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by

Richard E. Heisey

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
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EMBEDDING PIECEWISE LINEAR R^∞-MANIFOLDS INTO R^∞

Richard E. Heisey

It is well known that a compact piecewise linear manifold of dimension n, $n \geq 2$, can be piecewise linearly embedded into R^{2n}. Here we establish an infinite-dimensional analogue. Let $R^\infty = \varinjlim R^n$, the countable direct limit of lines. We show that any separable, paracompact piecewise linear R^∞-manifold can be piecewise linearly embedded onto a closed piecewise linear submanifold of R^∞. As a consequence piecewise linear R^∞-manifolds may be regarded as "polyhedra" in R^∞.

I. Definitions and Statement of the Main Theorem

Let $R^\infty = \varinjlim R^n$, the countable direct limit of lines. We think of R^∞ as $\{(x_i): \text{all but finitely many } x_i \text{ are } 0\}$ and identify R^n with $R^n \times \{0\} \times \{0\} \times \{0\} \times \cdots \subset R^\infty$. A straightforward observation, e.g. see Lemma III-6 of [1], shows that any compact subset of R^∞ is contained in some R^n. Let U and V be open subsets of R^∞. A map $f: U \to V$ is R^∞-*piecewise linear*, hereafter R^∞-p.l., if for every compact polyhedron $C \subset U$ and for every choice of n such that $f(C) \subset V \cap R^n$, the restriction $f|_C: C \to V \cap R^n$ is piecewise linear in the usual sense. (By *polyhedron* we mean a subset $P \subset R^n$ such that every point $x \in P$ has a cone neighborhood xL, where L is compact. For this and other basic definitions and results from piecewise linear topology see [3].)

A *piecewise linear* R^∞*-atlas for a space* M is a collection of pairs $\{(U_\alpha,\phi_\alpha)\}$ where $\{U_\alpha\}$ is an open cover of M by nonempty sets, $\phi_\alpha\colon U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism onto an open subset of R^∞, and where, if $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\beta\phi_\alpha^{-1}\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is R^∞-p.l. A *piecewise linear* R^∞*-structure* for M is a maximal p.l. R^∞-atlas for M. Since any p.l. R^∞-atlas for the space M is contained in a unique maximal p.l. R^∞-atlas, a p.l. R^∞-atlas for M determines a p.l. R^∞-structure for M. A *piecewise linear* R^∞*-manifold* is a paracompact space M together with a p.l. R^∞-structure. A *piecewise linear* R^∞*-atlas for the* p.l. R^∞*-manifold* M is any p.l. R^∞-atlas for the space M which is contained in the p.l. R^∞-structure for M. An element (U,ϕ) of some p.l. R^∞-atlas for the p.l. R^∞-manifold M is a *piecewise linear* R^∞*-chart* for M. If (U,ϕ) is such a chart and if $\phi'\colon U' \to \phi'(U')$ is the restriction of ϕ to a nonempty open subset of U then, clearly, (U',ϕ') is such a chart.

A map $f\colon M \to N$ between two p.l. R^∞-manifolds is R^∞*-piecewise linear* if for each $x \in M$ there is a p.l. R^∞-chart (U,ϕ) for M and a p.l. R^∞-chart (V,ψ) for N such that $x \in U$, $f(x) \in V$ and $\psi f\phi^{-1}\colon \phi(U \cap f^{-1}(V)) \to \psi(V)$ is R^∞-p.l. It follows then that if $f\colon M \to N$ is R^∞-p.l. and (U,ϕ) and (V,ψ) are any given R^∞-p.l. charts with $x \in U$ and $f(x) \in V$ then $\psi f\phi^{-1}\colon \phi(U \cap f^{-1}(V)) \to \psi(V)$ is R^∞-p.l. An R^∞-p.l. map $f\colon M \to N$ is an R^∞-p.l. *isomorphism* if f is a homeomorphism and $f^{-1}\colon N \to M$ is R^∞-p.l.

Let $\tau\colon R^\infty \times R^\infty \to R^\infty$ be the natural linear homeomorphism $\tau((x_i),(y_i)) = (x_1,y_1,x_2,y_2,x_3,y_3,\cdots)$. (That τ is a homeomorphism follows since R is locally compact [1, Corollary III-1].) We identify $R^\infty \times R^\infty$ with R^∞ as p.l. R^∞-manifolds via τ. Thus, for a p.l. R^∞-manifold M we may identify any given p.l. R^∞-chart with image in R^∞ with one whose image is in $R^\infty \times R^\infty$. Let N be a subset of the p.l. R^∞-manifold M such that for each $x \in N$ there is a p.l. R^∞-chart (U,ϕ) for M with $x \in U$ such that $\phi(U) = U_1 \times U_2$, U_i open in R^∞, and such that $\phi(U \cap N) = U_1 \times \{\mathbf{0}\}$. (Here $\mathbf{0} = (0,0,0,\cdots)$.) If we identify $R^\infty \times \{\mathbf{0}\}$ with R^∞, then, for such a chart (U,ϕ), $\phi|(U \cap N): U \cap N \to U_1$ is a homeomorphism. Thus, charts of the form $(U',\phi') = (U \cap N, \phi|(U \cap N))$ form a p.l. R^∞-atlas for N inducing a p.l. R^∞-structure for N. With this p.l. R^∞-structure we call N a p.l. *R^∞-submanifold of* M (of infinite codimension).

We may now state our main theorem.

Theorem. If M *is a separable, paracompact* p.l. *R^∞-manifold then there is an* R^∞-p.l. *isomorphism* $f: M \to N$, N *a closed* p.l. *R^∞-submanifold of* R^∞.

The proof of this theorem is given in §III.

There is a natural definition for R^∞-polyhedra and p.l. maps between them.

Definition. A subset X of R^∞ is an *R^∞-polyhedron* if for each compact polyhedron C in R^∞, $C \cap X$ is a polyhedron. A map $f: X \to Y$ between two R^∞-polyhedra is *R^∞-piecewise linear* if for each compact polyhedron $C \subset X$ and any choice of n such that $f(C) \subset Y \cap R^n$, $f|C: C \to Y \cap R^n$ is p.l.

We conclude our paper by showing, in §IV, that any p.l. R^∞-submanifold of R^∞ is an R^∞-polyhedron and that for maps between two such submanifolds the two definitions of R^∞-piecewise linear agree. Thus, the study of p.l. R^∞-manifolds and R^∞-p.l. maps between them is a special case of the study of R^∞-polyhedra and R^∞-p.l. maps between them.

II. Preliminary Results

Lemma 1 below is the crucial auxilliary result we will need. In addition we establish some helpful elementary results about R^∞-p.l. maps. First, a useful definition.

Definition. If U is an open subset of R^∞ and P is a finite-dimensional polyhedron we say a map $f: U \to P$ is *piecewise linear* (p.l.) if for every compact polyhedron $C \subset U$, $f|C: C \to P$ is p.l. If M is a p.l. R^∞-manifold we say that $f: M \to P$ is p.l. if $f\phi^{-1}: \phi(U) \to P$ is p.l. for each p.l. R^∞-chart (U,ϕ) for M.

Lemma 1. *Let* U *be an open subset of* R^∞. *Let* $A \subset W \subset U$ *where* A *is closed in* U *and* W *is open in* U. *Then there is a* p.l. *map* $\lambda: U \to I$ *such that* $\lambda|A = 0$ *and* $\lambda|(U-W) = 1$.

Proof. The proof proceeds in the spirit of the proof of Proposition IV.2 in [2]. Let $c = \{c_i\} = (c_1,c_2,\cdots, c_{n_0},0,0,\cdots) \in R^\infty$, and let $V = [(c_1-\varepsilon_1,c_1+\varepsilon_1) \times (c_2-\varepsilon_2, c_2+\varepsilon_2) \times \cdots \times (c_{n_0}-\varepsilon_{n_0},c_{n_0}+\varepsilon_{n_0}) \times (-\varepsilon_{n_0+1},\varepsilon_{n_0+1}) \times \cdots] \cap R^\infty$, where $\varepsilon_i > 0$. Define $V(2) = [(c_1-2\varepsilon_1,c_1+2\varepsilon_1) \times (c_2-2\varepsilon_2, c_2+2\varepsilon_2) \times \cdots] \cap R^\infty$. Let $\alpha_i: R \to I$ be a p.l. map such that $\alpha_i|[c_i-\varepsilon_i,c_i+\varepsilon_i] = 1$ and $\alpha_i|[R\setminus(c_i-2\varepsilon_i,c_i+2\varepsilon_i)] = 0$. Define $\psi_i\colon R^\infty \to I$ by $\psi_i((x_i,x_2,\cdots)) = \alpha_i(x_i)$ and then $\psi(x) = \min\{\psi_1(x)|i = i,2,3\cdots\}$. If $x \in R^n$, $n \geq n_0$, then $\psi_i(x) = 1$, $i > n$, and $\psi(x) = \min\{\psi_1(x),\cdots,\psi_n(x)\}$. Thus, ψ is continuous, $\psi|R^n$ is p.l., $n \geq 1$, $\psi|V = 1$, and $\psi|[R\setminus V(2)] = 0$. Note that sets of the form of V form a basis for R^∞ [1, Proposition II.1(a)].

Let A, W, and U be as in Lemma 1. By elementary reasoning $U = \varinjlim C_n$ where $C_n \subset R^n$ is a compact polyhedron and $C_n \subset \mathrm{Int}_{R^{n+1}} C_{n+1}$. Let $A_k = A \cap C_k$. Choose finitely many basic open sets of the type in the preceding paragraph, $V_{i,1},\cdots,V_{1,k_1}$, covering compact A_1 and such that $V_{1,i}(2) \subset W$, $i = 1,2,\cdots,k_1$. For each $x \in A_2\setminus C_1$, choose a basic open set $V_{2,x}$ such that $x \in V_{2,x} \subset V_{2,x}(2) \subset W\setminus C_1$. Then $V_{1,1},\cdots,V_{1,k_1}$ together with $\{V_{2,x}\colon x \in A_2\setminus C_1\}$ form an open cover of A_2, so we may select a finite subcover $V_{1,1},\cdots,V_{1,k_1},V_{2,1},\cdots,V_{2,k_2}$. Continuing, we obtain a sequence $V_{1,1},\cdots,V_{1,k_1},V_{2,1},\cdots,V_{2,k_2},V_{3,1},\cdots,V_{3,k_3},\cdots$ covering A such that $V_{i,j}(2) \subset W\setminus C_{i-1}$, $i > 1$. By the work in the preceding paragraph, for each (i,j) there is a p.l. map $\phi_{i,j}\colon U \to I$ such that $\phi_{i,j}|V_{i,j} = 0$ and $\phi_{i,j}|[U\setminus V_{i,j}(2)] = 1$. Let $\phi_i = \min\{\phi_{i,1},\cdots,\phi_{i,k_i}\}$ and $\phi = \min\{\phi_i|i = 1,2,\cdots\}$. Let $x \in C_n$. Then $x \notin V_{k,j}(2)$, $k > n$, so that $\phi_{k,j}(x) = 1$, $k > n$. Therefore, $\phi|C_n = \min\{\phi_1,\cdots,\phi_n\}$. Thus, $\phi|C_n$ is p.l., $n \geq 1$, and it follows that ϕ is p.l. Also, $\phi|A = 0$ and $\phi|(U\setminus W) = 1$.

Lemma 2. (a) *The composition of* R^∞-p.l. *maps is* R^∞-p.l. *Also, if* $f\colon M \to N$ *is* R^∞-p.l. *and* $g\colon N \to P$, *P a*

finite-dimensional polyhedron, is p.l., *then* gf *is* p.l.
(b) *A map* $f = (f_1, f_2, f_3, \cdots): M \to R^\infty$, M *a* p.l. R^∞*-manifold, is* R^∞-p.l. *if and only if each* f_i *is* p.l.

Proof. The proof of (a) is straightforward, and we omit it. For (b) regard a p.l. map (on a finite-dimensional polyhedron) as one that is locally conical [3, p. 5]. Given a p.l. R^∞-chart (U, ϕ) for M and a compact polyhedron $C \subset \phi(U)$, $f\phi^{-1}(c) \subset R^n$ some n. Thus, it is clear that if each $f_i\phi^{-1}$ is locally conical so is $f\phi^{-1}|C$. Thus, $f\phi^{-1}$ is R^∞-p.l. Conversely, if f is R^∞-p.l. then $f_i = \pi_i f$ where $\pi_i: R^\infty \to R$ is the projection onto the ith-coordinate. Since π_i is p.l., f_i is p.l. by (a).

Lemma 3. *If* $\{(U_\alpha, \phi_\alpha) | \alpha \in \mathcal{A}\}$ *is a* p.l. R^∞*-atlas for the* p.l. R^∞*-manifold* M, *then there is a* p.l. R^∞*-atlas* $\{(U_\alpha, \psi_\alpha) | \alpha \in \mathcal{A}\}$ *for* M *such that* $\psi_\alpha(x) \in (-1,1)^\infty = \varinjlim(-1,1)^n$, *all* $\alpha \in \mathcal{A}$, $x \in U_\alpha$.

Proof. Let $\beta: R \to (-1,1)$ be a p.l. homeomorphism taking 0 to 0. Then $\beta': R^\infty \to (-1,1)^\infty$, defined by $\beta'(x_1, x_2, x_3, \cdots) = (\beta(x_1), \beta(x_2), \beta(x_3), \cdots)$, is an R^∞-p.l. isomorphism. Letting $\psi_\alpha = \beta'\phi_\alpha$ gives the desired atlas.

Lemma 4. *Let* (U, ϕ) *be a* p.l. R^∞*-chart for the* p.l. R^∞*-manifold* M *such that* $\phi(U) \subset (-1,1)^\infty$. *Let* A *be a closed subset of* U, V *an open subset of* U *such that* $A \subset V \subset \overline{V} \subset U$, $\overline{V}$ *the closure of* V *in* M. *Then there is a* p.l. *map* $\lambda: M \to I$ *and an* R^∞-p.l. *map* $\psi: M \to (-1,1)^\infty \subset R^\infty$ *such that* $\lambda|A = 1$, $\lambda|M\backslash V = 0$, $\psi|\lambda^{-1}(1) = \phi|\lambda^{-1}(1)$, *and* $\psi|(U\backslash V) = \mathbf{0}$.

Proof. By Lemma 1 there is a p.l. map $\lambda': \phi(U) \to I$ such that $\lambda'|\phi(A) = 1$, $\lambda'|\phi(U\backslash\overline{V}) = 0$. Define p.l. $\lambda: M \to I$

by $\lambda|U = \lambda'\phi$ and $\lambda|(M\setminus\overline{V}) = 0$. Write $\phi(x) = (\phi_1(x), \phi_2(x), \cdots)$. Define $\psi = (\psi_1, \psi_2, \cdots)$ by

$$\psi_i(x) = \begin{cases} \max\{-\lambda(x), \min\{\lambda(x), \phi_i(x)\}\}, & x \in U \\ 0, & x \in M\setminus\overline{V}. \end{cases}$$

That ψ is R^∞-p.l. follows from Lemma 2(b). The other desired properties are clear.

Finally, we will use the following.

Proposition 5. *Let* M *and* N *be* p.l. R^∞*-manifolds. Let* $f: M \to N$ *be an* R^∞-p.l. *map which is also a homeomorphism. Then* f *is an* R^∞-p.l. *isomorphism.* I.e. f^{-1} *is also* R^∞-p.l.

Proof. Let (U,ϕ) be a p.l. R^∞-chart at $x \in M$, (V,ψ) a p.l. R^∞-chart at $f(x)$ in N such that $f^{-1}(V) \subset U$. It suffices to show that $\phi f^{-1}\psi^{-1}: \psi(V) \to \phi(f^{-1}(V))$ is R^∞-p.l. Let $C \subset \psi(V)$ be a compact polyhedron. Then $\phi f^{-1}\psi^{-1}(C)$ is compact. Hence, we may choose n and then a compact polyhedron P such that $\phi f^{-1}\psi^{-1}(C) \subset P \subset \phi(f^{-1}(V)) \cap R^n$. On P, $\psi f\phi^{-1}$ is a p.l. homeomorphism, so $Q = \psi f\phi^{-1}(P)$ is a compact polyhedron [3, p. 13] and $\phi f^{-1}\psi^{-1}|Q$ is p.l. [3, p. 6]. Since C is a subpolyhedron of Q, $\phi f^{-1}\psi^{-1}|C$ is also p.l., as required.

In relation to the above proposition we remark that if $f: M \to N$ is an R^∞-p.l. map such that $f(M) \subset Q$ where Q is a p.l. R^∞-submanifold of N then $f: M \to Q$ is also (clearly) R^∞-p.l. Thus, if f is also a topological embedding onto Q, then $f: M \to Q$ is a p.l. R^∞-isomorphism.

III. Proof of the Theorem

Let M be as in the theorem. Let $\rho: R^\infty \to (R^\infty)^\infty = \varinjlim (R^\infty)^n$ be the map obtained by Cantor diagonalization.

That is, $\rho((x_1,x_2,x_3,\cdots)) = ((x_1,x_3,x_6,x_{10},\cdots),(x_2,x_5,x_9,\cdots),(x_4,x_8,\cdots),\cdots)$. Then ρ is a linear homeomorphism [1, Corollary III-3]. Identify R^∞ with $(R^\infty)^\infty$ as p.l. R^∞-manifolds via ρ. It suffices, then, to show that there is an R^∞-p.l. isomorphism $f: M \to (R^\infty)^\infty$ onto a closed R^∞-submanifold of $(R^\infty)^\infty$. Note that, by Lemma 2(b), with our identification a map $f: M \to (R^\infty)^\infty$ is R^∞-p.l. if and only if each of its projections to R is p.l. and, hence, if and only if each of its projections to R^∞ is R^∞-p.l.

Let $m \in M$. By Lemma 3 there is a p.l. R^∞-chart (U_m,ϕ_m) with $m \in U_m$ and $\phi_m(U_m) \subset (-1,1)^\infty$. If we choose an open set G such that $\phi(m) \in G \subset \overline{G} \subset \phi(U_m)$, $\overline{G}$ the closure of G in R^∞, then for any U with $\overline{U} \subset \phi^{-1}(G)$ we have $\phi(\overline{U}) = \overline{\phi(U)}$. Thus, there is a p.l. R^∞-atlas $\{(U_\alpha,\phi_\alpha)\}$ for M such that each $\phi_\alpha(U_\alpha) \subset (-1,1)^\infty$ and each ϕ_α extends to a closed embedding $\phi_\alpha: \overline{U}_\alpha \to \overline{\phi_\alpha(U_\alpha)}$ into R^∞. Since M is paracompact and Lindelöf we thus obtain a countable, locally finite, p.l. R^∞-atlas $\{(U_i,\phi_i)\}$ for M such that, for each i, $\phi_i(U_i) \subset (-1,1)^\infty$ and ϕ_i extends to a closed embedding into R^∞.

Let $\{W_i\}$, $\{V_i\}$ be precise open refinements of U_i such that

$$\phi \neq W_i \subset \overline{W}_i \subset V_i \subset \overline{V}_i \subset U_i$$

(the closures in M). By Lemma 4 there is, for each i, a p.l. map $\lambda_i: M \to I$ and an R^∞-p.l. map $\psi_i: M \to R^\infty$ such that $\lambda_i|\overline{W}_i = 1$, $\lambda_i|(M - V_i) = 0$, $\psi_i|\lambda_i^{-1}(1) = \phi_i$ and $\lambda_i|(M - V_i) = 0$.

Choose a nonzero point $e \in R^\infty$. Define $f: M \to (R^\infty)^\infty$ by

$$f(m) = (\textstyle\sum_{i=1}^\infty i\lambda_i(m)e,\psi_1(m),\lambda_1(m)e,\psi_2(m),\lambda_2(m)e,\psi_3(m),\lambda_3(m)e,\cdots).$$

We will show that f is the desired R^∞-p.l. isomorphism. The local finiteness of $\{U_i\}$ guarantees that the sum is finite and that f is well defined, i.e. that $f(m) \in (R^\infty)^\infty$. Let f_i be the projection of f onto the i-th copy of R^∞. Since each f_i is R^∞-p.l., f is an R^∞-p.l. map. Let x and y be distinct elements of M. Choose i such that $x \in W_i$. If $\lambda_i(x) \neq \lambda_i(y)$, then clearly $f(x) \neq f(y)$. Otherwise $\lambda_i(y) = \lambda_i(x) = 1$ which implies $\psi_i(x) = \phi_i(x) \neq \phi_i(y) = \psi_i(y)$. Thus, f is one-to-one.

To see that f is a closed topological embedding let $f(m_\alpha) \to y = (y_1, y_2, \cdots) \in (R^\infty)^\infty$, $\{m_\alpha | \alpha \in \mathcal{A}\}$ a net in A, a closed subset of M. Then $f_1(m_\alpha) \to y_1$ implies that for some n and some $\beta \in \mathcal{A}$, $\sum_{i=1}^\infty i\lambda_i(m) \leq n$, $\alpha > \beta$. Since $M \subset \cup_{i=1}^\infty \lambda_i^{-1}(1)$, it follows that $\{m_\alpha | \alpha > \beta\} \subset \lambda_1^{-1}(1) \cup \cdots \cup \lambda_n^{-1}(1)$. Thus, for some cofinal $\mathcal{D} \subset \mathcal{A}$ and some k, $\{m_\alpha | \alpha \in \mathcal{D}\} \subset \lambda_k^{-1}(1)$. But on $\lambda_k^{-1}(1)$, $f_{2k} = \phi_k$ is a closed embedding into R^∞. Thus, since $f_{2k}(m_\alpha) = \phi_k(m_\alpha) \to y_{2k}$ we have that $y_{2k} = \phi_k(m)$ some $m \in A$ and that $\{m_\alpha | \alpha \in \mathcal{D}\} \to m$. Thus, $\{f(m_\alpha) | \alpha \in \mathcal{D}\} \to f(m)$ so that $y = f(m) \in f(A)$. We have shown that f(A) is closed, and it follows that f is a closed topological embedding.

Let $N = f(M)$. To see that N is a p.l. R^∞-submanifold of $(R^\infty)^\infty$ let $m_0 \in M$. Find j such that $m_0 \in W_j$ and then a neighborhood $\mathcal{O}$ of m_0 such that $\mathcal{O} \subset W_j$. Then on $\mathcal{O}$, $f_{2j} = \phi_j$. Let $Z = (R^\infty \times R^\infty \times \cdots \times R^\infty \times \phi_j(\mathcal{O}) \times R^\infty \times R^\infty \times \cdots) \cap (R^\infty)^\infty$, where $\phi_j(\mathcal{O})$ occurs in the 2j factor. Define $\gamma: Z \to Z$ by $\gamma((x_i)) = (y_i)$ where $y_i = x_i - f_i\phi_j^{-1}(x_{2j})$, $i \neq 2j$, and $y_{2j} = x_{2j}$. Then γ is an R^∞-p.l. isomorphism, and $\gamma(Z \cap f(M)) = \mathbf{0} \times \mathbf{0} \times \cdots \times \mathbf{0} \times \phi_j(\mathcal{O}) \times \mathbf{0} \times \mathbf{0} \cdots$. Define $\delta: Z \to \phi_j(\mathcal{O}) \times R^\infty$ by $\delta((x_i)) = (x_{2j}, \rho^{-1}(x_1, \cdots, x_{2j-1}, x_{2j+1}, x_{2j+2}, \cdots))$

where $\rho: R^\infty \to (R^\infty)^\infty$ is the homeomorphism given at the beginning of this proof. Then δ is a p.l. R^∞-isomorphism. Thus, $(Z,\delta\gamma)$ is a p.l. R^∞-chart for $(R^\infty)^\infty$ with $f(m_0) \in Z$, $\gamma\delta(Z) = \phi_j(\mathcal{O}) \times R^\infty$, and $\delta\gamma(Z \cap f(M)) = \phi_j(\mathcal{O}) \times \{\mathbf{0}\}$. Thus, $N = f(M)$ is an R^∞-p.l. submanifold of $(R^\infty)^\infty$.

We thus have an R^∞-p.l. map $f: M \to (R^\infty)^\infty$ which is a topological embedding onto a closed p.l. R^∞-submanifold of $(R^\infty)^\infty$. While it is relatively easy to see directly that f^{-1} is R^∞-p.l., for this we refer, instead, to the remark following Proposition 5. Thus, f is the desired R^∞-p.l. isomorphism.

IV. R^∞-Polyhedra

In this section we show that any p.l. R^∞-submanifold of R^∞ is an R^∞-polyhedron (see definition in §I) and relate the two definitions of R^∞-p.l. maps on such spaces.

Lemma 6. *Let* M *be a* p.l. R^∞*-manifold. Let* $C \subset M$ *be compact. Than any* p.l. R^∞*-atlas for* M *contains finitely many* p.l. R^∞*-charts* $\{(U_i,\phi_i) \mid i = 1,\cdots,n\}$ *such that there are cubes* $D_1,\cdots,D_n$ *in* $R^{k_1},\cdots,R^{k_n}$*, respectively, with* $D_i \subset \phi_i(U_i)$ *and* $C \subset \cup_{i=1}^n \phi_i^{-1}(\mathrm{Int}_{R^{k_i}}(D_i))$.

Proof. Let $c \in C$. Let (U,ϕ) be a p.l. R^∞-chart from the given atlas. Choose an open set V such that $c \in V \subset \overline{V} \subset U$. Then compact $\phi(C \cap \overline{V}) \subset R^n$ some n. Let $y = (y_i) = \phi(c)$ and choose $D' = (\Pi_{i=1}^\infty [y_i-\varepsilon_i, y_i+\varepsilon_i]) \cap R^\infty$ such that $D' \subset \phi(U)$. Then, if $D = \Pi_{i=1}^n [y_i-\varepsilon_i, y_i+\varepsilon_i]$, $\phi^{-1}(\mathrm{Int}_{R^n} D) \supset \phi^{-1}((\Pi_{i=1}^n (y_i-\varepsilon_i, y_i+\varepsilon_i)) \cap \phi(C \cap V))$, a neighborhood of c in $C \cap V$ and, hence, also in C. The lemma now follows from the compactness of C.

Proposition 7. *If* M *is a closed* p.l. R^∞*-submanifold of* R^∞*, then* M *is an* R^∞*-polyhedron.*

Proof. Let $C \subset R^\infty$ be a compact polyhedron. From the definition of submanifold for each $x \in M$ there is a p.l. R^∞-chart (U,ϕ) with $x \in U$ and such that $\phi(U) = U_1 \times U_2$, $\phi(U \cap M) = U_1 \times \{\mathbf{0}\} \equiv U_1$. The corresponding charts $(U',\phi') = (U \cap M, \phi|(U \cap M))$ form a p.l. R^∞-atlas for M. By the preceding lemma there are finitely many such charts $\{(U_i',\phi_i') | i = 1,\cdots,n\}$ and compact cubes $D_i \subset \phi_i'(U_i')$ such that $C \cap M \subset \cup_{i=1}^n (\phi_i')^{-1}(D_i)$. Thus, $C \cap M = C \cap \cup_{i=1}^n (\phi_i')^{-1}(D_i)$. Since finite unions and intersections of compact polyhedra are again polyhedra it suffices to show that each $(\phi_i')^{-1}(D_i)$ is a polyhedron in R^∞. But $(\phi_i')^{-1}(D_i) = \phi_i^{-1}(D_i \times \mathbf{0})$, a compact polyhedron since $\phi_i^{-1}: \phi_i(U) \to U$ is an R^∞-p.l. isomorphism.

Proposition 8. *If* M *and* N *are closed* p.l. R^∞*-submanifolds of* R^∞ *then* $f: M \to N$ *is* R^∞-p.l. *in the manifold sense if and only if* f *is* R^∞-p.l. *in the polyhedral sense.*

Proof. First note that if (U,ϕ) is any p.l. R^∞-chart in R^∞, and if $Q \subset U$ is a compact polyhedron then $\phi|Q = (id_{R^\infty})^{-1}\phi|Q$ is p.l. and, hence, $\phi(Q)$ is a compact polyhedron.

Now let f be R^∞-p.l. in the manifold sense. Let $C \subset M$ be a compact polyhedron, $x \in C$. Let (U,ϕ), (V,ψ) be charts for M and N, respectively, with $x \in U$, $f(x) \in V$ and such that $(U',\phi') = (U \cap M, \phi|(U \cap M))$ and $(V',\psi') = (V \cap N, \psi|(V \cap N))$ are p.l. R^∞-charts for M and N. Choose a compact polyhedral neighborhood P of x in $C \cap U' \cap f^{-1}(V')$. Then

$\phi'(P) = \phi(P)$ is a compact polyhedron, and, since f is R^∞-p.l. in manifold sense, $\psi' f(\phi')^{-1}|\phi(P)$ is p.l. Thus, $f|P = \psi^{-1}(\psi' f(\phi')^{-1})\phi|P$ is p.l. Thus, $f|C$ is locally p.l. and, hence, p.l. as required.

Conversely, if $f: M \to N$ is R^∞-p.l. in the polyhedral sense and $x \in M$, let (U,ϕ), (V,ψ), (U',ϕ'), (V',ψ') be as above. If $P \subset \phi'(U' \cap f^{-1}(V'))$ is a compact polyhedron then $(\phi')^{-1}(P)$ is a compact polyhedron in M. Thus $f|(\phi')^{-1}(P)$ is p.l. Hence $\psi' f(\phi')^{-1}|P$ is p.l. as required.

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Vanderbilt University
Nashville, Tennessee 37235