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It is not difficult to find examples of self-homeomorphisms of a metric space which are not uniformly continuous but which have some uniformly continuous powers.

My purpose is to raise the question of what variety of powers of a non-uniformly continuous homeomorphism may be uniformly continuous. My particular interest is in self-homeomorphisms of the reals. The following theorem gives some information.

Theorem. Corresponding to any integer $n > 1$ *there exists a self-homeomorphism,* f*, of the reals such that* $f, f^2, f^3, \cdots, f^{n-1}$ *are not uniformly continuous but* f^n *is uniformly continuous.*

Clearly, for such an f it follows that f^{-1} is not uniformly continuous. Also, it is trivial that a homeomorphism and all of its positive powers may be uniformly continuous but the negative iterates are non-uniformly continuous, etc. It will be clear from the proof of the theorem that the same theorem holds for any Euclidean space.

The question posed here is to describe all subsets, z, of $\mathbb{Z}$ for which one may find a self-homeomorphism, f, of the reals which is not uniformly continuous but if $j \in z$ then f^j is uniformly continuous.

An answer to the question would be of interest in discrete dynamical systems.

Proof of the theorem. We will take the positive reals as our model and will give an example of an orientation preserving homeomorphism. It will be clear that this convenience is not vital.

Let $n > 1$ be specified. Let $x_1 = 1$. If the positive integer s is of the form

$$nk, nk-1, \cdots, nk-n+2 \quad (k = 1,2,3,\cdots)$$

then define $x_{s+1} - x_s = 1/s$ and define $x_{s+1} - x_s = 2$ for s of the form $nk-n+1$.

For example, for $n = 4$, the values of $x_{s+1} - x_s$ are

$$1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ 2,\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ 2,\ \frac{1}{10},\ \frac{1}{11},\ \frac{1}{12},\ 2,\cdots$$

Define $f(x_s) = x_{s+1}$, $f(0) = 0$. Define f to be linear on each interval $[x_s, x_{s+1}]$ and, also, on $[0, x_1]$.

The homeomorphisms $f, f^2, \cdots, f^{n-1}$ are not uniformly continuous because in each instance a null sequence of intervals maps into intervals of length 2. However, f^n is uniformly continuous since it is piecewise linear and the slope of each segment is less than or equal to 1.

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