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PROBLEM SECTION

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PROBLEM SECTION

CONTRIBUTED PROBLEMS

The contributor's name is in parentheses immediately following the statement of the problem. In most cases, there is an article by the contributor in this volume containing material related to the problem. Other names appearing later in parentheses are of researchers who can give additional information about the facts and problems mentioned.

A. Cardinal Invariants

11. (*Grabner*) Suppose that X is a wrb space. Does $\chi(X) = t(X)$?

12. (*Nyikos*) Does there exist, for each cardinal κ, a first countable, locally compact, countably compact space of cardinality $\geq\kappa$? [Yes if $2^\kappa = \kappa^+$ and $\Box_\kappa$ for all singular cardinals κ of countable cofinality: Juhász, Nagy, Weiss, Period. Math. Hung. 10 (1979), 193-206.]

13. (*van Douwen*) Let $\exp_Y X$ stand for the least cardinal κ (if it exists) such that X can be embedded as a closed subspace in a product of κ copies of Y. Does there exist an N-compact space X such that $\exp_{\mathbf{N}} X \neq \exp_{\mathbf{R}} X$? [Such a space cannot be strongly zero-dimensional.]

14. (*van Douwen*) Is every compact Hausdorff space a continuous image of some zero-dimensional compact space of the same cardinality? of the same character? [The answer is known to be yes for weight.]

See also the problems under the next two headings.

B. Generalized Metric Spaces and Metrization

21. (*Stephenson*) Is every regular, feebly compact, symmetrizable space first countable (equivalently, developable)?

C. Compactness and Generalizations

37. (*Nyikos*) Is initial κ-compactness productive if and only if κ is a singular strong limit cardinal? [For "if" the answer is affirmative in ZFC (Saks and Stephenson, AMS Transactions 196 (1974), 177-189) and for "only if" there is an affirmative answer under GCH (van Douwen) and in numerous other models of set theory (van Douwen, Nyikos).]

38. (*van Douwen*) Is there a (preferably separable locally compact) first countable pseudocompact space that is $\aleph_1$-compact (i.e. has no uncountable closed discrete subset) but is not countably compact? [Yes if $\mathfrak{b} = \omega_1$ (Nyikos) or $\mathfrak{b} = \mathfrak{c}$ (van Douwen), where $\mathfrak{b}$ denotes the least cardinality of an unbounded family in ${}^{\omega}\omega$ with respect to "eventually greater than."]

39. (*van Douwen*) Let μ be the least cardinality of a compact space that is not sequentially compact. It is known that $2^{\mathfrak{t}} \leq \mu \leq 2^{\mathfrak{s}}$. What else can be said about μ? [Here $\mathfrak{t}$ denotes the least cardinality of a chain of subsets of ω (with respect to almost-containment: $A \subset^* B$ iff $A \setminus B$ is finite) such that no infinite subset of ω is almost contained in every one, while $\mathfrak{s}$ is the least cardinality of a splitting family $\mathcal{S}$ of subsets of ω: a family such that for each infinite $A \subset \omega$, there exists $S \in \mathcal{S}$ such that $A \cap S$ and $A \setminus S$ are both infinite.]

See also A12, A14, L4, R2, and R3.

D. Paracompactness and Generalizations

28. (*de Caux*) Is every Lindelöf space a D-sapce? [Compare D8, vol. 2.]

29. (*Grabner*) Suppose that X is a regular wrb-space. Are the following equivalent?

(1) X is paracompact.

(2) X is irreducible and $\aleph$-preparacompact.

(3) X is θ-refinable and $\aleph$-preparacompact.

L. Topological Algebra

3. (*McCoy*) Let X be a completely regular k-space. If C(X) with the compact-open topology is a k-space, must X be hemicompact? (This would imply that C(X) is completely metrizable.)

4. (*van Douwen*) Must every locally compact Hausdorff topological group contain a dyadic neighborhood of the identity? (This would imply the result, whose proof has never appeared in English, that every compact Hausdorff topological group is dyadic.)

5. (*van Douwen*) A quasi-group is a set G with three binary operations $\cdot$, / and \ such that a/b and b\a are the unique solutions to xb = a and bx = a for all a,b $\in$ G. A topological quasi-group is a quasi-group with a topology with respect to which these operations are jointly continuous. (a) Is there a (preferably compact) zero dim. topological quasi-group whose underlying set cannot be that of a topological group? The quasi-group of Cayley numbers of value 1 (=S^7) is a well-known connected example. (b) Is there a quasi-group

which is also a (preferably compact) space such that the $\cdot$ is jointly continuous $/$ and $\backslash$ are separately continuous but are not jointly continuous? (c) Is there a quasi-group which is a (preferably compact) space as in (b) but whose underlying set cannot be that of a semitopological group?

P. Products, Hyperspaces, Remainders, and Similar Constructions

14. (*R. Pol*) Let H be the hyperspace of the Hilbert cube. The set $\{X \in H: X \text{ is countable-dimensional}\}$ is PCA but not analytic; is it true that this set is not coanalytic?

15. (*Nyikos*) Is it consistent that $\beta\omega-\omega$ is the union of a chain of nowhere dense sets? [This cannot happen under MA since then $\beta\omega-\omega$ cannot be covered by c or fewer nowhere dense sets (Hechler).]

See also A13.

R. Dimensions Theory

2. (*R. Pol*) Let $\mathcal{D}$ be an upper semi-continuous decomposition of a compactum X into countable-dimensional compacta. Is it true that $\sup\{\text{ind } S: S \in \mathcal{D}\} < \omega_1$?

3. (*R. Pol*) Let $\alpha < \omega_1$. What is the ordinal number $\mu(\alpha) = \min\{\text{ind } X: X$ is a countable-dimensional compactum containing topologically all compacta S with $\text{ind } S \leq \alpha\}$?

See also A14.

S. Problems Closely Related to Set Theory

9. (*Telgarsky*) Let X belong to the σ-algebra generated by analytic subsets of an uncountable Polish space Y. Is the game G(X,Y) determined?

10. (*Telgarsky*) Let X be a Lusin set on the real line. Does Player II have a winning strategy in the game G(X,**R**)?

11. (*van Douwen*) For a space X let $\mathcal{K}(X)$ denote the poset (under inclusion) of compact subsets of X and let cof $(\mathcal{K}(X))$ denote the cofinality of $\mathcal{K}(X)$, i.e.

$$\min\{|\mathcal{L}|: \mathcal{L} \subset \mathcal{K}(X), \forall K \in \mathcal{K}(X), \exists L \in \mathcal{L}(K \subset L)\}$$

If X is separable metrizable, and analytic (or at least absolutely Borel) but not locally compact, is cof $\mathcal{K}(X) = d$? [Here d denotes the cofinality of $({}^{\omega}\omega, <^*)$ where $<^*$ means "eventually less than." The answer is known to be yes if X is absolutely $F_{\sigma\delta}$ (in particular, σ-compact or completely metrizable). Also, if X is analytic but not σ-compact, then cof $(\mathcal{K}(X)) \geq d$ since then $d = \min\{|\mathcal{L}|: \mathcal{L} \subset \mathcal{K}(X), \cup\mathcal{L} = X\}$.]

See also C39 and P15.

T. Algebraic and Geometric Topology

7. (*B. Clark*) Does longitudinal surgery on a knot k always yield a manifold of maximal Heegard genus among those that can be obtained by surgery on k?

U. Uniform Spaces

1. (*Levy*) Which star-like subsets of $\mathbf{R}^2$ are U-embedded?

V. Geometric Problems

1. (*Meyerson*) Can a square table be balanced on all hills (perhaps with negative heights) of compact convex support?

2. (*Meyerson*) Can a cyclic quadrilateral table be balanced on all non-negative hills with compact convex support?

3. (*Meyerson*) Does every planar simple closed curve contain the vertices of a square?

INFORMATION ON EARLIER PROBLEMS

Volume 1

Classic Problem III, Related problems: (a) Is a screenable normal space collectionwise normal? *Partial results:* (*M. E. Rudin*) If there is a screenable normal space that is not paracompact, there is one that is not collectionwise normal. (*F. Tall*): Every screenable normal space is collectionwise normal with respect to paracompact subsets.

(b) Is a screenable, collectionwise normal space paracompact? *Consistency result:* (*M. E. Rudin*) There is a counterexample under $\diamondsuit$.

Volume 2

C5 (*Comfort*) Is it a theorem of ZFC that $\omega_1^* \not\approx \omega_0^*$? [$\kappa$* denotes the Stone-Čech remainder of a discrete space of cardinality κ.] *Note.* Glazer's claim of an affirmative answer (AMS Abstracts 2, p. 180; Topology Proceedings 5, p. 263) remains unconfirmed.

Classic Problem VI, Related Problem A. Is there a hereditarily separable, countably compact, noncompact space? *Solution.* For Hausdorff spaces this is independent: there are regular (even hereditarily normal) examples under CH

(*Hajnal and Juhász*) and perfectly normal ones under ◊ (*Ostaszewski*) or CH + "there exists a Souslin tree" (*Dahroug*) but not even Hausdorff examples if PFA (*Hajnal*). On the other hand, there are easy T_1 examples in ZFC.

Classic Problem VII, Related problems. Is there a pseudonormal space which is not countably metacompact but is first countable, of cardinality $\aleph_1$, separable and locally compact? *Solution.* Yes (*Davies*) AMS Proceedings 77 (1979), 276-278. *Note.* Davies's example X is not separable but can be modified to a separable example by a standard trick, cf. the proof of the theorem on p. 76 of Topology Proceedings vol. 1, by de Caux.

Volume 3

C21 (*van Douwen*) Is it true that for all infinite cardinals κ we have: κ is singular iff initial κ-compactness is productive iff initial κ-compactness is finitely productive? *Solution.* Yes if GCH but no if MA + $c > \aleph_\omega$ (van Douwen, The product of two normal initially κ-compact spaces, to appear). Moreover, there is no known model in which initial κ-compactness is finitely productive for any cardinals other than singular strong limit cardinals. (Compare Problem C37.)

C22 (*van Douwen*) Is initial κ-compactness productive if κ is singular? *Solution.* See above.

Volume 4

C24 (*Saks, attributed to Comfort*) Does there exist a family of spaces $\{X_i : i \in I\}$ with $|I| = 2^c$, $\Pi_{i \in I} X_i$ is not countably compact, and $\Pi_{i \in J} X_i$ is countably compact, whenever $J \subset I$ and $|J| < 2^c$? *Consistency results.* Yes if $2^c = \aleph_2$: The product of $\aleph_1$ sequentially compact spaces is countably compact (Scarborough and Stone, AMS Transactions 124 (1966), 131-147) and if CH then there is a family of 2^c sequentially compact spaces whose product is not countably compact (Rajagopalan, Springer Lecture Notes #540 (1975), 501-577). The proofs and constructions generalize to models of MA + $2^c = c^+$.

Volume 5

C31 (*Watson*) Is there a pseudocompact, para-Lindelöf space which is not compact? *Solution.* No, Burke and Davis.

C36 (*van Douwen*) Does there exist in ZFC a separable, normal, countably compact, noncompact space? *Solution.* Yes, Franklin and Rajagopalan, AMS Transactions 155 (1971), 304-314. Their example is also locally compact and scattered, hence sequentially compact.

D24 (*Watson, attributed to Arhangel'skij and Tall*) Is every normal, locally compact, metacompact space paracompact? *Note.* Peg Daniels has shown that a counterexample cannot be boundedly metacompact; in other words, a counterexample would have an open cover such that for each refinement and each integer n there is a point which is in at least n members of the refinement.

D27 (*Wicke*) Is every collectionwise normal meta-Lindelöf space paracompact? *Consistency result:* (*M. E. Rudin*) No if $\diamond^{++}$.

N3 (*Steprans*) Is there a measure zero subset X of **R** such that every measure zero subset of **R** is contained in the union of countably many translates of X? *Solution.* No. Todorčević, Galvin, and Fremlin have independently given general theorems which imply that the answer is negative.

S7 (*Przymusinski*) Does there exist a σ-set of cardinality $\aleph_1$? (A σ-set is a separable metric space in which every F_σ is a G_δ.) *Solution.* This is ZFC-independent. As is well known, MA implies every subset of **R** of card $<c$ is a Q-set (every subset is a G_δ). On the other hand (Miller, Ann. Math. Log. 16 (1979), 233-267) it is also consistent that every separable, uncountable metric space contains subsets that are arbitrarily far up in the Borel hierarchy.

S8 (*Przymusinski*) A q-set is a metrizable space in which every subset is a G_δ. Is every q-set strongly zero-dimensional? linearly orderable? *Consistency result.* Yes to both if V = L, because then every q-set is σ-discrete (*Reed*). Every σ-discrete normal space X satisfies dim X = 0 by the countable sum theorem, and every strongly zero-dimensional metric space is linearly orderable (*Herrlich*).