}$. Let $C_n = \{n\} \times N \cup \{n\}$ for each n. Since C_1 is a convergent sequence, there is X_{n_1} $(n_1 > n_o)$ with $C_1 \subseteq X_{n_1}$. Since X_{n_1} is closed and Fréchet, we can choose C_{n_2} $(n_2 > 1)$ and X_{n_3} $(n_3 > n_2)$ such that $C_{n_2} \cap X_{n_1}$ is at most finite and $C_{n_2} \subseteq X_{n_3}$. So, we can assume that $C_{n_2} \subseteq X_{n_3} - X_{n_1}$. In this way, we can choose C_{n_k} and $X_{n_{k+1}}$ $(n_{k+1} > n_k > n_{k-1})$ with $C_{n_k} \subseteq X_{n_{k+1}} - X_{n_{k-1}}$. Let $M' = \bigcup_{k=1}^{\infty} C_{n_k} \cup \{n_k;\ k \in N\} \cup \{0\}$. Then, for each $\alpha \in \Lambda$, $M' \cap C_\alpha$ is closed in X. Thus M' is a closed subset of X. Then M' is sequential, hence M' has the sequential closure topology. Thus M' is a copy of S_2. Hence X contains a copy of S_2. This is a contradiction.

(2) We prove only the "if" part. For $x \in X$ let $\{C_\alpha;\ C_\alpha \ni x\}$ be $\{C_{\alpha_1}, C_{\alpha_2}, \cdots\}$. Put $X_n = \bigcup_{i=1}^{n} C_{\alpha_i}$. Suppose

$x \in \overline{X - X_n}$ for each n. Since X is strongly Fréchet, there exist $x_n \notin X_n$ such that $x_n \to x$.

Let $C = \{x_n;\ n \in N\} \cup \{x\}$. Then the compact subset C has the weak topology with respect to a countable covering $\{C \cap C_\alpha;\ C \cap C_\alpha \neq \emptyset\}$ of C. Then C is contained in some finite union of C_α. Thus some C_{α_0} must contain infinitely many x_n's, hence $C_{\alpha_0} \ni x$. Then C_{α_0} is contained in some X_{n_0}. But this is a contradiction, for $X_{n_0} \not\ni x_n$ for $n \geqq n_0$. Thus $x \notin \overline{X - X_n}$ for some n, hence $x \in \operatorname{int} X_n$. This implies that X is locally metrizable. Hence X is metrizable, for X is paracompact.

Lemma 2.2. *Let* X *be a* CW-*complex with the cells* $\{e_\gamma\}$. *If* X *contains no closed copy of* S_α, *then for each* $x \in X$ *the cardinality of* $\Gamma_x = \{\gamma;\ \overline{e}_\gamma \ni x\}$ *is less than* α.

Proof. For some $x_0 \in X$, suppose $|\Gamma_{x_0}| \geqq \alpha$. Since $\overline{e}_\gamma \in x_0$ for $\gamma \in \Gamma_{x_0}$, there exist $x_{\gamma n}$ such that $x_{\gamma n} \to x_0$ and $x_{\gamma n} \in e_\gamma$. Let $C_\gamma = \{x_{\gamma n};\ n \in N\} \cup \{x_0\}$ and let $S = \cup\{C_\gamma;\ \gamma \in \Gamma_{x_0}\}$. Suppose L is any finite subcomplex of X. Then $S \cap L$ is closed in X. Thus S is closed in X. Moreover S has the weak topology with respect to $\{C_\gamma;\ \gamma \in \Gamma_{x_0}\}$. Indeed, for $F \subseteq S$, let $F \cap C_\gamma$ be closed in S for each $\gamma \in \Gamma_{x_0}$. Then $F \cap L = \{F \cap C_\gamma;\ e_\gamma \subseteq L \text{ and } \gamma \in \Gamma_{x_0}\}$. Thus $F \cap L$ is closed in S. Hence, F is closed in S. This implies that X contains a closed copy of S_α. This is a contradiction.

In [6], S. P. Franklin and B. V. Smith Thomas proved that a k_ω-space with metrizable "pieces" is metrizable if and only if it contains no copy of S_2 and no sequential fan S_ω.

Analogously to this result, we have

Proposition 2.3. *Let* X *be* (a) *a* CW-*complex* (resp. Whitehead complex), *or* (b) *a paracompact space having the weak topology with respect to a point-countable closed covering of metric spaces. Then the following are equivalent.*

(1) X *is metrizable.*

(2) X *contains no copy of* S_2 *and no* S_ω (resp. no copy of S_2).

(3) $t(X \times S_c) \leqq \omega$.

Proof. (1) ⇒ (2) is easy. We have (3) ⇒ (2) from Proposition 1.1.(2). (1) ⇒ (3) follows from [2; Corollary 4].

(2) ⇒ (1). In case of (b), we have this implication from Lemmas 1.2 and 2.1.

So, we suppose X is a CW-complex. First we prove that X is Fréchet. To see this, since $t(X) \leqq \omega$, it is sufficient to show that every closed separable subset $S = \overline{D}$ with D countable, is Fréchet. Clearly, D is contained in some countable union L of finite subcomplexes L_n. Since L is closed in X, S is a closed subset of L. Thus S has the weak topology with respect to a countable covering of compact metric subsets $L_n \cap S$ of S. Since S contains no copy of S_2, by Lemma 2.1(1), S is Fréchet. Then X is Fréchet. Second we prove that X is metrizable. Since X contains no copy of S_ω, by Lemma 2.2, X has the cells $\{e_\lambda\}$ such that

$\{\bar{e}_\lambda\}$, $\bar{e}_\lambda = \text{cl } e_\lambda$, is point finite. For $x \in X$, let $\{\bar{e}_\lambda; \bar{e}_\lambda \ni x\}$ be $\{\bar{e}_{\lambda_1}, \bar{e}_{\lambda_2}, \cdots, \bar{e}_{\lambda_\ell}\}$. Put $E = \bigcup_{i=1}^{\ell} \bar{e}_{\lambda_i}$. Suppose $x \in \overline{X - E}$. Since X is Fréchet, there is a convergent sequence $\{x_n; n \in N\}$ such that $x_n \to x$ and $x_n \notin E$. Since the convergent sequence is contained in a finite union of cells $\bar{e}_\lambda$, some $\bar{e}_{\lambda_{i_0}}$ must contain an infinitely many x_n's. Hence $x \in \bar{e}_\lambda$. Thus $\bar{e}_\lambda = \bar{e}_{\lambda_{i_0}}$ for some $i_0 \leqq \ell$. But this is a contradiction, because $x_n \notin \bar{e}_\lambda$ for all n. Then $x \notin \overline{X - E}$, which implies $x \in \text{int } E$. Since E is compact metric, X is locally metrizable. Then X is metrizable, for X is paracompact. Since a point-finite Whitehead complex is locally finite, the parenthetic part is proved similarly.

Let I_α be the space obtained from disjoint union of α closed unit intervals [0,1] by identifying all zero points. Then each I_α is a Whitehead complex. C. H. Dowker [4] showed that $I_\omega \times I_c$ is not a Whitehead complex.

From Proposition 2.3 and Lemma 2.2, we have the following generalization of the Dowker's example.

Corollary 2.4. *Let* $X \times Y$ *be a* CW-*complex and* $\{e_\lambda; \lambda\}$ *be the cells of* Y. *Then* X *is metrizable, or each cardinality of* $\{\lambda; \bar{e}_\lambda \ni y\}$ *is less than* c.

Proposition 2.5. *Suppose that* X_1 *and* X_2 *are* CW-*complexes* (resp. Whitehead complexes). *Then the following are equivalent.*

(1) $t(X_1 \times X_2) \leqq \omega$.

(2) $X_1 \times X_2$ *is a* k-*space.*

(3) $X_1 \times X_2$ *is a* CW-*complex* (resp. Whitehead complex).

Proof. (1) → (2). Since $t(X_1 \times X_2) \leqq \omega$, $X_1 \times X_2$ has the weak topology with respect to the closed covering of all closed, separable subsets of $X_1 \times X_2$. Each subset S of $X_1 \times X_2$ is clearly contained in $\Pi_1(S) \times \Pi_2(S)$, where $\Pi_i\colon X_1 \times X_2 \to X_i (i = 1,2)$ are projections. Thus $X_1 \times X_2$ has the weak topology with respect to a covering $\{F_1 \times F_2;$ F_i is a closed separable subset of $X_i\}$. As is seen in the proof of Proposition 2.3, (2) → (1), each F_i is a k_ω-space. Hence, by [11; (7.5)] each $F_1 \times F_2$ is a k-space. This implies $X_1 \times X_2$ is a k-space.

(2) → (3). Let $\{e_\gamma\}$; $\{e_\delta\}$ be the cells of X_1; X_2 respectively. Since X_1 and X_2 are complexes; affine complexes, $X_1 \times X_2$ is a complex; affine complex with cells $\{e_\gamma \times e_\delta\}$ respectively. Moreover, if X_1 and X_2 are CW-complexes, then $X_1 \times X_2$ is closure finite. Thus, to prove that $X_1 \times X_2$ is a CW-complex (also, a Whitehead complex), we only show that $X_1 \times X_2$ has the weak topology with respect to a collection $\{\bar{e}_\gamma \times \bar{e}_\delta\}$. Each compact subset of $X_1 \times X_2$ is contained in a compact subset of $X_1 \times X_2$ with type A × B. Then, each compact subset of $X_1 \times X_2$ is contained in a finite union of $\bar{e}_\gamma \times \bar{e}_\delta$. Since $X_1 \times X_2$ is a k-space, this implies that $X_1 \times X_2$ has the weak topology with respect to the collection $\{\bar{e}_\gamma \times \bar{e}_\delta\}$.

We have (3) → (1) from that every CW-complex is sequential, hence $t(X_1 \times X_2) \leqq \omega$.

Let X be a CW-complex with the cells $\{e_\gamma\}$. Then we shall call X *point-finite; point-countable; locally* *countable*, if the covering $\{\bar{e}_\gamma\}$ of X is so respectively.

Lemma 2.6. *Let* X *be a Fréchet* CW-*complex or a Whitehead complex. If* X *is a point-countable, then it is locally countable.*

Proof. Since every point-countable Whitehead complex is locally countable, then we suppose that X is a Fréchet CW-complex. Let $\{e_\gamma\}$ be the cells of X such that $\{\bar{e}_\gamma\}$ is point-countable. For $x \in X$, let $\{\bar{e}_\gamma;\ e_\gamma \ni x\}$ be $\{\bar{e}_{\gamma_1}, \bar{e}_{\gamma_2}, \cdots\}$. Put $E = \bigcup_{i=1}^{\infty} \bar{e}_{\gamma_i}$. Since X is Fréchet, by the proof of Proposition 2.3, (2) → (1), we have $x \notin \overline{X - E}$. This implies $x \in \text{int } E$. Since each $\bar{e}_{\gamma_i}$ is compact, by the proof of [17;(D)], each $\bar{e}_{\gamma_i}$ meets at most finitely many e_γ's, so that int E meets at most countably many $\bar{e}_\gamma$'s. This implies that X is locally countable. The parenthetic part is proved similarly.

Proposition 2.7. *Let* X *be a Fréchet* CW-*complex* (resp. a Whitehead complex). *Then the following are equivalent.*

(1) X *is point-countable.*

(2) X *is locally countable.*

(3) X^2 *is a* CW-*complex* (resp. Whitehead complex).

Proof. (1) → (2) follows from Lemma 2.6.

(2) → (3). Every locally countable CW-complex is a k_ω-space, and every product of two locally k_ω-spaces is a k-space. Thus (2) → (3) follows from Proposition 2.5.

(3) → (1). Suppose that X is not point-countable. Then, by Lemma 2.2, X contains a closed copy of S_{ω_1}.

Thus X^2 is a k-space which contains a closed copy of $S^2_{\omega_1}$. Hence $S^2_{\omega_1}$ is a k-space. However, by [7; Lemma 5], $S^2_{\omega_1}$ is not a k-space. This is a contradiction.

In terms of a set-theoretic axiom $BF(\omega_2)$ below, we shall consider the product $X \times Y$ of CW-complexes X and Y.

$BF(\omega_2)$: If $F \subseteq \{f;\ f\colon N \to N$ is a function$\}$ has cardinality less than ω_2, then there is a function $g\colon N \to N$ such that $\{n \in N;\ f(n) > g(n)\}$ is finite for all $f \in F$. Hence CH implies $BF(\omega_2)$ is false.

In [7], Gary Gruenhage proved the following result (*):

(*) $S_\omega \times S_{\omega_1}$ is a k-space if and only if $BF(\omega_2)$ holds.

From this result (*), if the assertion of Proposition 1.1 by replacing "S_c" by "S_{ω_1}" holds, then $BF(\omega_2)$ is false.

Lemma 2.8. *$I_\omega \times I_{\omega_1}$ is a Whitehead complex if and only if $BF(\omega_2)$ holds.*

Proof. "If." Since $BF(\omega_2)$ holds, by the proof of [7; Lemma 1] it turns out that $I_\omega \times I_{\omega_1}$ is sequential. Hence $I_\omega \times I_{\omega_1}$ is a Whitehead complex by Proposition 2.5. "Only if." $I_\omega \times I_{\omega_1}$ is a k-space and it contains a closed copy of $S_\omega \times S_{\omega_1}$, so that $S_\omega \times S_{\omega_1}$ is a k-space. Thus by the result (*), $BF(\omega_2)$ holds.

Proposition 2.9. *If X and Y are Fréchet CW-complexes* (resp. Whitehead complexes), *then the following are equivalent.*

(1) $X \times Y$ *is a CW-complex* (resp. Whitehead complex)

if and only if X *or* Y *is locally finite, otherwise* X *and* Y *are locally countable.*

(2) BF(ω_2) *is false.*

Proof. (1) → (2) follows from Lemma 2.8.

(2) → (1). The "if" part of (1) does not use (2). Suppose that X or Y is a locally finite CW-complex. Then X or Y is locally compact. Thus X × Y is a k-space. Suppose that X and Y are locally countable. Then they are locally k_ω-spaces, hence X × Y is a k-space. In any case, X × Y is a k-space. Hence X × Y is a CW-complex by Proposition 2.5. The parenthetic part is proved similarly. Next we prove the "only if" part. Suppose that Y is not locally countable. Then by Lemma 2.6, Y is not a point-countable CW-complex. Then by Lemma 2.2, Y contains a closed copy of S_{ω_1}. To show X is point-finite, suppose not. Then X contains a closed copy of S_ω by Lemma 2.2. Thus X × Y contains a closed copy of $S_\omega \times S_{\omega_1}$. Since BF($\omega_2$) is false, $S_\omega \times S_{\omega_1}$ is not a k-space by the result (*). But, since X × Y is a CW-complex, $S_\omega \times S_{\omega_1}$ is a k-space. This is a contradiction. Thus X is point-finite, hence is locally finite by Lemma 2.6. Similarly, Y is locally finite if X is not locally countable. This finishes the proof.

The following questions (a) and (b) remain, the latter is related to Proposition 2.7.

Questions. (a) For every CW-complexes X and Y, does (1) ↔ (2) of the previous proposition hold?

(b) Is X locally countable if X^2 is a CW-complex?

Supplement

Quite recently, through Zhou Hao-xuan, the author learned of the following result due to Liu Ying-ming "A necessary and sufficient condition for the product of CW-complexes," Acta Mathematica Sinica, 21 (1978), 171-175 (Chinese).

[CH] Let X and Y be CW-complexes. Then X × Y is a CW-complex if and only if either X or Y is locally finite, or X and Y are locally countable.

Referring to the above paper and G. Gruenhage [7], we can prove that the answers to the questions (a) and (b) are affirmative.

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