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SEQUENTIAL ORDER OF HOMOGENEOUS AND PRODUCT SPACES

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In this paper we will present some examples of sequential spaces with properties related to their sequential order [1]. In section 1 we will show that homogeneous spaces may have any prescribed sequential order. In section 2 we will show that the sequential order of the sequential coreflection of a product of spaces of "small" sequential order may have "large" sequential order and will answer a question posed by Michael [4] when we give an example of a sequential \aleph_0 -space Z so that the sequential coreflection of Z^2 is not regular. Our constructions use sets of sequences as underlying sets and will be facilitated by the following notation. If X is a set, we denote finite sequences in X by $\langle x_0, x_1, \dots, x_k \rangle$, where $k \in \omega$ and every $x_j \in X$. If $\{x_0, x_1, \dots, x_k\} \subset X$ and $S \subset X$, we let $\langle x_0, x_1, \dots, x_k, S \rangle = \{\langle x_0, x_1, \dots, x_k, x_{k+1} \rangle : x_{k+1} \in S\}$ and let $\langle x_0, x_1, \dots, x_k, S, \dots \rangle$ be the set of all finite sequences in X which extend a member of $\langle x_0, x_1, \dots, x_k, S \rangle$.

1. Homogeneous Spaces

The authors of [1] asked whether their space S_ω is the only countable, Hausdorff, homogeneous, sequential space which is not first countable. A technique which produces many spaces with these properties was given in [3]; this technique produces spaces which, like S_ω , have sequential

order ω_1 . While [5] gives a non-regular example with the listed properties and has sequential order 2, the situation among regular spaces is clarified by the following.

A. *Examples.* For every $\alpha \leq \omega_1$ there is a countable, regular, homogeneous, weakly first countable space X_α with sequential order α .

Following [1], if A is a subset of a topological space X then $A^0 = A$; if $\alpha = \beta + 1$, then $A^\alpha =$ the set of limits of sequences in A^β ; if α is a limit ordinal, then $A^\alpha = \cup_{\beta < \alpha} A^\beta$.

We will construct our spaces by induction on α , letting X_0 be a countable discrete space. Suppose we have constructed X_β for all $\beta < \alpha$.

1. α is not a limit ordinal.

Write $\alpha = \beta + 1$; let B be a wfc system [2] for X_β ; pick a distinguished point x^* in X_β . Let X_α be the set of finite sequences $\langle x_0, x_1, \dots, x_{2k} \rangle$ in $X_\beta \cup \omega$ so that

$$\begin{aligned} x_j &\in X_\beta \text{ if } j \text{ is even,} \\ x_j &\in \omega \text{ if } j \text{ is odd.} \end{aligned}$$

We define a wfc system B' for X_α as follows. If $n < \omega$ and $\sigma = \langle x_0, x_1, \dots, x_k \rangle \in X_\alpha$, then

$$\begin{aligned} B'(n, \sigma) = \{ &\sigma \} \cup \langle x_0, \dots, x_{k-1}, B(n, x_k) \setminus \{x_k\}, \dots \rangle \\ &\cup \langle x_0, \dots, x_k, \omega \setminus n, x^*, \dots \rangle, \end{aligned}$$

where $\omega \setminus n = \{n, n+1, n+2, \dots\}$.

We may then show that the sets

$$\begin{aligned} &\{ \sigma \} \cup \langle x_0, x_1, \dots, x_{k-1}, \cup \{x_k\}, \dots \rangle \cup \\ &\cup_{n \geq n_0} \langle x_0, x_1, \dots, x_k, n, \cup \{x_k\}, \dots \rangle \end{aligned}$$

where $\sigma = \langle x_0, x_1, \dots, x_k \rangle \in X_\alpha$, U is a neighborhood of x_k in X_β , $n_0 < \omega$, and every V_n is a neighborhood of x^* in X_β , make up a local neighborhood base at σ .

Inductively, X_α may be shown to have a base of clopen sets.

For each $\sigma = \langle x_0, x_1, \dots, x_k \rangle \in X_\alpha$ the clopen neighborhood $\langle x_0, x_1, \dots, x_{k-1}, X_\beta, \dots \rangle$ of σ is canonically homeomorphic to $X_\alpha = \langle X_\beta, \dots \rangle$ under a homeomorphism carrying σ to $\langle x^* \rangle$. It follows that distinct points σ and τ lie in disjoint clopen sets which admit a homeomorphism carrying σ to τ ; hence X_α is homogeneous.

We wish to show that X_α has sequential order α . Suppose $A \subset X_\alpha$ and $\langle x^* \rangle \in \text{cl}A \setminus A$. Then as $\{\langle x^* \rangle\} \cup \langle X_\beta \setminus \{x^*\}, \dots \rangle \cup \langle x^*, \omega, X_\beta, \dots \rangle$ is a neighborhood of $\langle x^* \rangle$, we may assume that either (1) $A \subset \langle X_\beta \setminus \{x^*\}, \dots \rangle$, or (2) $A \subset \langle x^*, \omega, X_\beta, \dots \rangle$.

(case 1) $A \subset \langle X_\beta \setminus \{x^*\}, \dots \rangle$. Let $\pi: \langle X_\beta \setminus \{x^*\}, \dots \rangle \rightarrow \langle X_\beta \setminus \{x^*\} \rangle$ be the natural "trimming" function. Since $\langle x^* \rangle \in \text{cl} A$, we may use the given neighborhood base for $\langle x^* \rangle$ to show that $\langle x^* \rangle \in \text{cl} \pi(A)$. As $\pi(A) \subset \langle X_\beta \rangle$ and $\langle X_\beta \rangle$ is closed and homeomorphic to X_β , we deduce that $\langle x^* \rangle \in [\pi(A)]^\beta$. That $\langle x^* \rangle \in A^\beta$, follows from (*).

(*) For every $\gamma \in [\pi(A)]^\gamma \setminus \pi(A) \subset A^\gamma$.

If $\gamma = 1$ and $\langle x_0 \rangle \in \text{seq cl } \pi(A) \setminus \pi(A)$, then there is a sequence $\langle \sigma_n \rangle_{n < \omega}$ in A so that $\langle \pi(\sigma_n) \rangle_{n < \omega}$ converges to $\langle x_0 \rangle$. Recalling the definition of B' , we see that in fact $\langle \sigma_n \rangle_{n < \omega}$ converges to $\langle x_0 \rangle$. Assume now that (*) holds for all $\delta < \gamma$. If $\gamma = \delta + 1$, let $\langle x_0 \rangle \in [\pi(A)]^\gamma \setminus \pi(A)$; we may

assume $\langle x_0 \rangle \notin [\pi(A)]^\delta$. Then there is a sequence $\langle \sigma_n \rangle_{n < \omega}$ in $[\pi(A)]^\delta \setminus \pi(A)$ converging to $\langle x_0 \rangle$, the induction hypothesis yielding that $\langle x_0 \rangle \in A^\gamma$. On the other hand, if γ is a limit ordinal, then $[\pi(A)]^\gamma \setminus \pi(A) = \bigcup_{\delta < \gamma} [\pi(A)]^\delta \setminus \pi(A) \subset \bigcup_{\delta < \gamma} A^\delta = A^\gamma$. This establishes (*).

(case 2) $A \subset \langle x^*, \omega, X_\beta, \dots \rangle$. Let $\pi: \langle x^*, \omega, X_\beta, \dots \rangle \rightarrow \langle x^*, \omega, X_\beta \rangle$ be the trimming function. Again, $\langle x^* \rangle \in \text{cl } A$ implies that $\langle x^* \rangle \in \text{cl } \pi(A)$. Because $\pi(A) \subset \langle x^*, \omega, X_\beta \rangle \cup \{\langle x^* \rangle\}$ and because the latter set is closed and homeomorphic to the sequential sum [1] of \aleph_0 copies of X_β (thus has sequential order $\beta + 1$), we get $\langle x^* \rangle \in [\pi(A)]^{\beta+1}$. We may verify that (*) holds for π , and thus $\langle x^* \rangle \in A^{\beta+1}$.

Thus in either case $\langle x^* \rangle \in A^{\beta+1} = A^\alpha$. Hence the sequential order of X_α is no greater than α . As X_α contains a closed copy of the sequential sum of \aleph_0 copies of X_β , the sequential order is precisely α .

2. α is a limit ordinal.

Write $\alpha = \sup\{\beta_i : i < \omega\}$ so that the β_i 's are not limit ordinals. For every $i < \omega$, let $X_i = X_{\beta_i}$; distinguish a point x^i in X_i , and let $X_i^* = X_i \setminus \{x^i\}$. X_α is the set of all finite sequences $\langle x_0, x_1, \dots, x_k \rangle$ in $\bigcup_{i < \omega} X_i^*$ so that for all $j < k$

if $x_j \in X_i^*$, then $x_{j+1} \notin X_i^*$.

Let B_i be a wfc system for X_i so that for every $x \in X_i$ and every $n \leq i$, $B_i(n, x) = X_i$. Now define a wfc system B for X_α as follows: For $n < \omega$ and $\sigma = \langle x_0, x_1, \dots, x_k \rangle \in X_\alpha$ with $x_k \in X_{i_0}^*$ let

$$B(n, \sigma) = \{\sigma\} \cup \langle x_0, x_1, \dots, x_{k-1}, B_{i_0}(n, x_k) \setminus \{x_k\}, \dots \rangle \\ \cup \cup_{i \neq i_0} \langle x_0, x_1, \dots, x_k, B_i(n, x^i) \setminus \{x^i\}, \dots \rangle.$$

A neighborhood base at σ is formed by the sets of the form

$$\{\sigma\} \cup \langle x_0, x_1, \dots, x_{k-1}, U \setminus \{x_k\}, \dots \rangle \cup \\ \cup_{i \neq i_0} \langle x_0, x_1, \dots, x_k, V^i \setminus \{x^i\}, \dots \rangle,$$

where U is a neighborhood of x_k in X_{i_0} , for all $i \neq i_0$ V^i is a neighborhood in X_i of x^i , and for all but finitely many i $V^i = X^i$. X_α may be shown to have a base of clopen sets.

Fix $x'_0 \in X^*$. Let $\sigma = \langle x_0, x_1, \dots, x_k \rangle \in X_\alpha$ with $x_k \in X^*_i$. We will find clopen neighborhoods of $\langle x'_0 \rangle$ and σ that are homeomorphic under a mapping carrying σ to $\langle x'_0 \rangle$, giving homogeneity as before. If $i = 0$, then we can find a homeomorphism f on X_0 so that $f(x_k) = x'_0$ and a clopen neighborhood V of x_k in X_0 so that $x^0 \notin V \cup f(V)$; thus $\langle f(V), \dots \rangle$ and $\langle x_0, x_1, \dots, x_{k-1}, V, \dots \rangle$ are the desired clopen neighborhoods. If on the other hand $i \neq 0$, find homeomorphisms f_0 on X_0 and f_i on X_i so that $f_0(x^0) = x'_0$ and $f_i(x_k) = x^i$; also find clopen neighborhoods V_0 of x^0 and V_i of x_k so that $x^0 \notin f_0(V_0)$ and $x^i \notin V_i$. The natural map defined piecewise from the clopen neighborhood $\{\sigma\} \cup \langle x_0, x_1, \dots, x_{k-1}, V_i \setminus \{x_k\}, \dots \rangle \cup \langle x_0, x_1, \dots, x_k, V_0 \setminus \{x^0\}, \dots \rangle \cup \cup_{j \neq i, 0} \langle x_0, x_1, \dots, x_k, X^*_j, \dots \rangle$ of σ onto the clopen neighborhood $\{\langle x'_0 \rangle\} \cup \langle x'_0, f_i(V_i) \setminus \{x^i\}, \dots \rangle \cup \langle f_0(V_0) \setminus \{x^0\}, \dots \rangle \cup \cup_{j \neq i, 0} \langle x'_0, X^*_j, \dots \rangle$ of $\langle x'_0 \rangle$ is a homeomorphism.

Suppose $A \subset X_\alpha$ and $\langle x'_0 \rangle \in \text{cl } A \setminus \text{seq cl } A$. Since

$\langle x'_0 \rangle \notin \text{seq cl } A$, there is an $i_0 < \omega$ so that A misses $\bigcup_{i > i_0} \langle x'_0, X^*_i, \dots \rangle$. We may assume that either $A \subset \langle X^*_0 \setminus \{x'_0\}, \dots \rangle$ or that $A \subset \langle x'_0, X^*_i, \dots \rangle$ for some $i \leq i_0$. If $A \subset \langle x'_0, X^*_i, \dots \rangle$, let $\pi: \langle x'_0, X^*_i, \dots \rangle \rightarrow \langle x'_0, X^*_i \rangle$, be the trimming function. Then $\langle x'_0 \rangle \in \text{cl } \pi(A)$, and since $\pi(A) \subset \langle x'_0, X^*_i \rangle$ and $\langle x'_0, X^*_i \rangle \cup \{\langle x'_0 \rangle\}$ is closed and homeomorphic to X_i , we have that $\langle x'_0 \rangle \in [\pi(A)]^{\beta_i} \subset [\pi(A)]^\alpha$. One may show that π satisfies (*); thus $\langle x'_0 \rangle \in A^\alpha$. The proof in the case $A \subset \langle X_0 \setminus \{x'_0\}, \dots \rangle$ is similar. Hence X_α has sequential order no larger than α . The subsets $\{\langle x'_0 \rangle\} \cup \langle x'_0, X^*_i \rangle$ of X_α are closed and homeomorphic to X_i when $x'_0 \notin X^*_i$, so the sequential order of X_α is at least $\sup_{i < \omega} \beta_i = \alpha$. This completes the proof.

2. Product Spaces

While the product of two sequential spaces need not be sequential, if X and Y are sequential there is a natural sequential space topology for the set $X \times Y$: decree that a sequence $\langle (x_n, y_n) \rangle_{n \in \omega}$ "converges" to (x, y) if and only if $\langle x_n \rangle_{n \in \omega}$ converges to x in X and $\langle y_n \rangle_{n \in \omega}$ converges to y in Y ; define U to be open in $X \times Y$ if it is sequentially open with respect to these "convergent" sequences. This is the sequential coreflection of the usual product topology, denoted henceforth by $\sigma(X \times Y)$.

Our next examples show that there is no natural bound on the sequential order of $\sigma(X \times Y)$ based on the sequential orders of X and Y .

B. *Examples.* There are countable regular spaces X and Y so that X is Fréchet and Y is weakly first countable with sequential order 2 so that σX^2 and σY^2 have sequential order ω_1 .

Let X be the set of all finite sequences in ω of even (possibly 0) length. A set U is open in X if and only if for every $\langle n_1, n_2, \dots, n_{2k} \rangle \in U$ and every $i \in \omega$, there is a $j \in \omega$ so that $\langle n_1, n_2, \dots, n_{2k}, i, \omega \setminus j, \dots \rangle \subset U$.

The space σX^2 contains a closed copy of S_ω . Let $s(\phi) = (\phi, \phi)$ and for $n_1 \in \omega$ let $s(n_1) = (\langle 0, n_1 \rangle, \phi)$. Generally, $s(n_1, n_2, \dots, n_{2k}) = (\langle 0, n_1, n_2, \dots, n_{2k-1} \rangle, \langle n_1, n_2, \dots, n_{2k} \rangle)$ and $s(n_1, n_2, \dots, n_{2k+1}) = (\langle 0, n_1, n_2, \dots, n_{2k+1} \rangle, \langle n_1, n_2, \dots, n_{2k} \rangle)$. Observe that $\langle s(n_1, n_2, \dots, n_{k+1}) \rangle_{n_{k+1} \in \omega}$ converges to $s(n_1, \dots, n_k)$. We will show that these are essentially the only sequences in $S = \{s(\sigma) : \sigma \text{ a finite sequence in } \omega\}$ converging to a point of X^2 , hence S is a sequentially closed copy of S_ω in X^2 .

Suppose with us that there is a sequence σ in S converging to $(\langle r_1, r_2, \dots, r_{2k} \rangle, \langle s_1, s_2, \dots, s_{2\ell} \rangle) \in X^2$ which is not eventually constant in either factor. Then there is such a $\sigma = \langle (\langle 0, n_1^p, n_2^p, \dots, n_{i_p}^p \rangle, \langle n_1^p, n_2^p, \dots, n_{j_p}^p \rangle) \rangle_{p \in \omega}$ so that $|i_p - j_p| = 1$, $i_p \geq 2k + 1$, $j_p \geq 2\ell + 2$ for all $p \in \omega$; $\{n_{2k+1}^p : p \in \omega\}$ and $\{n_{2\ell+2}^p : p \in \omega\}$ are infinite; $\{n_{2k}^p : p \in \omega\}$ and $\{n_{2\ell+1}^p : p \in \omega\}$ are finite. Now if $j \leq 2\ell$, $\langle n_j^p \rangle_{p \in \omega}$ is eventually constant ($=s_j$), so $2k + 1 > 2\ell$; also $2\ell + 1 \neq 2k + 1$, so $2k + 1 > 2\ell + 2$. We also have that if $j \leq 2k - 1$, $\langle n_j^p \rangle_{p \in \omega}$ is eventually constant ($=r_{j+1}$), so $2\ell + 2 > 2k - 1$. With $2k \neq 2\ell + 2$, we get $2\ell + 2 > 2k + 1$, a contradiction.

So every convergent sequence in S is eventually constant in one of the factors. If σ is a sequence in S which is constant in the second factor, $\sigma \setminus \langle \langle 0, n_1, n_2, \dots, n_{2k-1} \rangle, \langle n_1, n_2, \dots, n_{2k} \rangle \rangle = \langle \langle 0, n_1, n_2, \dots, n_{2k}, n_{2k+1}^p \rangle, \langle n_1, n_2, \dots, n_{2k} \rangle \rangle_{p \in \omega}$ for some $\langle n_1, n_2, \dots, n_{2k} \rangle \in X$. Likewise if σ is constant in the first factor, $\sigma \setminus \langle \langle 0, n_1, n_2, \dots, n_{2k-1} \rangle, \langle n_1, n_2, \dots, n_{2k-2} \rangle \rangle = \langle \langle \langle 0, n_1, n_2, \dots, n_{2k-1} \rangle, \langle n_1, n_2, \dots, n_{2k-1}, n_{2k}^p \rangle \rangle_{p \in \omega}$, showing that every sequence in S converging to a point in X^2 is eventually constant or a subsequence of one of our canonical convergent sequences, as desired.

Let Y be the set of all non-void finite sequences of positive rationals with wfc system given by

$$B(m, \langle q_0, q_1, \dots, q_k \rangle) = \langle q_0, q_1, \dots, q_{k-1}, S_m(q_k) \rangle \cup \langle q_0, q_1, \dots, q_k, S_m(0), \dots \rangle, \text{ where } S_m(q) = \{r \in \mathbf{Q}^+ : |r-q| < 1/m\}.$$

Now let $\langle q(j) \rangle_{j < \omega}$ be a sequence in \mathbf{Q}^+ converging monotonically to 0 and $\langle q(j, k) \rangle_{k < \omega}$ be a sequence in $(q(j+1), q(j)) \cap \mathbf{Q}^+$ converging monotonically to $q(j)$. We will show that the set $S = \{s(\sigma) : \sigma \text{ a finite sequence in } \omega\}$ is a closed copy of S_ω in σY^2 , where $s(\phi) = \langle \langle 1 \rangle, \langle 1 \rangle \rangle$, $s(n_1, n_2, \dots, n_{2k-1}) = \langle \langle 1, q(n_1, n_2), \dots, q(n_{2k-3}, n_{2k-2}), q(n_{2k-1}) \rangle, \langle 1+q(n_1), q(n_2, n_3), \dots, q(n_{2k-2}, n_{2k-1}) \rangle \rangle$, $s(n_1, n_2, \dots, n_{2k}) = \langle \langle 1, q(n_1, n_2), \dots, q(n_{2k-1}, n_{2k}) \rangle, \langle 1+q(n_1), q(n_2, n_3), \dots, q(n_{2k-2}, n_{2k-1}), q(n_{2k}) \rangle \rangle$.

We will show that a sequence $\langle \sigma_p \rangle_{p < \omega} = \langle s(n_1^p, n_2^p, \dots, n_{i_p}^p) \rangle_{p < \omega}$ in S cannot converge to a point $\langle \langle r_1, r_2, \dots, r_k \rangle, \langle s_1, s_2, \dots, s_\ell \rangle \rangle$ of Y^2 if for infinitely many $p < \omega$ both the first coordinate $(= \sigma_p^1)$ has length $> k$ and the second

coordinate $(=\sigma_p^2)$ has length $> \ell$. For if such a sequence did converge we could, by finding a subsequence, assume that the σ_p^1 's extend $\langle r_1, \dots, r_k \rangle$ and converge to 0 in the $k+1$ position, while the σ_p^2 's extend $\langle s_1, \dots, s_\ell \rangle$ and converge to 0 in the $\ell+1$ position. That is, $\{n_i^p : p < \omega\}$ is finite for all $i < 2k - 1$ and $\{n_{2k-1}^p : p < \omega\}$ is infinite, while $\{n_i^p : p < \omega\}$ is finite for all $i < 2\ell$ and $\{n_{2\ell}^p : p < \omega\}$ is infinite; this contradiction establishes our claim.

If $\langle \sigma_p \rangle_{p < \omega} = \langle s(n_1^p, n_2^p, \dots, n_i^p) \rangle_{p < \omega}$ is a sequence in S converging to $(\langle r_1, r_2, \dots, r_k \rangle, \langle s_1, s_2, \dots, s_\ell \rangle)$ so that the σ_p^1 's have length k , then σ_p^1 is eventually constant $(=\langle r_1, \dots, r_{k-1} \rangle)$ in the first $k-1$ positions, i.e. for appropriate $n_1, n_2, \dots, n_{2k-4}$ and large $p \langle 1, q(n_1^p, n_2^p), \dots, q(n_{2k-5}^p, n_{2k-4}^p) \rangle = \langle 1, q(n_1, n_2), \dots, q(n_{2k-5}, n_{2k-4}) \rangle$. Further, $\langle \sigma_p^1 \rangle_{p < \omega}$ converges to $r_k \neq 0$ in the k position, so that $\langle n_{2k-3}^p \rangle_{p < \omega}$ is eventually constant $(=n_{2k-3})$ and $\{n_{2k-2}^p : p < \omega\}$ is infinite. Consequently $\langle \sigma_p \rangle_{p < \omega}$ is a subsequence of $\langle s(n_1, n_2, \dots, n_{2k-3}, n_{2k-2}) \rangle_{n_{2k-2} < \omega}$.

Similarly, if $\langle \sigma_p \rangle_{p < \omega}$ is a sequence in S converging to $(\langle r_1, r_2, \dots, r_k \rangle, \langle s_1, s_2, \dots, s_\ell \rangle)$ so that the σ_p^2 's have length ℓ , then $\langle \sigma_p \rangle_{p < \omega}$ is eventually a subsequence of $\langle s(n_1, n_2, \dots, n_{2\ell-2}, n_{2\ell-1}) \rangle_{n_{2\ell-1} < \omega}$.

Since $\langle s(n_1, n_2, \dots, n_j, n_{j+1}) \rangle_{n_{j+1} < \omega}$ converges to $s(n_1, n_2, \dots, n_j)$ and these are essentially the only convergent sequences in S , S may be viewed as a closed copy of S_ω in σY^2 .

C. *Example.* There is a regular space Z with a countable weak base [6] so that σZ^2 is not regular.

Let P be a countable set of irrationals which is dense in I , where $\phi: P \rightarrow \mathbf{N}$ is one-to-one. Let Z be the set of all non-void sequences (finite or infinite) in P with wfc system defined as follows.

$$B(n, \langle p_0, p_1, \dots, p_k \rangle) = \langle p_0, p_1, \dots, p_{k-1}, S_n(p_k) \rangle \cup \langle p_0, p_1, \dots, p_k, S_n(0), \dots \rangle,$$

$$B(n, \langle p_i \rangle_{i \in \omega}) = \langle p_0, p_1, \dots, p_{n-1}, p_n, \dots \rangle, \text{ where}$$

$S_n(x) = \{y \in P: |y-x| < \frac{1}{n}\}$ and $\langle p_0, p_1, \dots, p_k, T, \dots \rangle$ is the set of all sequences in P , finite or infinite, which extend a member of $\langle p_0, p_1, \dots, p_k, T \rangle$. Z is regular and has a countable weak base.

For $k \in \omega$ let

$$W_{2k} = \cup \{B(\phi(q_k), \langle p_0, p_1, \dots, p_k \rangle) \times B(1, \langle q_0, q_1, \dots, q_k \rangle):$$

$$p_0, p_1, \dots, p_k, q_0, q_1, \dots, q_k \in P\}$$

$$W_{2k+1} = \cup \{B(1, \langle p_0, p_1, \dots, p_{k+1} \rangle) \times B(\phi(p_{k+1}), \langle q_0, q_1, \dots, q_k \rangle):$$

$$p_0, p_1, \dots, p_{k+1}, q_0, q_1, \dots, q_k \in P\}.$$

It is straightforward to check that a sequence converging to a member of W_k must eventually be in $W_k \cup W_{k+1}$, and hence $W = \cup_{k \in \omega} W_k$ is a sequentially open set in Z^2 .

Let $\{p_0, q_0'\} \subset P$ so that $\phi(p_0) > (q_0')^{-1}$. We will show that every sequentially open set U in Z^2 with $(\langle p_0 \rangle, \langle q_0' \rangle) \in U$ contains a sequence converging to a point not in W , hence that σZ^2 is not regular.

Assume we have found p_i ($i \leq k$), q_i ($i < k$), and q_k' so that

1. $(\langle p_0, \dots, p_i \rangle, \langle q_0, \dots, q_i \rangle) \in U$ if $i < k$.
2. $\phi(q_i) > (p_{i+1})^{-1}$ and $\phi(p_i) > q_i^{-1}$ if $i < k$.

- 3. $\phi(p_k) > (q'_k)^{-1}$.
- 4. $(\langle p_0, \dots, p_k \rangle, \langle q_0, \dots, q_{k-1}, q'_k \rangle) \in U$

Because of (4) we can find an $m \in \omega$ such that

$B(m, \langle p_0, \dots, p_k \rangle) \times B(m, \langle q_0, \dots, q_{k-1}, q'_k \rangle) \subset U$; choose $p'_{k+1} \in P$ so that $\langle p_0, \dots, p_k, p'_{k+1} \rangle \in B(m, \langle p_0, \dots, p_k \rangle)$ and $q_k \in P$ so that $\langle q_0, \dots, q_k \rangle \in B(m, \langle q_0, \dots, q_{k-1}, q'_k \rangle)$, $q_k^{-1} < (q'_k)^{-1} < \phi(p_k)$, and $\phi(q_k) > (p'_{k+1})^{-1}$. Note that $(\langle p_0, \dots, p_k \rangle, \langle q_0, \dots, q_k \rangle) \in U$.

Since $(\langle p_0, \dots, p_k, p'_{k+1} \rangle, \langle q_0, \dots, q_k \rangle) \in U$, there is an $n < \omega$ so that $B(n, \langle p_0, \dots, p_k, p'_{k+1} \rangle) \times B(n, \langle q_0, \dots, q_k \rangle) \subset U$. So there is a $q'_{k+1} \in P$ so that $\langle q_0, \dots, q_k, q'_{k+1} \rangle \in B(n, \langle q_0, \dots, q_k \rangle)$ and a $p_{k+1} \in P$ such that $\langle p_0, \dots, p_{k+1} \rangle \in B(n, \langle p_0, \dots, p_k, p'_{k+1} \rangle)$, $\phi(q_k) > (p_{k+1})^{-1}$, and $\phi(p_{k+1}) > (q'_{k+1})^{-1}$. This finishes the induction.

The sequence $\{(\langle p_0, \dots, p_k \rangle, \langle q_0, \dots, q_k \rangle) : k \in \omega\}$ in U converges to $\zeta = (\langle p_i \rangle_{i < \omega}, \langle q_i \rangle_{i < \omega})$ in Z^2 . To see that $\zeta \notin W$, note that if $\zeta \in B(\phi(s_k), \langle r_0, r_1, \dots, r_k \rangle) \times B(1, \langle s_0, s_1, \dots, s_k \rangle)$ for some $k \geq 0$, then, since every infinite sequence in $B(\phi(s_k), \langle r_0, r_1, \dots, r_k \rangle)$ is an extension of $\langle r_0, r_1, \dots, r_k \rangle, \langle p_0, \dots, p_k \rangle = \langle r_0, \dots, r_k \rangle$, and for the same reason $\langle s_0, \dots, s_k \rangle = \langle q_0, \dots, q_k \rangle$. Thus $\zeta \in B(\phi(q_k), \langle p_0, p_1, \dots, p_k \rangle) \times B(1, \langle q_0, \dots, q_k \rangle)$, which would mean $\phi(q_k) < (p_{k+1})^{-1}$, violating (2). A like argument shows that ζ is not in any of the sets $B(1, \langle r_0, r_1, \dots, r_{k+1} \rangle) \times B(\phi(r_{k+1}), \langle s_0, \dots, s_k \rangle)$. Thus $\zeta \notin W$ as claimed.

We note that Z is an \aleph_0 -space (Theorem 1.15 in [6]), thereby answering Michael's question in [4].

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