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by

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CARDINAL INVARIANTS, PSEUDOCOMPACTNESS AND MINIMALITY: SOME RECENT ADVANCES IN THE TOPOLOGICAL THEORY OF TOPOLOGICAL GROUPS

W.W. Comfort[1] and Douglass L. Grant[2]

0. Introduction

This work contains some new results, and some new proofs of old results, and some points of view which we hope serve to simplify or unify certain portions of the topological theory of topological groups. This is not a comprehensive survey. We give no description of the two most significant examples to have appeared in recent years--Shelah's example [Sh2], assuming CH, of a group of cardinality $\aleph_1$ with no Hausdorff, non-discrete group topology, and van Douwen's example [vD], assuming MA, of two countably compact groups whose product is not countably compact--and we ignore entirely those results which appear to belong more properly to harmonic analysis, or to the applications of category theory to the theory of topological groups, or to free topological (semi-) groups generated by topological spaces. (We note in passing that these latter fields have

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been systematically surveyed within the past few years; see for example [S-T1], [S-T2] and [F].) We confine our attention instead to the study of (certain) cardinal functions defined on groups, to some remarks concerning pseudocompact groups, and to a comprehensive survey of the recent literature concerning minimal groups.

Every topological group considered here is assumed to satisfy the T_0 separation axiom. As is well-known [HR1] (8.4), this ensures that our groups are completely regular, Hausdorff spaces, i.e., Tychonoff spaces.

Infinite cardinals are denoted by symbols like α, β, κ and λ, while ω denotes specifically the least infinite cardinal; ordinals are denoted by ξ, η, ζ and the like. For an infinite cardinal α, the symbol α^+ denotes the least cardinal greater than α.

For a topological space X we set

$$\mathcal{J}(X) = \{U \subset X\colon U \text{ is open}\}, \text{ and}$$

$$\mathcal{J}^*(X) = \mathcal{J}(X)\backslash\{\emptyset\}.$$

We usually denote by the symbol e, together with a subscript when this is appropriate, the identity element (that is, the neutral element) of a group.

We denote by Z, T, Q and R the integers, the circle, the rationals and the reals, respectively, each with its usual algebraic operations and each with its usual topology. The symbol $\oplus$ denotes weak sum; that is,

$$\oplus_{i\in I}G_i = \{x \in \prod_{i\in I}G_i\colon |\{i \in I\colon x_i \neq e_i\}| < \omega\}.$$

Part A: Some Cardinal Functions

For a topological space $X = \langle X,\mathcal{J}\rangle$ and $x \in X$, we define

$$w(X) = \min\{|\mathcal{B}|: \mathcal{B} \text{ is a base for } \mathcal{J}\},$$

$$(x,X) = \min\{|\mathcal{B}|: \mathcal{B} \text{ is a local base at } x\},$$

$$\psi(x,X) = \min\{|\mathcal{U}|: \mathcal{U} \subset \mathcal{J}, \cap\mathcal{U} = \{x\}\},$$

$$\chi(X) = \sup\{\chi(x,X): x \in X\},$$

$$\psi(X) = \sup\{\psi(X): x \in X\},$$

$$d(X) = \min\{|D|: D \text{ is dense in } X\},$$

$$\kappa(X) = \min\{|\mathcal{K}|: K \in \mathcal{K} \text{ is compact, and } X = \cup\mathcal{K}\}, \text{ and}$$

$$o(X) = |\mathcal{J}|.$$

When X is a homogeneous space--in particular, when X is a topological group--the two functions $\chi(\cdot,X)$, $\psi(\cdot,X)$ are constant on X and hence $\chi(X) = \chi(x,X)$, $\psi(X) = \psi(x,X)$ for every $x \in X$.

1. The Relation $W = \chi\cdot\kappa$

1.1. *Lemma. Let* G *be an infinite, locally compact group. Then* $w(G) = \chi(G)\cdot\kappa(G)$.

Proof. ($\geq$) That $\chi(G) \leq w(G)$ is clear. We show $\kappa(G) \leq w(G)$. Let H be an infinite, open, σ-compact subgroup of G. Then $w(H) \geq \omega$, and since each coset of H is σ-compact we have

$$\kappa(G) \leq \kappa(H)\cdot|G/H| \leq \omega\cdot|G/H| \leq w(H)\cdot|G/H| = w(G).$$

Since $\chi(G) \leq w(G)$ and $\kappa(G) \leq w(G)$ and $w(G) \geq \omega$, we have $\chi(G) \cdot \kappa(G) \leq w(G)$.

($\leq$) Let $\{U_\xi: \xi < \chi(G)\}$ be a local base at e with $U_\xi = U_\xi^{-1}$, and for $\xi < \chi(G)$ let A_ξ be a subset of G with $|A_\xi| \leq \kappa(G)$ such that $G = A_\xi U_\xi$. That $\{xU_\xi: \xi < \chi(G), x \in A_\xi\}$ is a base for G is now clear: if V is open in G and $p \in V$ then there is $\xi < \chi(G)$ such that $pU_\xi^{-1}U_\xi \subset V$ and there is $x \in A_\xi$ such that $p \in xU_\xi$; from $x \in pU_\xi^{-1}$ we have

$$p \in xU_\xi \subset pU_\xi^{-1}U_\xi \subset V,$$

as required.

For a locally compact Abelian group G, the dual group $\hat{G}$ of G is (by definition) the set of continuous homomorphisms from G to the circle group T, with the compact-open topology. A base at $1 \in \hat{G}$ is given by sets of the form

$$\{f \in \hat{G}\colon |f(x) - 1| < 1/n \text{ for } x \in K\}$$

with $n < \omega$ as K runs through the family of compact subsets of G. It is enough to restrict K to a family $\mathcal{K}$ of compact sets such that every compact subset of G is a subset of an element of $\mathcal{K}$. Since such $\mathcal{K}$ exist with $|\mathcal{K}| = \kappa(G)$ we have $\chi(\hat{G}) = \kappa(G)\cdot\omega = \kappa(G)$ for infinite G and hence by Pontrjagin duality $\chi(G) = \kappa(\hat{G})$ also. This observation allows [C2] a simple proof of a useful result (see [HR1] (24.14)).

1.2. *Theorem. Let* G *be a locally compact Abelian group. Then* $w(G) = w(\hat{G})$.

Proof. If G is finite then G and $\hat{G}$ are isomorphic. If G is infinite then from 1.1 above we have

$$w(G) = \chi(G)\cdot\kappa(G) = \kappa(\hat{G})\cdot\chi(\hat{G}) = w(\hat{G}),$$

as required.

We note in passing that the equality $d(G) = d(\hat{G})$ fails for many (locally compact, Abelian) groups G. For example if $G = T^{2^\alpha}$ with $\alpha \geq \omega$ then $d(G) \leq \alpha$ by the Hewitt-Marczewski-Pondiczery theorem; but $\hat{G}$ is the weak sum $\oplus_{\xi<2^\alpha} Z_\xi$ in the discrete topology, and hence $d(\hat{G}) = |\hat{G}| = 2^\alpha$.

We continue with a consequence of Lemma 1.1.

1.3. *Corollary. Let* G *be a non-discrete,* σ*-compact, locally compact group. Then*

(a) $w(G) = \chi(G)$;

(b) $|G| = 2^{w(G)}$; *and*

(c) $oG = 2^{w(G)}$.

Proof. (a) Since $\chi(G) \geq \omega$ and $\kappa(G) \leq \omega$, this is immediate from 1.1.

(g) The relation $|G| \leq o(G) \leq 2^{w(G)}$ holds for every T_1 space. That

$$2^{w(G)} = 2^{\chi(G)} = 2^{\psi(G)} \leq |G|$$

follows from the Čech-Pospisil theorem ([HR2] (28.58) or [En] (Problem 3G)).

(c) $2^{w(G)} = |G| \leq o(G) \leq 2^{w(G)}$.

We remark that not every non-discrete, locally compact group G can be shown to satisfy $|G| = 2^{w(G)}$--indeed for every (infinite) cardinal $\lambda \geq 2^{\omega}$, including those λ not of the form 2^{κ}, there is a non-discrete, locally compact group G such that $|G| = \lambda$. To see this let A be a discrete group of cardinality $\lambda \geq 2^{\omega}$ and take $G = T \times A$.

2. The Number of Open Subsets

Hajnal and Juhász [HJ1, HJ2] showed that a question of de Groot [deG], whether or not $o(X)$ has the form 2^{κ} for every (Hausdorff) space X, is not settled by the usual axioms of set theory. That is, there are models of ZFC in which de Groot's question is answered "Yes," and models where it is answered "No." More recently Shelah [Sh1], using a result of Kunen and Roitman [KR], has defined a model of ZFC in which there is an infinite Hausdorff space

X such that $o(X) \neq (o(X))^{\omega}$; indeed, Shelah [Sh1] can arrange even $cf(o(X)) = \omega$. It is reasonable to ask whether this pathology can extend to the context of topological groups; the situation is as follows.

For locally compact groups G the relation $o(G) = 2^{w(G)}$ always holds. We prove this result, improving 1.3(c) above, in 2.1. On the other hand Juhász has indicated, for example in a seminar in April, 1980 at Wesleyan University, that the construction of [HJ2] can be modified, using a technique of Roitman [Ro], to produce (in a suitable model of $2^{\aleph_0} = \aleph_1 < 2^{\aleph_1} = \aleph_3$) a topological group G for which $o(G) = \aleph_2$; the group G may be chosen hereditarily separable and non-Lindelöf. Finally, the relation $o(G) = (o(G))^{\omega}$ holds for all infinite topological groups G. This result is given in [J] (4.9) and in [C2]; the proof given below in 2.2 provides, at least formally, a bit more information than that of [J].

2.1. *Theorem. Let* G *be a locally compact group. Then* $o(G) = 2^{w(G)}$.

Proof. If G is discrete we have $w(G) = |G|$ and $o(G) = 2^{|G|}$. Let us assume, then, that G is not discrete. Let H be a subgroup of G generated by a compact neighborhood of e and let $\{x_\xi H: \xi < \alpha\}$ be a faithful enumeration of the coset space G/H. Each of the cosets $x_\xi H$ is open in G and the function

$$U \to \langle U \cap x_\xi H: \xi < \alpha \rangle$$

is one-to-one from $\mathcal{J}(G)$ onto $\prod_{\xi<\alpha} \mathcal{J}(x_\xi H)$; hence

$$o(G) = |\mathcal{J}(G)| = \prod_{\xi<\alpha} |\mathcal{J}(x_\xi H)| = (o(H))^{\alpha}.$$

Since $w(G) = \alpha \cdot w(H)$ we have

$$o(G) = (o(H))^{\alpha} = (2^{w(H)})^{\alpha} = 2^{w(G)},$$

as required.

Definition. Let β be a cardinal number and G a topological group. Then G is *totally* β-*bounded* if for every non-empty open subset U of G there is $A \subset G$ with $|A| < \beta$ such that $G = AU$. The *total boundedness number* of G, denoted $\alpha(G)$, is the cardinal

$$\alpha(G) = \min\{\beta\colon G \text{ is totally } \beta\text{-bounded}\}.$$

Notation. For G a topological group, set

$$\gamma(G) = \min\{o(U)\colon U \in \mathcal{J}^*(G)\}.$$

2.2. *Theorem. Let* G *be a non-discrete topological group. Then* (a) $o(G) = \gamma(G)^{\omega}$; *and* (b) *if* $\alpha < \alpha(G)$ *then* $o(G) = \gamma(G)^{\alpha}$.

Proof. We first prove (b), assuming $\alpha(G) \geq \omega^+$. Let U_1 and U_2 be neighborhoods of e such that $o(U_1) = \gamma(G)$ and such that if $A \subset G$ with $G = AU_2$ then $|A| \geq \alpha$. Let $U = U_1 \cap U_2$, let V be a neighborhood of e with $VV^{-1} \subset U$, and let $A \subset G$ be *maximally* V-*dispersed* in the sense that

(1) if $a, a' \in A$ with $a \neq a'$ then $aV \cap a'V = \emptyset$, and

(2) A is maximal with respect to (1).

For $x \in G$ there is $a \in A$ such that $xV \cap aV \neq \emptyset$ and hence $x \in aVV^{-1} \subset AU \subset AU_2$. It follows that $G = AU_2$ and hence $|A| \geq \alpha$.

The functions $\prod_{a\in A}\mathcal{J}(aV) \to \mathcal{J}(G)$ and $\mathcal{J}(G) \to \prod_{a\in A}\mathcal{J}(aU)$ defined by the rules

$$f = \langle f(a)\colon a \in A\rangle \to \cup_{a\in A} f(a) \text{ and } W \to \langle W \cap aU\colon a \in A\rangle$$

respectively are one-to-one. It follows that

$$(\gamma(G))^{|A|} = (o(V))^{|A|} \leq o(G) \leq (o(U))^{|A|} = (\gamma(G))^{|A|}$$

and hence $o(G) = (o(G))^{\alpha}$.

It remains only to prove (a), assuming $\alpha(G) = \omega$. Let U be a neighborhood of e with $o(U) = \gamma(G)$, let $\{U_n: n < \omega\}$ be a cellular family in G with each $U_n \subset U$, and let $G = AU$ with $|A| < \omega$. Then, as before,

$$(\gamma(G))^{\omega} = \prod_{n<\omega} o(U_n) \leq o(G) \leq \prod_{a\in A} o(aU) = (\gamma(G))^{|A|} \leq (\gamma(G))^{\omega}.$$

3. Concerning Closed Subgroups

The comprehensive monograph of Hewitt and Ross [HR1], [HR2] contains *inter alia* a great deal of information on topological properties of closed subgroups and their quotient groups. With no attempt here to be complete or comprehensive, we cite from [HR1] several results describing properties which, if enjoyed by both H and G/H, are enjoyed by G itself.

The restriction in 3.1 to closed subgroups H of G is dictated in part by our convention that the coset space G/H, which is of course a group if and only if H is normal in G, is to satisfy the T_0 separation axiom; see in this connection [HR1] (5.21). In fact, this restriction is unnecessary in parts of 3.1.

Parenthetical references in 3.1 are to sections of [HR1].

3.1. *Theorem. Let* G *be a topological group and* H *a closed subgroup, and let* P *be any one of the properties listed below. If both* H *and* G/H *have* P, *then* G *has* P.
(a) *compact* (5.25); (b) *locally compact* (5.25); (c) *compactly generated* (5.39(i)); (d) *metrizable* (5.38(e));

(e) $d \leq \alpha$ (*fixed* $\alpha \geq \omega$) (5.38(f)); (f) *connected* (7.14).

Let us note that to this list of properties might be added the property $\psi \leq \alpha$ (fixed $\alpha \geq \omega$). That is: if $\{e\}$ is the intersection of $\leq\alpha$ relatively open subsets of H and if $\{H\}$ is the intersection of $\leq\alpha$ open subsets of G/H, then $\{e\}$ is the intersection of $\leq\alpha$ open subsets of G.

It is natural to ask, in the face of 3.1(e), whether it is possible to find a topological group G and a closed subgroup H with $d(H) > d(G)$. Before describing the answer to this question we record an important theorem of Kuz'minov [Kz], widely used but not widely understood, which is in fact one of the few deep and difficult results in the topological theory of topological groups.

Definition. A space X is *dyadic* if for some cardinal number α there is a continuous function from $\{0,1\}^{\alpha}$ onto X.

3.2. *Theorem* (Kuz'minov [Kz]). *Every compact group is dyadic.*

Using the full power of Pontrjagin duality theory, Vilenkin [V] in 1958 had already proved Theorem 3.2 for compact Abelian groups. A readable English-language adaptation of Vilenkin's proof is given in [HR1] (25.35), but apparently there is no valid proof written in English of Kuz'minov's theorem.

There is an extensive literature on dyadic spaces. For our present expository purposes we need only, in addition to Theorem 3.2, the simple fact (see for example [EP]

(Theorem 1)) that a dyadic space X is the continuous image of $\{0,1\}^{\alpha}$ with $\alpha = w(X)$.

Let us return now to the question whether $d(H) > d(G)$ is possible for H a closed subgroup of a topological group G. For locally compact groups G, the matter is easily settled in the negative.

3.3. *Lemma* [CI]. *Let* G *be a locally compact group. Then*

(a) $w(G) \leq 2^{d(G)}$, *and* $\kappa(G) \leq d(G)$; *and*

(b) $d(G)$ *is minimal with respect to* (a)--*that is, if* $\alpha \geq \omega$ *and* $w(G) \leq 2^{\alpha}$, $\kappa(G) \leq \alpha$, *then* $d(G) \leq \alpha$.

Proof. Statements (a) are elementary. We prove (b). Let H be a compact, normal G_{δ} subgroup of G, so that G/H is a metric space and

$$d(G/H) \leq \kappa(G/H) \leq \kappa(G) \leq \alpha.$$

Since H is a compact group with $w(H) \leq w(G) \leq 2^{\alpha}$ there is by Kuz'minov's theorem a continuous function from $\{0,1\}^{2^{\alpha}}$ onto H, and since $d(\{0,1\}^{2^{\alpha}}) \leq \alpha$ by the Hewitt-Marczewski-Pondiczery theorem we have $d(H) \leq \alpha$ as well. It then follows from Theorem 3.1(e) that $d(G) \leq \alpha$, as required.

3.4. *Theorem* [CI]. *Let* G *be a locally compact group and* H *a closed subgroup. Then* $d(H) \leq d(G)$.

Proof. From 3.3 we have $w(H) \leq w(G) \leq 2^{d(G)}$ and $\kappa(H) \leq \kappa(G) \leq d(G)$; hence $d(H) \leq d(G)$.

It has been shown by Ginsburg, Rajagopalan and Saks [GRS], using the free topological group generated by a topological space, that there are topological groups G (not locally compact) and closed subgroups H of G with $d(H) > d(G)$. Indeed, according to [CI] (Theorem 3.3), this anomaly can arise to the maximal extent consistent with the usual cardinal inequalities in topology governing density character of subspaces: For every pair $\langle \alpha, \beta \rangle$ of infinite cardinals with $\beta \leq 2^{\alpha}$, there are a topological group G and a closed subgroup H of G such that $d(G) = \alpha$ and $d(H) = \beta$.

Let us remark finally that $d(H) > d(G)$ is possible for H a dense subgroup of a compact group G, even with G the Stone-Čech compactification of H. The following argument extends trivially that of [Cl].

3.5. *Let* α *be a infinite cardinal,* $G = \{0,1\}^{2^{\alpha}}$ *and*

$$H = \{x \in G: |\{\xi < 2^{\alpha}: x_{\xi} \neq 0\}| \leq \omega\}.$$

Then (a) $d(G) \leq \alpha$*;* (b) $d(H) = 2^{\alpha}$*; and* (c) $G = \beta(H)$.

Proof. (a) is again a special case of the Hewitt-Marczewski-Pondiczery theorem, and (c) is a special case of Glicksberg [Gs] (Theorem 2), of Corson [Co] (Theorem 2), and of Kister [Ki]. For (b) we note first that

$$d(H) \leq |H| = (2^{\alpha})^{\omega} = 2^{\alpha},$$

and second that if $D \subset H$ with $|D| = \gamma < 2^{\alpha}$ then

$$|\{\xi < 2^{\alpha}: x_{\xi} \neq 0 \text{ for some } x \in D\}| \leq \gamma \cdot \omega < 2^{\alpha}$$

and hence there is $\eta < 2^{\alpha}$ such that the open set $\pi_{\eta}^{-1}(\{1\})$ misses D. Thus D is not dense in H if $|D| < 2^{\alpha}$ and hence $d(H) \geq 2^{\alpha}$.

Part B: Concerning Pseudocompact Groups

A topological group is said to be *pre-compact* if its completion relative to either of its two natural uniform structures (it doesn't matter which one) is compact. It is a theorem of Weil [We] that when this occurs the completion $\overline{G}$ of G carries naturally the structure of a topological group of which G is a subgroup. Further, G is pre-compact if and only if G is *totally bounded*, i.e., for every $U \in \mathcal{J}^*(G)$ there is finite $F \subset G$ such that $G = FU$. It is not difficult to see that a pseudocompact group, and *a fortiori* a countably compact group, is totally bounded. A pseudocompact group G is G_δ-dense in its Weil completion $\overline{G}$ (in the sense that every non-empty G_δ of $\overline{G}$ meets G) and conversely: given a compact group K, a dense subgroup G of K is pseudocompact if and only if G is G_δ-dense in K [cf. [CRs2] (Theorem 1.2)]. It is occasionally useful, and completely justified by the foregoing remarks, to replace the study of pseudocompact groups by the study of G_δ-dense subgroups of compact groups.

There is an extensive literature devoted in part to pseudocompact groups; see for example [CRs2], [I], [W1], [W2], [CSo], [So], [CRb1], [CRb2]. Somewhat capriciously, we here select for discussion just two or three questions of interest.

4. Small Dense Subgroups

Given a compact group K, how small a dense, pseudocompact subgroup G does K possess? If $w(K) = \alpha \geq \omega$ then K has a cofinal G_δ-family of cardinality α^ω and the selection of one point from each of these gives a set which generates a G_δ-dense subgroup G of K satisfying $|G| \leq \alpha^\omega$. This straightforward reasoning appears so tight that one is tempted to believe the upper bound α^ω to be optimal. In fact, however, Itzkowitz [I] for compact Abelian groups and Wilcox [W1], [W2] for compact groups K in general showed that if $w(K) \le 2^\alpha$ then there is a dense, pseudocompact subgroup G of K with $|G| \le \alpha^\omega$. In the following argument taken from [CSa] we improve this statement.

4.1. *Theorem* [CSa]. *Let* K *be an infinite compact group with* $w(K) \le 2^\alpha$. *Then there is a dense, countably compact subgroup* G *of* K *such that* $|G| \le \alpha^\omega$.

Proof. There is by 2.5(a) and 2.2 a dense subset D_0 of K with $|D_0| \le \alpha \le \alpha^\omega$; let G_0 be the subgroup of K generated by D_0. If $\xi < \omega^+$ and a family $\{G_\eta: \eta < \xi\}$ of subgroups of K has been defined satisfying

$$G_0 \subset G_{\eta'} \subset G_\eta \text{ for } \eta' < \eta < \xi \text{ and}$$
$$|G_\eta| \le \alpha^\omega \text{ for } \eta < \xi,$$

let D_ξ be a set formed by choosing one accumulation point in K from each countably infinite subset of $\cup_{\eta<\xi} G_\eta$ and let G_ξ be the subgroup of K generated by $(\cup_{\eta<\xi} G_\eta) \cup D_\xi$. The subgroup $G = \cup_{\xi<\omega^+} G_\xi$ is clearly as required.

Let p be a uniform ultrafilter over the countably infinite, discrete set ω, that is, let $p \in U(\omega) = \beta(\omega)\backslash\omega$. Following Allen Bernstein [Be] and Saks [Sa] we say that a (completely regular, Hausdorff) space X is p-*compact* if for every function $f: \omega \to X$ the continuous Stone extension $\overline{f}: \beta(\omega) \to \beta(X)$ satisfies $\overline{f}(p) \in X$. (Actually the definition of [Be] and [Sa] differ formally from that just given, but for present purposes the formulation above seems optimal.)

As Bernstein noted [Be], every p-compact space is countably compact, and the product of p-compact spaces is p-compact. In work following [CSa], Ginsburg and Saks [GS] noted that (if $p \in U(\omega)$ is chosen in advance) then for every countably infinite subset $f[\omega]$ in $\cup_{\eta<\xi} G_\eta$ with $f\colon \omega \to K = \beta(K)$ one might as well choose $\overline{f}(p) \in K$ for the required accumulation point. The resulting group G is then p-compact and one has the following improvement of 4.1.

4.2. *Theorem* [GS]. *Let* $p \in U(\omega)$ *and let* K *be an infinite, compact group with* $w(K) \leq 2^\alpha$. *Then there is a dense,* p-*compact subgroup* G *of* K *such that* $|G| \leq \alpha^\omega$.

As the proofs of 4.1 and 4.2 make clear, the groups G may be chosen to contain any subset S of K specified in advance such that $|S| \leq \alpha^\omega$.

When $\langle K,G \rangle$ is a pair as in 4.1 with $w(K) = 2^\alpha$ we have from 1.3(b) above that $|K| = 2^{2^\alpha}$ and hence

$$|G| \leq \alpha^\omega \leq 2^\alpha < |K|.$$

Does a given compact, Abelian group K with $w(K) > \omega$ contain a dense, pseudocompact subgroup G such that $|G| < |K|$? It is shown in [CRb2], using a combinatorial result of Cater, Erdös and Galvin [CEG], that the answer is "Yes" for all groups K except (perhaps) some or all of those for which the cardinal number $\alpha = w(K)$ satisfies $\beth_\upsilon \leq \alpha < \beth_{\upsilon+1}$ with υ a limit ordinal such that $\mathrm{cf}(\upsilon) = \omega$. (Here as usual the beth cardinals $\beth_\upsilon$ are defined recursively by the rules $\beth_0 = \omega$, $\beth_{\upsilon+1} = 2^{\beth_\upsilon}$, and $\beth_\upsilon = \Sigma_{\xi<\upsilon}\beth_\xi$ for non-zero limit ordinals υ.)

Assuming GCH, the answer is "No" precisely for such groups, and these are precisely the groups K with $\mathrm{cf}(w(K)) = \omega$. In any event the answer depends only on the cardinal number $w(K)$, not on the algebraic structure of K [CRb2].

The following more delicate question, raised in [CSo], has turned out to be undecidable in ZFC: Do there exist a compact, Abelian group K and a pseudocompact, *totally dense* subgroup G of K such that $|G| < |K|$? (Recall that a subgroup G of K is totally dense in K if $G \cap H$ is dense in H for every closed subgroup H of K.) In summary, the situation is as follows.

4.3. *Theorem* [CRb2]. *Let* $\alpha \geq \omega$. *There are a compact, Abelian group* K *with* $w(K) = \alpha$ *and a totally dense subgroup* G *of* K *such that* $|G| < |K|$ *if and only if* $\alpha = \log(2^\alpha)$ *and* $\mathrm{cf}(\alpha) = \omega$.

Assuming GCH, the cardinal numbers α of 4.3 with $\alpha > \omega$ are exactly those for which $\beth_\upsilon \leq \alpha < \beth_{\upsilon+1}$ with $\mathrm{cf}(\upsilon) = \omega$, and one has the following result.

4.4. *Theorem* [CRb2]. *Assume* GCH. *There is no pair* $\langle K,G \rangle$ *with* K *a compact, Abelian group,* G *a totally dense, pseudocompact subgroup of* K, *and* $|G| < |K|$.

On the other hand, as is noted in [CRb2], it is easy to produce such pairs $\langle K,G \rangle$ in suitably bizarre models of ZFC. For example, let

$$\beth_1 < \alpha = \aleph_\omega < \beth_2$$

in a model in which $\{2^{\aleph_k}: k < \omega\}$ has a strictly increasing subsequence. (The existence of such models is assured by Easton [Ea]; see also [CEG].) The torsion subgroup G_1 of $K = \prod(Z(p_k)^{\aleph_k})$ is totally dense in K [CSo] and satisfies

$$|G_1| = \Sigma_k 2^{\aleph_k} < \prod_k 2^{\aleph_k} = |K|;$$

and there is by 4.1 above a dense, pseudocompact (even countably compact) subgroup G_2 of K such that

$$|G_2| \leq \alpha^\omega \leq (\beth_2)^\omega = \beth_2 < 2^\alpha = |K|.$$

The pair $\langle K,G \rangle$, with G the group generated by $G_1 \cup G_2$, is as required.

5. Concerning Products

It is known that the product of any set of pseudocompact groups is pseudocompact [CRs2]. The corresponding question for countably compact groups remained open for many years, indeed until the appearance (alluded to in our Introduction) of van Douwen's example [vD], defined using MA, of two countably compact groups whose product is not countably compact. As is shown carefully in [vD], the construction given there cannot be carried out without MA. It is not now known whether there are models of ZFC in which every finite product--indeed perhaps every product--of countably compact groups is countably compact. We now indicate a result, noted some years ago by one of us [C2], which appeared at the time likely to provide a (large) family of countably compact groups whose product is not countably compact. To administer the coup de grâce to the problem it remained only to show that for every $p \in U(\omega)$

there is a countably compact group not p-compact. This project has proved less tractable than anticipated.

The equivalence (a) $\Leftrightarrow$ (b) of Theorem 5.1 for $|\mathcal{C}| = 1$ is given by Ginsburg and Saks [GS]. For less restricted classes $\mathcal{C}$, see [C3] and [Sa].

5.1. *Theorem. For a* (possibly proper) *class* $\mathcal{C}$ *of Tychonoff spaces, the following statements are equivalent.*

(a) *Every product (repetitions permitted) of a set of elements of* $\mathcal{C}$ *is countably compact;*

(b) *every product (repetitions permitted) of* $\leq 2^{2^\omega}$ *elements of* $\mathcal{C}$ *is countably compact; and*

(c) *there is* $p \in U(\omega)$ *such that every element of* $\mathcal{C}$ *is* p-*compact.*

Proof. That (a) $\Rightarrow$ (b) is clear. We have remarked above, with Bernstein [Be], that the product of p-compact spaces is p-compact; hence (c) $\Rightarrow$ (a). We show (b) $\Rightarrow$ (c).

If (c) fails then for $p \in U(\omega)$ there are $X_p \in \mathcal{C}$ and $f_p\colon \omega \to X_p \subset \beta(X_p)$ such that $\overline{f}_p(p) \in \beta(X_p)\backslash X_p$. For ease of exposition we assume the function $p \to X_p$ is one-to-one from $U(\omega)$ into $\mathcal{C}$. We define

$$f\colon \omega \to X = \prod_{p\in U(\omega)} X_p \subset \prod_{p\in U(\omega)} \beta(X_p)$$

by the rule $(f(n))_p = f_p(n)$ and we note from (b) that since $|U(\omega)| \leq 2^{2^\omega}$ there is $q \in U(\omega)$ such that $\overline{f}(q) \in X$. It then follows, with π_q denoting the projection from X onto X_q, that

$$\overline{f}_q(q) = \overline{\pi_q \circ f}(q) \in X_q,$$

a contradiction.

5.2. *Corollary* [C2]. *The following statements are equivalent.*

(a) *Every product of countably compact groups is countably compact;*

(b) *there is* $p \in U(\omega)$ *such that every countably compact group is* p-*compact.*

The suggestion that there might exist models of ZFC in which condition (b) of 5.1 above holds is perhaps not so ridiculous as it at first appears. Van Douwen [vD] (Theorem 7.3) notes in effect that if some $p \in U(\omega)$ satisfies $\chi(p, U(\omega)) = \omega^+$ then every initially ω^+-compact space is p-compact. (That such p exist is Kunen's Axiom; it follows, of course, from CH, and it is consistent with $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ [Ku].) Not every countably compact group, however, is initially ω^+-compact; the simplest example to this effect is probably $\{x \in \{0,1\}^{\omega^+} : |\{\xi < \omega^+ : x_\xi \neq 0\}| \leq \omega\}$.

6. Pseudocompact Refinement Topologies

It is a common and honorable endeavor, much prosecuted by topologists, to find or characterize those topologies on a set which are extremal (i.e., minimal or maximal) with respect to a specified topological property. It is known, for example, that every Abelian group admits a maximal totally bounded topological group topology [CRs1]; the extensive literature concerning minimal (Hausdorff) group topologies is the subject of §§7-8 of the present article. We show here that, except in the trivial (that is, the metrizable) case, a compact topology on an Abelian group is not maximal among pseudocompact topologies. This result is closely related to the statement that every non-metrizable, compact Abelian group admits a proper, dense, pseudocompact subgroup. Theorem 6.3 below, which continues work initiated in [CSo], is taken from [CRb1].

6.1. *Lemma* [CSo]. *Let* $\langle G,\mathcal{J}\rangle$ *be a totally disconnected, compact Abelian group with* $w(G) > \omega$. *Then there is a proper, dense, pseudocompact subgroup* H *of* G *such that* $|G/H| < \omega$.

Proof. Let $\hat{G}_p$ be the p-primary subgroup of the dual group $\hat{G}$ of G, so that $\hat{G} = \oplus_p \hat{G}_p$. Since $|\hat{G}| = w(G) > \omega$ there is p such that $|\hat{G}_p| > \omega$, and the socle S of $\hat{G}_p$ (consisting of all elements of order $\leq p$) satisfies $S = \oplus(Z(p))^\alpha$ for some $\alpha > \omega$. The annihilator A of S in G and the canonical homomorphism ϕ satisfy

$$\phi\colon G \to G/A = \hat{S} = (Z(p))^\alpha,$$

and the group $\{x \in \hat{S}\colon |\{\xi < \alpha\colon x_\xi \neq 0\}| \leq \omega\}$, which is countably compact, extends to a maximal, proper subgroup $\tilde{H}$ of $\hat{S}$. Then $H = \phi^{-1}(\tilde{H})$ is as required.

6.2. *Lemma. Let* G *be a compact Abelian group and* C *the component of the identity. If* $w(C) = \alpha > \omega$ *then there is a continuous homomorphism of* G *onto* T^α.

Proof. It is enough to show that the dual group $\hat{G}$ of G contains a subgroup S isomorphic to $\oplus(Z^\alpha)$, for then the annihilator A of S in G and the canonical homomorphism ϕ will satisfy

$$\phi\colon G \to G/A = \hat{S} = \hat{Z}^\alpha = T^\alpha.$$

Let I be a maximal independent subset of $\hat{C}$, for $\chi \in I$ let $\tilde{\chi} \in \hat{G}$ with $\tilde{\chi}|C = \chi$, and let $\tilde{I} = \{\tilde{\chi}: \chi \in I\}$. Denoting by H and S the subgroups of $\hat{C}$ and $\hat{G}$ generated by I and $\tilde{I}$, respectively, we have up to isomorphism the inclusions

$$\hat{G} \supset S = \oplus_{i \in I} Z_i \text{ and } \hat{C} \supset H = \oplus_{i \in I} Z_i.$$

Then with W the injective hull of $\hat{C}$ we have from

$$|\hat{C}| = w(G) = \alpha > \omega \text{ and } \oplus_{i \in I} Z_i \subset \hat{C} \subset W = \oplus_{i \in I} Q_i$$

that $|I| = \alpha$, as requried.

6.3. *Theorem* [CRbl]. *Let* G *be an Abelian group and* $\mathcal{J}$ *a compact group topology with* $w\langle G, \mathcal{J}\rangle > \omega$. *Then* $\mathcal{J}$ *is not maximal among pseudocompact group topologies for* G.

Proof. Let C be the component of the identity in G. From Lemma 1.1, the comment following Theorem 3.1 and the fact that $\psi = \chi$ for compact groups, we have

$$\omega < w(G) = \psi(e,G) \leq \psi(e,C) \cdot \psi(C,G/C) = w(C) \cdot w(G/C)$$

and hence $w(C) > \omega$ or $w(G/C) > \omega$. We consider these non-exclusive, but exhaustive, cases separately.

Case 1. $w(G/C) > \omega$. There is by Lemma 6.1 a proper, dense, pseudocompact subgroup $\tilde{H}$ of G/C such that $|(G/C)/\tilde{H}| < \omega$. It is not difficult to check that with ϕ the canonical homomorphism of G onto G/C and with $H = \phi^{-1}(\tilde{H})$, the group H is a proper, dense, pseudocompact subgroup of G with $|G/H| < \omega$ (see [CSo] for details). The topology $\mathcal{J}'$ for G generated by $\mathcal{J} \cup \{xH: x \in G\}$ is then a pseudocompact group topology for G such that $\mathcal{J}' \supsetneq \mathcal{J}$.

Case 2. $w(C) = \alpha > \omega$. There is by Lemma 6.2 a continuous homomorphism ϕ of $G = \langle G, \mathcal{J}\rangle$ onto T^α. Let ψ be a homomorphism of T^α onto T such that $\psi(t) = \prod\{t_\xi: \xi < \alpha\}$

for $t = \langle t_\xi : \xi < \alpha \rangle \in \oplus T^\alpha$ (a finite product) and define

$$H = \operatorname{graph}(\psi \circ \phi) \subset G \times T.$$

We claim that the group H is G_δ-dense in the compact group $G \times T$ (and hence pseudocompact [CRs2]). Indeed let F be a non-empty G_δ of G, let $p \in T$, and let A be a non-empty subset of $\phi[F]$ for which there is $\xi < \alpha$ with $A = (\pi_{\alpha \setminus \{\xi\}}[A]) \times T$. Let $x \in F$ with $\phi(x) \in A$ and choose any element y of F such that $\phi(y) \in A$ and

$$\phi(y)_{\alpha \setminus \{\xi\}} = \phi(x)_{\alpha \setminus \{\xi\}} \text{ and}$$
$$\phi(y)_\xi = \phi(x)_\xi \cdot p \cdot (\psi(\phi(x)))^{-1}.$$

Then $\phi(x^{-1}y) \in \oplus T^\alpha$ and $\psi(\phi(x^{-1}y)) = \phi(x^{-1}y)_\xi = p \cdot (\psi(\phi(x)))^{-1}$ and hence

$$\psi(\phi(y)) = \psi(\phi(x^{-1}y)) \cdot \psi(\phi(x)) = p,$$

i.e., $\langle y, p \rangle \in (F \times \{p\}) \cap H$.

The projection from H onto G is a one-to-one function, continuous when H has the (pseudocompact) topology inherited from $G \times T$ with $G = \langle G, \mathcal{J} \rangle$. The required pseudocompact topology $\mathcal{J}' \supsetneq \mathcal{J}$ for G is now defined by the requirement that this projection be a homeomorphism to $\langle G, \mathcal{J}' \rangle$.

6.4. *Remarks.* (a) It is well-known and easy to prove that a pseudocompact metric space is countably compact and hence compact. Thus in Theorem 6.3 the hypothesis $w(G, \mathcal{J}) > \omega$ cannot be omitted.

(b) The pseudocompact topology $\mathcal{J}' \supsetneq \mathcal{J}$ constructed in the proof of Theorem 6.3 is, like every totally bounded topology on an Abelian group, the topology induced on G by some group of homomorphisms from G to T [CRs1]. The following argument, taken from [CSa] and due in effect to

Lewis C. Robertson, shows that the subgroup $\mathcal{H}$ of $\mathrm{Hom}\langle G,T\rangle$ which induces $\mathcal{J}'$ is proper. Restated: the finest totally bounded topology on an (infinite) Abelian group G--that is, the topology induced by all of $\mathrm{Hom}\langle G,T\rangle$--is not pseudocompact. Indeed there is in this topology a (necessarily closed) subgroup H of G with $|G/H| = \omega$, and then G/H is a countably infinite, pseudocompact and Lindelöf (hence compact) topological (Hausdorff) group. The existence of such a group is incompatible with Theorem 1.3(b) above.

Part C: Minimality in Topological Groups

We reiterate for emphasis our standing convention: all topological groups referred to are to satisfy the Hausdorff separation axiom.

7. Permanence Properties and Sufficient Conditions

Definition. A topological group $\langle G,\mathcal{J}\rangle$ is *minimal* if there is no coarser (Hausdorff) topological group topology for G; and $\langle G,\mathcal{J}\rangle$ is *totally minimal* if its (Hausdorff) quotient groups are minimal.

Clearly, compact groups are totally minimal.

Equivalently, one might define the two concepts by noting that every continuous, bijective (resp., continuous, surjective) homomorphism from a minimal group to a (Hausdorff) group is open. Further defining a homomorphism to be *almost open* if the image of a unit neighborhood is dense in a unit neighborhood, we see that the two properties imply, respectively, the following two properties introduced by Husain [H] in the context of the open mapping theorem: $(G,\mathcal{J})$ is $B_r(A)$ (resp., $B(A)$) if every continuous almost open bijective (resp., surjective) homomorphism from G to a (Hausdorff) group is open. It is an easy exercise to show locally compact groups are B(A), and Husain established that complete metrizable groups are, as well. Brown [Br] later showed that Čech-completeness is a sufficient condition.

In the same context, Sulley [Sy] proved for Abelian groups the following criterion for inheritance of both Husain's properties by closures and dense subgroups, which the second author [Gl] showed holds for arbitrary groups.

7.1. *Theorem. Let* H *be a dense subgroup of a group* G.

(a) *If* H *is* $B_r(A)$ (resp., B(A)), *then* G *is* $B_r(A)$ (resp., B(A)).

(b) *If* G *is* $B_r(A)$ *and* H *has non-trivial intersection with every non-trivial closed normal subgroup of* G, *then* H *is* $B_r(A)$.

(c) *If* G *is* B(A) *and* H *intersects every closed normal subgroup* N *of* G *in a dense subgroup of* N, *then* H *is* B(A).

In 1972, R. M. Stephenson [St1] established, for G compact, the precise analog of Theorem 7.1(b) with "minimal" replacing "$B_r(A)$". For H pre-compact, it therefore follows that minimality is equivalent to the $B_r(A)$ property, and total minimality to B(A). The assumption that G be compact was removed by Banaschewski [Ba]; a statement analogous to (c) for totally minimal groups was established independently by Schwanengel [Sc] and by Dikranjan and Prodanov [DP1].

Let U denote the multiplicative group of complex roots of unity, the torsion subgroup of the circle group T.

Since U contains all proper closed subgroups of T, it is totally minimal and so B(A), the latter fact having been established independently by Sunyach [Su]. In fact one of us [G2] has shown arbitrary powers of U to be totally minimal; the same result has been established independently by Dikranjan and Stojanov [DSj1], and by Eberhardt and Schwanengel [ES] for countable products. Indeed Stojanov [Sj1] has shown that every product of totally minimal periodic groups is totally minimal.

Let G_m denote the group given by $\{x \in U$: order of x is not divisible by any m-th power except $1\}$. Sulley [Sy] shows G_2 to be $B_r(A)$ and hence minimal, but not B(A) and hence not totally minimal, by observing that, since every torsion element has some power of squarefree order, G_2 intersects every closed subgroup of T, but $G_2 \cap \{1,-1,i,-i\} = \{1,-1\}$.

The next result, inspired by Prem Sharma, and its corollary, due originally to Stephenson, is a clear indication that the analogy between the two pairs of properties does not extend much farther.

7.2. *Theorem. Let* G *be an Abelian topological group with sufficiently many continuous characters to separate points. If* G *is minimal, then* G *is pre-compact.*

Proof. The weak topology induced on G by all its continuous characters is a Hausdorff topology since points are separated; it is pre-compact by [CRs1]. By the minimality of G, this topology coincides with the original topology.

7.3. *Corollary* [Stl]. *A locally compact Abelian group which is also minimal is compact.*

More generally, we have this simple extension of 7.3.

7.4. *Corollary. Let* $G = \prod_{i \in I} G_i$ *be a minimal group, with each* G_i *a locally compact Abelian group. Then each* G_i *(and hence* G*) is a compact group.*

Proof. G is pre-compact and hence each group $G_i = \pi_i[G]$ is pre-compact and locally compact; hence each group G_i is compact.

The assumption of commutativity is essential in Theorem 7.2: Dierolf and Schwanengel [DS2] have shown that the non-Abelian group

$$\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0,\ b \in R\right\},$$

with the natural (locally compact, non-compact) topology inherited from R^4, is a minimal group.

Other properties close to compactness, such as countable compactness, seem to be of little relevance in this context, however. If G is a group, e its identity element, and α an uncountable cardinal, we set

$$\Sigma_\omega G^\alpha = \{x \in G^\alpha : |\{\xi < \alpha : x_\xi \neq e\}| \leq \omega\}.$$

It is known [Gl] that if K is a compact group with non-trivial centre, then $\Sigma_\omega K^\alpha$ is dense in K^α but the diagonal subgroup of $(\text{Cent } K)^\alpha$ has trivial intersection with $\Sigma_\omega K^\alpha$. The latter is then countably compact, but not $B_r(A)$ and hence not minimal. Suppose now that D is a finite, discrete, simple, non-Abelian group (such as A_5), so that the only closed normal subgroups of D^{α} are of form $\Pi\{E_{\xi}: \xi < \alpha\}$, where each $E_{\xi} = \{e\}$ or D. Then $\Sigma_{\omega}D^{\alpha}$ intersects all such subgroups densely, and hence $\Sigma_{\omega}D^{\alpha}$ is countably compact and totally minimal (but not compact). This example appears in [G3] and, in a somewhat different contest, in [EDS].

The permanence properties of the groups in our several classes have also been studied extensively. For instance, all four properties are inherited by closed central subgroups (cf. [G1]). However, every discrete group can be embedded as a (closed) subgroup of a locally compact, minimal group [DS2].

The situation concerning products is at least as complex. Let Z(p) denote the integers with (pre-compact) group topology having as a fundamental system of unit neighborhoods the family $\{p^{n}Z: n \geq 1\}$, and let $\overline{Z(p)}$ denote as usual the Weil completion of Z(p). The following result is from [DSj2].

7.5. *Theorem. For an Abelian topological group* G, *the following are equivalent:*

(1) *all subgroups of* G *are minimal;*

(2) *all subgroups of* G *are totally minimal;*

(3) G *is one of the following types of groups:* (i) *a subgroup of* $\overline{Z(p)}$ *for some* p; (ii) $\oplus F_{p}$, *with each* F_{p} *a finite Abelian* p-*group;* (iii) $X \times F_{p}$, *with* X *a rank* 1 *subgroup of* $\overline{Z(p)}$ *and with* F_{p} *a finite Abelian* p-*group.*

The methods of [DSj2] are derived in part from [P1], where it is shown that the groups $\overline{Z(p)}$ are the only infinite compact Abelian groups for which all subgroups are minimal.

Each group $Z(p)$ is minimal (in fact, totally minimal and hence $B(A)$), but its square is not minimal [Dv] and hence, since pre-compact, not $B_r(A)$. (That $Z(p) \times Z(p)$ is not minimal is established by Doïtchinov by the direct and elegant observation that with H_k the subgroup of $Z \times Z$ generated by the three elements $\langle p^{k^2}, 0 \rangle$, $\langle 0, p^{k^2} \rangle$ and $\langle 1, \Sigma_{s=1}^{k-1} p^{s^2} \rangle$, the family $\{H_k: 1 \leq k < \omega\}$ is the unit neighborhood filter of a Hausdorff topology on $Z \times Z$ strictly coarser than that of $Z(p) \times Z(p)$.)

Another result of Doïtchinov [Dv], that the product of a minimal group with a compact one is minimal, was generalized in one direction by the second author [G3] who replaced "minimal" by "$B_r(A)$", and more recently and extensively by Eberhardt, Dierolf and Schwanengel, who established the following result. (A group is *sup-complete* if it is complete in its two-sided uniformity.)

7.6. *Theorem* [EDS]. *The product of two minimal (resp., totally minimal) groups, one of which is sup-complete, is minimal (resp., totally minimal).*

While it is somewhat beyond the scope of the paper, we note for contrast the result of Banaschewski [Ba] to the effect that any product of pre-compact minimal rings or R-modules is minimal. This result was obtained independently by Dikranjan [D] for rings with unit.

Eberhardt, Dierolf and Schwanengel also consider the so-called "Three Space Problem": if N is a closed normal subgroup of G and both N and G/N are minimal (resp., totally minimal), is G then minimal (resp. totally minimal)? Their result, given in 7.7 below, may be compared with the statements of Theorem 3.1. (A group is *totally sup-complete* if all its Hausdorff quotients are sup-complete.)

7.7. *Theorem* [EDS]. *Let* N *be a (totally) sup-complete normal subgroup of* G. *If both* N *and* G/N *are (totally) minimal, then* G *is (totally) minimal.*

A more specialized result from the same paper is the following.

7.8. *Theorem* [EDS]. *An Abelian torsion group* G *is totally minimal if and only if it contains a closed normal subgroup* N *such that* N *and* G/N *are totally minimal.*

Somewhat surprisingly, minimality is less well behaved in this respect than its stronger counterpart, as the following example shows.

7.9. *Example* [EDS]. Let $N = \Sigma_{\omega} A_5^{\alpha}$, let L be a two-element subgroup of A_5, define s: $L \to \mathrm{Aut}\, N$ by $s(y)(\langle x_{\xi} \rangle_{\xi<\alpha}) = \langle y x_{\xi} y^{-1} \rangle_{\xi<\alpha}$, and let $G = N \circledS L$ be the semi-direct product as defined in [Bo, Sect. III.2.10]. Then G is a non-minimal torsion group with a closed, normal, totally minimal subgroup N such that G/N is the two-element group.

We describe briefly a recent, significant advance in the theory of minimal groups achieved by Dikranjan and Stojanov [DSj1], [Sj1].

Definition. An *A-class* is a class of minimal Abelian groups closed under the operations of taking products, closed subgroups and quotients, and containing all compact Abelian groups.

For a pre-compact Abelian group G with Weil completion $\overline{G}$, and for p a prime, let $td_p(G)$ denote the subgroup of G generated by those elements x of G such that $cl_{\overline{G}}\{x^n: n \in Z\}$ is a compact $\overline{Z_p}$-module; and let wtd(G) denote the smallest subgroup of G containing $td_p(G)$ for all primes p.

7.10. *Theorem* [Sj1]. *The class of all Abelian pre-compact groups* G *such that* $wtd(\overline{G}) \subseteq G$ *is an* A*-class containing all others.*

Also from [Sj1], we have the following results concerning powers of minimal and totally minimal groups. For an Abelian topological group G, let the *socle* of G be the subgroup of G consisting of those torsion elements whose order is not divisible by the square of any prime.

7.11. *Theorem. Let* G *be a pre-compact Abelian group. The following are equivalent:*

(1) G^{α} *is totally minimal for every cardinal* α;

(2) $G^{\overline{G}}$ *is totally minimal;*

(3) $wtd(\overline{G}) \subseteq G$.

7.12. *Theorem. Let* G *be a pre-compact Abelian group and let* S *be the socle of* $\overline{G}$. *The following are equivalent:*

(1) G^{α} *is minimal for all cardinals* α;

(2) $G^{\overline{G}}$ *is minimal;*

(3) *for some sequence* $\{k_p\colon p \text{ prime}\}$ *of non-negative integers,* $S + \sum_p p^{k_p} \cdot td_p(\overline{G}) \subseteq G$.

Moreover, if G is a totally minimal Abelian group, then G^{α} is totally minimal for all α if and only if G^{α} is minimal for all α.

It turns out, however, that even the strongest result concerning the powers of the groups in a countable set guarantees little about the products of groups within the set. The following example is again due to Stojanov [Sj1].

7.13. *Example* [Sj1]. Let p be a (fixed) prime, let $Z_{(p^n)} = \overline{Z}_p/p^n\overline{Z}p$, and let G_n be the subgroup of $(\overline{Z}_{(p^n)})^{\omega}$ generated by $\oplus(Z_{(p^n)})^{\omega}$ and $p^{n-1}((\overline{Z}_{(p^n)})^{\omega})$. Then G_n^{α} is minimal for all cardinals α, but $\prod_{n \in N} G_n$ is not minimal.

8. Conditions Necessary for Minimality

The results recorded in §7 above have been directed toward sufficient conditions for minimality. Until the ground-breaking papers of Prodanov [P2], [P3], in which pre-compactness assumed a pivotal role, necessary conditions had proved more elusive. Except as noted, the following five results are from [P2], [P3].

8.1. *Theorem. Let* G *be a minimal Abelian group and* S *its socle. Then for every non-empty open subset* V *of* G *there is an integer* n *such that finitely many translates of* V *cover* $nG + S$.

Thus divisible subgroups, the socle, and finitely generated subgroups of G are all pre-compact.

8.2. *Theorem. Every Abelian totally minimal group is pre-compact.*

8.3. *Corollary. The product of a minimal group and a complete minimal Abelian group is minimal.*

8.4. *Corollary* [DP2]. *If* G *is a periodic divisible Abelian group, then* G *admits minimal topologies if and only if* $G = U^n$ *for some* $n < \omega$.

8.5. *Theorem. Let* G *be a complete minimal Abelian group and* $H = \cap_{n<\omega} \mathrm{cl}(nG)$. *Then* H *is compact, and for each neighborhood* V *of the identity there is* $n < \omega$ *such that* $V + H \supset \mathrm{cl}(nG)$.

The paper [DP2] also contains a number of further structure theorems for periodic Abelian groups which admit minimal or totally minimal topologies.

It is an attractive conjecture that commutativity is not necessary for the truth of Theorem 8.2. However, Dierolf and Schwanengel [DS1] produced the following pathological example of a totally minimal group which is not pre-compact and, which, incidentally, has a non-minimal subgroup.

8.6. *Example*. Let A be an infinite discrete space, X the permutation group on A with the finite-open topology, and

$$Y = Y(A) = \{x \in X: \{a \in A: x(a) \neq a\} \text{ is finite}\}.$$

Then Y is a dense subgroup of X and can be shown to be minimal, so X is minimal, by Stephenson's analog of Theorem 7.1(b). Since X has no proper, non-trivial, closed normal subgroups, X is then (vacuously) totally minimal. But by Exercise X.3.19 of [Bo], X is not pre-compact, since it has no completion. Moreover, if $A_1 = \{a_n: n \in Z\}$ is a faithfully indexed subset of A and $f: A \to A$ is defined by $f(a_n) = a_{n+1}$, $f(a) = a$ if $a \in A\backslash A_1$, then the group generated by f is an infinite, discrete, closed subgroup of X, and so non-minimal.

Dikranjan and Stojanov found further conditions which imply pre-compactness for a minimal Abelian group.

8.7. *Theorem* [DSj2]. *If every subgroup of an Abelian topological group is minimal, then the group is pre-compact.*

8.8. *Theorem* [Sj2]. *If* G *is a minimal Abelian group and there is a pre-compact subgroup* H *of* G *such that every subset* S *of* G/H *independent over* Z *satisfies* $|S| < 2^{\omega}$, *then* G *itself is pre-compact.*

8.9. *Theorem* [Sj2]. *If* G *is a torsion-free Abelian group without non-zero divisible subgroups, then all complete, minimal group topologies on* G *are compact.*

Another example due to Schwanengel [Sc] shows that a totally minimal group may be pre-compact but contain nevertheless a closed, non-minimal subgroup. Let p be a prime, $C = Z/pZ$, and $A = \text{Aut } C = \{m^*: 1 \leq m < p - 1\}$, where $m^*: x \to mx$, and let $\alpha \geq \omega$. Define a semi-direct product of $C^\alpha \times A^\alpha$ with itself by defining, in each coordinate,

$$((m,n^*),(s,t^*)) = (m + ns,(nt)^*).$$

Now let $Y = \{((m_\xi),(n_\xi^*))_{\xi<\alpha} \in C^\alpha \times A^\alpha: \{\xi < \alpha: m_\xi \neq 0\}$ is finite$\}$. Then Y is a totally minimal group [Sc]. Let $C^{(\alpha)}$ be its projection onto C^α. Then $C^{(\alpha)} \times \{(1)\}$ is a closed normal subgroup of Y, which is not minimal since it has trivial intersection with the diagonal subgroup of the compact group C^α.

We note finally that a question posed in [G2]--whether or not an Abelian group all of whose finite powers are minimal must have all powers minimal--has been answered in the negative; examples appear in [GC], together with related results concerning cardinality. Among the problems of special interest in this area which remain open, however, are those following. If all finite powers of a group are totally minimal or B(A), do all powers of it have this property? Are minimal Abelian groups necessarily pre-compact? Do the character and pseudocharacter of minimal groups necessarily coincide [A]?

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