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Let M be a compact piecewise linear manifold and H(M) the space of all homeomorphisms of M onto itself with the supremum topology: $\rho(f,g) = \sup_{x \in M}\{d(f(x), g(x)\}$. For some years there has been considerable interest in the question of whether H(M) is an ℓ_2-manifold; i.e., a separable metric space which is locally homeomorphic to ℓ_2, the Hilbert space of square-summable sequences. It is known that H(M) is locally contractible [1,3]. There are other partial results along these lines including the fact that PLH(M), the subspace of H(M) consisting of all piecewise linear homeomorphisms, is an ℓ_2^f-manifold, where ℓ_2^f denotes the subspace of ℓ_2 consisting of those sequences having only finitely many nonzero entries [5]. Let $H_\delta(B^n)$ and $PLH_\delta(B^n)$ denote the subspaces of $H(B^n)$ and $PLH(B^n)$, respectively, that are equal to the identity on the boundary of the n-ball B^n. Using the fact that, for $n \neq 4$ and $\partial M = \phi$ if $n = 5$, the identity component of $PLH(M^n)$ is dense in the identity component of $H(M^n)$, the question of whether $H(M^n)$ is an ℓ_2-manifold for a fixed n can be reduced to the question of whether the elements of $H_\delta(B^n)$ can be canonically approximated by elements of $PLH_\delta(B^n)$: i.e., given $\varepsilon: H_\delta(B^n) \to (0,1)$ does there exist a map $\psi: H_\delta(B^n) \to PLH_\delta(B^n)$ with $\rho(\psi(h),h) < \varepsilon(h)$ for all $h \in H_\delta(B^n)$? [4].

Also unresolved is the question of whether H(M) has artibrarily small contractible neighborhoods. Cernavskii [2] proposed an outline of a proof. However, his outline depends upon the use of a canonical engulfing theorem; it is proposed that the canonical engulfing theorem be obtained by making use of canonical general position. It is at this point that the suggestion breaks down since general position arguments cannot be made in a canonical fashion. For example, let L be a set consisting of six points and $f: L \times [0,1] \to E^3$ be a continuous function with the property that $f_0(L)$ consists of six points which are the vertices of two linked 1-spheres and $f_1(L)$ consists of six points which are the vertices of two unlinked 1-spheres. Given any $\varepsilon > 0$ and $t \in [0,1]$, there is a function $f_t': L \to E^3$ with $\rho(f_t', f_t) < \varepsilon$ and with $f_t'(L)$ in general position. However, this approximation cannot be accomplished in a canonical manner. More specifically, for sufficiently small $\varepsilon > 0$ there does not exist a continuous function $f': L \times [0,1] \to E^3$ with $\rho(f',f) < \varepsilon$ and with the property that for each t, $f_t'(L)$ is in general position (general position would require that the 1-spheres have non-empty intersection for each t). Similar examples exist in other dimensions. However, using different techniques, the theorem of this paper demonstrates that H(M) does have arbitrarily small weakly contractible neighborhoods.

For any manifold M, 1_M will denote the identity homeomorphism and $N_\varepsilon(h)$ will denote the ε-neighborhood of h in

$H(M)$. For a complex L, $|L|$ will denote the underlying point set; for any simplex σ of L, $st(\sigma)$ will denote $\bigcup_{\sigma<\tau} |\tau|$. The n skeleta, L_n, of L consists of the subcomplex of L consisting of all simplices of dimension $\leq n$.

Let $\eta_0\colon (0,\infty) \to (0,\infty)$ be the identity. Since $PLH(M)$ is uniformly locally contractible we can define inductively monotonically increasing functions $\eta_n\colon (0,\infty) \to (0,\infty)$ with the property that for each positive integer n and any $\varepsilon > 0$, if A is any subset of $PLH(M)$ with diameter less than $3\eta_n(\varepsilon)$ then A is contractible in a set of diameter less than $\eta_{n-1}(\varepsilon)$.

Theorem. Let M^n *be a compact* PL *manifold,* $n \neq 4$*; if* $n = 5$ *suppose* ∂M *is empty. Let* $h \in H(M^n)$ *and* $\varepsilon > 0$ *be given. Then there exists an open neighborhood* U *of* h *with* U *contained in the* ε*-neighborhood of* h *and with the property that if* $\phi\colon |K| \to U$ *is any mapping of a locally finite complex* K *into* U*, then* ϕ *is homotopic in* U *to the constant map* $\phi(k) = k$.

Proof. Since $H(M)$ is a topological group it suffices to consider the case $h = 1_M$. Let $\varepsilon > 0$ be given. Since $PLH(M)$ is homeomorphic to ℓ_2^f, there is a contractible neighborhood V of 1_M in $PLH(M)$ with $V \subset N_\varepsilon(1_M) \cap PLH(M)$. For each $h \in V$, choose δ_h such that $N_{\delta_h}(h) \cap PLH(M) \subset V$. Let $U = \bigcup_{h\in V} N_{\delta_h}(h)$ and note that U is an open neighborhood of 1_M in $H(M)$ contained in the ε-neighborhood of 1_M and, due to the dimension restrictions, V is a dense subset of U. Let K be a locally finite complex and $\phi\colon |K| \to U$ be given.

Since V is contractible to 1_M it suffices to define a map $\bar{\psi}: |K| \times I \to U$ with $\bar{\psi}(k,0) = \phi(k)$ and $\bar{\psi}(k,t) \in V$ for all $k \in |K|$ and $t \neq 1$.

Since K is a locally finite complex, we can obtain a triangulation of $K \times (0,1]$ and a mapping $\gamma: |K \times (0,1]| \to (0,1]$ so that if $\psi: |K \times (0,1]| \to PLH(M)$ is any mapping with the properties that

i) for each vertex (k,t) of the triangulation, $\rho(\psi(k,t),\phi(k)) < \gamma(k,t)$ and

ii) for each simplex σ of the triangulation $\operatorname{diam}\psi(\sigma) < \inf_{p\in\sigma}\gamma(p)$,

then $\psi(|K \times (0,1]|) \subset V$ and $\bar{\phi}: |K \times [0,1]| \to U$ defined by

$$\bar{\phi}(k,t) = \begin{cases} \psi(k,t), & t \neq 0 \\ \phi(k), & t = 0 \end{cases} \quad \text{is continuous.}$$

Now further subdivide $K \times (0,1]$ to form $(K \times (0,1])'$ so that if (k,t) and (k',t') are vertices of the same simplex, σ, of $(K \times (0,1])'$ then $\rho(\phi(k),\phi(k')) < \eta_m(\inf_{p\in st\sigma}\gamma(p))$, where m is the dimension of σ. By the previous paragraph it suffices to define $\psi: |K \times (0,1]| \to PLH(M)$ with the properties that i) for each vertex (k,t) of $(K \times (0,1])'$, $\operatorname{diam}\psi(\sigma) < \inf_{p\in\sigma}\gamma(p)$. We will define ψ inductively on the skeleta of $(K \times (0,1])'$. If $v = (k,t)$ is a vertex of $(K \times (0,1])'$ choose $\psi_0(v) \in PLH(M)$ with $\rho(\psi_0(v),\phi(k)) < \eta_m(\inf_{p\in st(v)}\gamma(p))$, where m is the dimension of $st(v)$. Assume inductively that $\psi_n: (K \times (0,1])'_n \to PLH(M)$ is defined with the property that ψ_n extends ψ_{n-1} and if σ is an n-simplex of $(K \times (0,1])'$ and if m is the dimension of the largest dimensional simplex that contains σ then $\operatorname{diam}\psi_n(\sigma) < \eta_{m-n}(\inf_{p\in st(\sigma)}\gamma(p))$. Now, let σ be a $(n+1)$-dimensional simplex of $(K \times (0,1])'$ and let m be the dimension of the largest dimensional simplex that contains σ. Every simplex which contains σ contains each simplex in the boundary of σ and hence the boundary of $\psi_n(\sigma)$ has diameter less than $3\eta_{m-n}(\inf_{p\in st(\sigma)}\gamma(p))$. [In the case $n = 0$, this condition is satisfied by the choice of ψ_0 and the choice of the triangulation $(K \times (0,1))'$.] Therefore, ψ_n can be extended to ψ_{n+1} on the interior of σ in such a way that $\operatorname{diam}\psi_{n+1}(\sigma) < \eta_{m-(n+1)}(\inf_{p\in st(\sigma)}\gamma(p))$. This completes the inductive definition of ψ_n. Since $(K \times (0,1])'$ is a locally finite complex, $\psi = \lim_{n\to\infty}\psi_n$ is continuous. We note that for any simplex σ in $(K \times (0,1])'$, $\operatorname{diam}\psi(\sigma) < \inf_{p\in\sigma}\gamma(p)$.

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