TOPOLOGY PROCEEDINGS
Volume 6, 1981
Pages 345–349
http://topology.auburn.edu/tp/

δ-COMPLETENESS AND δ-NORMALITY

by

A. Garcia-Máynez

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

δ-COMPLETENESS AND δ-NORMALITY

A. Garcia-Máynez

1. Introduction

In this paper we introduce the notion of δ-normal cover. We prove that the collection of δ-normal covers of a (Tychonoff) space X is a compatible uniformity $\mathcal{U}_\delta$ of X and the completion δX of the uniform space $(X, \mathcal{U}_\delta)$ lies between the topological completion μX and the realcompactification νX of X. X is *δ-complete* if the uniformity $\mathcal{U}_\delta$ is complete. Any product of δ-complete spaces and any closed subspace of a δ-complete space happen to be δ-complete. We prove that δX may be seen also as the intersection of all paracompact open subspaces of βX containing X. We finally prove that a space X is δ-complete if and only if it is homeomorphic to a closed subset of a product of locally compact metric spaces.

2. All Spaces Considered in This Note Will Be Completely Regular and Hausdorff ($T_{3\frac{1}{2}}$)

A collection $\mathcal{A}$ of subsets of X is a *cozero family* if each element $A \in \mathcal{A}$ is a cozero set in X. $\mathcal{A}$ is *strongly cozero* if for each $\mathcal{A}' \subset \mathcal{A}$, the set $\cup\{L \mid L \in \mathcal{A}'\}$ is a cozero set. $\mathcal{A}$ is *star-countable* if for each $A \in \mathcal{A}$, $\lambda_A = \{L \in \mathcal{A} \mid L \cap A \neq \Phi\}$ is countable. An open cover $\mathcal{A}$ of X is said to be *δ-normal* if it has a star-countable cozero refinement. Recall a cover $\mathcal{A}_1$ *Δ-refines* a cover $\mathcal{A}_2$ if $\{St(x, \mathcal{A}_1) \mid x \in X\}$ refines $\mathcal{A}_2$. We express this fact symbolically as $\mathcal{A}_1^\Delta < \mathcal{A}_2$. An open cover $\mathcal{A}$ of X is *normal* if there exist open covers $\mathcal{A}_1, \mathcal{A}_2, \cdots$ of X such that $\mathcal{A}_1^\Delta < \mathcal{A}$ and $\mathcal{A}_{m+1}^\Delta < \mathcal{A}_m$ for each $m = 1,2,\cdots$. A non-empty collection $\mathcal{U}$ of covers of a set X is a *uniformity on* X if for each pair $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{U}$ there exists $\mathcal{A}_3 \in \mathcal{U}$ such that $\mathcal{A}_3^\Delta < \mathcal{A}_1$ and $\mathcal{A}_3^\Delta < \mathcal{A}_2$. Any uniformity $\mathcal{U}$ on a set X induces a topology $\tau_{\mathcal{U}}$ on X, namely, $V \in \tau_{\mathcal{U}}$ iff for each $x \in V$ there exists $\mathcal{A}_x \in \mathcal{U}$ such that $St(x,\mathcal{A}_x) \subset V$. A uniformity $\mathcal{U}$ on a topological space (X,τ) is *compatible* if $\tau = \tau_{\mathcal{U}}$.

βX denotes, as usual, the Stone-Čech compactification of X and we consider X as an actual subspace of βX. For each $A \subset X$, we define $A_* = \beta X - cl_{\beta X}(X - A)$. Observe A_* is open in βX, $A_* \cap X = int_X A$ and A_* contains every open subset of βX whose intersection with X is $int_X A$. For each family $\mathcal{A} \subset \mathcal{P}(X)$, we write $L(\mathcal{A}) = \cup\{A_* | A \in \mathcal{A}\}$.

We may now prove our first result:

2.1. *Let* $\mathcal{A}$ *be an open cover of* X. *Then* $\mathcal{A}$ *is* δ*-normal iff there exists a paracompact open subspace* L *of* βX *such that* $X \subset L \subset L(\mathcal{A})$.

Proof (Necessity). We may assume, with no loss of generality, that $\mathcal{A}$ is cozero and star-countable. For each $A \in \mathcal{A}$, let A' be a cozero set in βX such that $A' \cap X = A$ and let $L = \cup\{A' | A \in \mathcal{A}\}$. Clearly $X \subset L \subset L(\mathcal{A})$ and $\{A' | A \in \mathcal{A}\}$ is a star-countable cozero cover of L (if $A_1' \cap A_2' \neq \Phi$, then $A_1' \cap A_2' \cap X \neq \Phi$ and hence $A_1 \cap A_2 \neq \Phi$). Using the star-countable property, we may index $\mathcal{A}$ as follows:

$$\mathcal{A} = \{A_{jm} | j \in J,\ m \in \mathbf{N}\},$$

where $A_{jm} \cap A_{kn} = \Phi$ whenever $j,k \in J$, $j \neq k$ and $m,n \in \mathbf{N}$.

Hence, L is the free union of the locally compact Lindelöf spaces $\{L_j | j \in J\}$, where $L_j = \cup\{A'_{jm} | m \in N\}$. Therefore, L is paracompact and open in βX.

(Sufficiency). Being paracompact and locally compact, L may be expressed in the form $L = \cup\{L_j | j \in J\}$, where the L_j's are Lindelöf, open in L and mutually disjoint. Consequently, the cover $\{A_* \cap L | A \in \mathcal{A}\}$ of L has a cozero and star-countable refinement $\mathcal{A}_z$. The restriction of $\mathcal{A}_z$ to X is then a star-countable cozero refinement of $\mathcal{A}$.

Using the fact that every open cover of a paracompact locally compact space (or, more generally, of a strongly paracompact space) has a star-countable strongly cozero Δ-refinement, we obtain:

2.1.1. *Corollary. Every δ-normal cover of a space* X *has a star-countable, strongly cozero, Δ-refinement. Hence, every δ-normal cover is normal and the collection $\mathcal{U}_\delta$ of all δ-normal covers of* X *is a compatible uniformity on* X.

The following result will be needed later. We leave the details of the proof to the reader:

2.2. *Let $\mathcal{U}$ be a compatible uniformity on the space* X *that every finite cozero cover of* X *belongs to $\mathcal{U}$. Let* $L = \cap\{L(\mathcal{A}) | \mathcal{A} \in \mathcal{U}\}$ *and let $\mathcal{U}_L$ be the collection of covers* $\{\{A_* \cap L | A \in \mathcal{A}\} | \mathcal{A} \in \mathcal{U}\}$. *Then $\mathcal{U}_L$ is a compatible complete uniformity on* L *which extends $\mathcal{U}$.*

A space X is *δ-complete* if the uniformity $\mathcal{U}_\delta$ is complete. Observe any strongly paracompact (in particular, any paracompact locally compact) space is δ-complete. Combining 2.1 and 2.2, we obtain:

2.3. *The completion* δX *of* $(X, \mathcal{U}_\delta)$ *may be viewed as a subspace of* βX *containing* X, *namely, as the intersection of all paracompact open subspaces of* βX *containing* X. *Hence,* X *is* δ*-complete iff there exist paracompact and open subspaces* $\{L_j | j \in J\}$ *of* βX *such that* $X = \cap\{L_j | j \in J\}$.

We obtain another characterization of δ-complete spaces:

2.4. *A space* X *is* δ*-complete iff it is homeomorphic to a closed subset of a product of locally compact metric spaces. Hence, every closed subset of a product of* δ*-complete spaces is* δ*-complete.*

Proof. The class A of paracompact locally compact spaces is a subcategory of $T_{3\frac{1}{2}}$ which is inversely preserved by perfect maps. Hence, using 2.3 and the theorem in [Fr], we deduce that X is δ-complete iff it is homeomorphic to a closed subset of a product of members of A. The proof is completed observing that each $Y \in A$ is homeomorphic to a closed subset of a product of a compact T_2 space and a locally compact metric space.

Since every countable cozero cover is δ-normal and every δ-normal cover is normal, we obtain, with the help of 2.2, a final remark:

2.5. *For each space* X, $\mu X \subset \delta X \subset \nu X$. *Hence, every* δ*-complete space is topologically complete and every* *realcompact space is δ-complete.*

Bibliography

[Fr] S. Franklin, *On epi-reflective hulls*, Gen. Top. Appl. 1 (1971), 29-31.

Instituto de Matemáticas de la Unam