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by

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0. Introduction

A Hausdorff space X is H-*closed* if X is closed in every Hausdorff space containing X as a subspace. Compact Hausdorff spaces are obviously H-closed and by the well-known theorem of Arhangel'skiǐ, each first countable compact Hausdorff space has cardinality at most c. The *character* of a space $X, \chi(X)$, is the minimum cardinal such that each point of X has a neighborhood base of that cardinality. R. Pol [Po] modified Arkhangelskii's method to show that $|X| \leq 2^{\chi(X)}$ for a compact Hausdorff space X. We generalize this result by showing that the same inequality holds for H-closed spaces. The case for first countable H-closed spaces has been shown by a different technique by A. A. Gryzlow [G].

In the first section we show that any infinite H-closed space X can be embedded as the outgrowth of an H-closed extension Y of a discrete space D such that $\chi(Y) \leq \chi(X)$, $|Y| = |X|$, and $Y \backslash D$ is homeomorphic to X. We also show that the existence of certain H-closed extensions of discrete spaces is equivalent to partitioning compact spaces into closed sets.

The second section contains our main results. That is, for an H-closed extension Y of a discrete space, $|Y| \leq 2^{\chi(Y)}$. An immediate corollary to this result and the results in Section 1 is that $|X| \leq 2^{\chi(X)}$ for H-closed X.

In the third section we give an example of a first countable H-closed space with cardinality less than c and yet uncountable. Of course these examples are consistent and contrast the compact case where for a first countable compact Hausdorff space X, either $|X| \leq \omega$ or $|X| = c$. Also in this section we present a technique for constructing first countable quasi-H-closed T_1 spaces with arbitrarily large cardinality and dense discrete subspaces of large cardinality. This technique may prove useful for constructing "real" examples of first countable H-closed spaces with cardinality $\aleph_1$. In the fourth section we conclude by using this technique to construct special first countable κ^+-Lindelöf spaces.

Our use of terminology and notation is standard; we refer the reader to Willard [Wi] for undefined terms. We will use Greek letters for infinite ordinals and cardinals will be identified with initial ordinals. The cardinality of a set X is denoted $|X|$, and the successor of a cardinal κ is κ^+. When we speak of κ as a topological space it will be understood that κ has the discrete topology. For a Tychonoff space X, βX denotes the Cech-Stone compactification. For a cardinal κ, $\beta\kappa$ is the set of ultrafilters on κ.

A well known equivalence to X being H-closed is that every open cover of X has a finite subcollection with dense union. A subset $U \subset X$ is *regular closed* if U = cl int U and U is *regular open* if U = int cl U. A regular closed subset of an H-closed space is itself H-closed. The *semi-regularization*, X_s, of a space X is the underlying set X with the topology generated by the regular open subsets of X. It is straightforward to show that $\chi(X_s) \leq \chi(X)$. A function $f: X \to Y$ is *θ-continuous* if for each $x \in X$ and open $V \subset Y$ with $f(x) \in V$ there is an open $U \subset X$ with $x \in U$ and $f(cl_X U) \subset cl_Y V$, f is *perfect* if $f^{\leftarrow}(y)$ is compact for each $y \in Y$ and f is closed, f is *irreducible* if $f(F) \neq Y$ for each proper closed subset F of X. For $A \subset X$, define $f^{\#}(A) = \{y \in Y: f^{\leftarrow}(y) \subseteq A\}$. Note that $f^{\#}(A) = Y\backslash f(X\backslash A)$; in particular, if f is a closed map and A is open, then $f^{\#}(A)$ is open. We will need the following fact in the sequel.

0.1. *Let* Y *be an* H-*closed extension of* X *and suppose* Z *is a space such that there is a continuous bijection* $f: Z \to Y\backslash X$. *Then there is an* H-*closed extension* T *of* X *such that* $T\backslash X = Z$ *and* $\chi(T) \leq \max\{\chi(Z), \chi(Y)\}$.

Proof. Let Y^+ be the simple extension corresponding to Y, i.e., the underlying set of Y^+ is Y with $U \subset Y^+$ defined to be open if $U \cap X$ is open in X and if $p \in U\backslash X$, then there is an open neighborhood V of p in Y such that $V \cap X = U \cap X$. Y^+ is an H-closed extension of X (cf. [PV]) and $Y^+ \geq Y$. Let Y^* have the same underlying set as Y with $W \subset Y^*$ defined to be open if W is open in Y^+ and $f^{\leftarrow}(W)$ is open in Z. Clearly, Y^* is an extension of X. If U is open in Y, then U is open in Y^*; so Y^* is Hausdorff. Since Y^* is the continuous Hausdorff image of the H-closed space Y^+, Y^* is H-closed. Also, $f: Z \to Y^*\backslash X$ is a continuous bijection. If W is open in Z, then $f(W) \cup X$ is open in Y^*. So, f is open and Z is homeomorphic to $Y^*\backslash X$. It is straightforward to show that $\chi(Y^*) \leq \max\{\chi(Z), \chi(Y)\}$.

1. Dense Discrete Suffices

The main purpose of this paper is to find a relationship between the character of an H-closed space and its cardinality. In this section we will show that it suffices to find such a relationship for spaces with a dense set of isolated points. Specifically we will show that for any infinite H-closed space there is an H-closed space with a dense set of isolated points with the same cardinality and character as the first. We will begin by investigating H-closed extensions of discrete spaces.

Let κ be an infinite cardinal and suppose that $\beta\kappa\backslash\kappa$ is partitioned into closed sets $\{F_i: i \in I\}$. We will let $Y = \kappa \cup \{F_i: i \in I\}$ be the Hausdorff space with the following topology. A set $U \subset Y$ will be open iff $F_i \in U$ implies $F_i \subset cl_{\beta\kappa}(U \cap \kappa)$ for all $i \in I$. The following theorem due to Porter and Votaw [PV] will be essential.

Theorem 1.1. *For* $i \in I$ *let* F_i *be a closed subset of* $\beta\kappa\backslash\kappa$. *The space* $Y = \kappa \cup \{F_i: i \in I\}$ *is* H-*closed iff* $\{F_i: i \in I\}$ *is a partition of* $\beta\kappa\backslash\kappa$.

Proof. We will omit the proof that Y is Hausdorff iff the F_i's are disjoint. Suppose that Y is H-closed and that $p \in \beta\kappa$. Clearly p defines an open filter on Y. If $p \notin F_i$, then there is a $U \in p$ such that $cl_{\beta\kappa}U \cap F_i = \emptyset$. Therefore $\{F_i\} \cup \kappa\backslash U$ is a Y-neighborhood of F_i which misses U. Since Y is H-closed either $p \in \kappa$ or there is an $i \in I$ with $p \in F_i$. The converse is straightforward as κ is dense in Y.

Lemma 1.2. *Let* $\{F_i: i \in I\}$ *be pairwise disjoint subsets of* $\beta\kappa\backslash\kappa$. *Suppose that* λ *is the minimum cardinal such* *that each* F_i *can be written as the intersection of* λ *clopen subsets of* $\beta\kappa$. *Then* $\chi(Y) = \lambda$ *where* $Y = \kappa \cup \{F_i\colon i \in I\}$.

Proof. Let $i \in I$ and choose $A_\alpha \subseteq \kappa$ for $\alpha < \lambda$ such that $F_i = \cap_{\alpha<\lambda} cl_{\beta\kappa} A_\alpha$. Let $\mathcal{U}$ be a filter base of cardinality λ generated by $\{A_\alpha\colon \alpha < \lambda\}$. The set $\{\{F_i\} \cup U\colon U \in \mathcal{U}\}$ is a neighborhood base for F_i in Y. Indeed, suppose $V \subseteq \kappa$ is such that $F_i \subseteq cl_{\beta\kappa} V$. Since $\cap_{\alpha<\lambda} cl_{\beta\kappa} A_\alpha \subseteq cl_{\beta\kappa} V$ and $\beta\kappa \backslash cl_{\beta\kappa} V$ is compact, there is a $U \in \mathcal{U}$ with $U \subseteq V$. Therefore $\chi(Y) \leq \lambda$. It is clear that if $\mathcal{U}$ is a neighborhood base for F_i then $F_i = \cap\{cl_{\beta\kappa}(U \cap \kappa)\colon U \in \mathcal{U}\}$; hence, $\chi(Y) = \lambda$.

Recall that for a Hausdorff space X, RO(X) is the complete boolean algegra of regular open subsets of X. The absolute, EX, of X is the subspace of the Stone space of RO(X) consisting of maximal filters of RO(X) with adherent points in X. The canonical map $\pi\colon EX \to X$ is defined by $\pi(\mathcal{U}) = x$ where x is the adherent point of $\mathcal{U}$ and is onto, perfect, irreducible, θ-continuous. Now $\{cl_{EX}\pi^{\leftarrow}(U)\colon U$ open in X$\}$ is a clopen basis for EX and $\pi(cl_{EX}\pi^{\leftarrow}(U)) = cl\, U$ for an open $U \subseteq X$. If X is H-closed then EX is compact and EX is always extremally disconnected. The reader is referred to the excellent survey by Woods [Wo] for more details.

For a space X and a cardinal λ, $F \subseteq X$ is a G_λ-*subset* if F equals the intersection of λ open subsets of X. Note that if X is compact zero dimensional and F is closed, then this is equivalent to the intersection of λ clopen subsets of X.

Lemma 1.3. *Let* X *be an* H-*closed space and let* $\lambda = \chi(X)$.

The absolute EX *is partitioned by* $\{\pi^{\leftarrow}(x)\colon x \in X\}$ *which are all closed* G_λ*-sets in* EX.

Proof. Let $x \in X$ and let $\mathcal{U}$ be an open neighborhood base for x with $|\mathcal{U}| \leq \lambda$. We claim that $\pi^{\leftarrow}(x) = \bigcap\{cl_{EX}\pi^{\leftarrow}(U)\colon U \in \mathcal{U}\}$. Since $cl_{EX}\pi^{\leftarrow}(U)$ is clopen for each $x \in X$, this will prove the lemma. Suppose $y \notin \pi^{\leftarrow}(x)$. Then $\pi(y) = z \neq x$. Since X is Hausdorff, there is a $U \in \mathcal{U}$ such that $z \notin cl_X U$. Therefore $y \in cl_{EX}\pi^{\leftarrow}(X\backslash cl_X U)$ which is a clopen neighborhood of y which misses $cl_{EX}\pi^{\leftarrow}(U)$. Hence $y \notin cl_{EX}\pi^{\leftarrow}(U)$.

Our next result is the one which ties everything together. It is surprising in its simplicity.

Theorem 1.4. *Let* λ *and* κ *be infinite cardinals. If there is a compact Hausdorff space* X *which can be partitioned into* κ *nonempty closed* G_λ*-subsets, then* $\beta\kappa\backslash\kappa$ *can be partitioned into* κ *nonempty closed* G_λ*-subsets of* $\beta\kappa$.

Proof. Let $\{A_\alpha = \alpha < \kappa\}$ be a partition of X where A_α is a nonempty closed G_λ-subset of X. Since $\kappa\cdot\omega = \kappa$, there is a function $f\colon \kappa \to X$ such that $|f^{\leftarrow}(A_\alpha)| = \aleph_0$. Extend f to a continuous function $\beta f\colon \beta\kappa \to cl_X f(\kappa) = Y$. For each $\alpha < \kappa$, let $Z_\alpha = (\beta f)^{\leftarrow}(A_\alpha)\backslash f^{\leftarrow}(A_\alpha)$; clearly, $Z_\alpha \subseteq \beta\kappa\backslash\kappa$. Also, Z_α is a G_λ-subset of $\beta\kappa$ since βf is continuous, A_α is a G_λ-subset of X, and $f^{\leftarrow}(A_\alpha)$ is countable. Now, $\phi \neq cl_{\beta\kappa}f^{\leftarrow}(A_\alpha)\backslash\kappa \subseteq Z_\alpha$ and $\{Z_\alpha\colon \alpha < \kappa\}$ is a partition of $\beta\kappa\backslash\kappa$ with nonempty closed sets.

Theorem 1.5. *The following are equivalent:*

(a) *There is an* H*-closed space* X *with* $|X| = \kappa$ *and*

$\chi(X) \leq \lambda$.

(b) *There is an H-closed extension* Y *of* κ *with* $|Y| = \kappa$ *and* $\chi(Y) \leq \lambda$.

(c) *There is a compact Hausdorff space which can be partitioned into* κ *many closed* G_λ*-subsets.*

Proof. The proof of (a) implies (b) follows from 1.3, 1.2, and 1.4. By 1.3, (b) implies (c). By 1.3, 1.4, and 1.2, (c) implies (b). Clearly (b) implies (a).

For an H-closed space X, by Theorem 1.5, there is an H-closed extension Y of κ where $|X| = \kappa$ such that $|Y| = \kappa$ and $\chi(Y) \leq \chi(X)$. This extension is constructed as follows:

(1) There is a function $f\colon \kappa \to EX$ such that $|f^{\leftarrow}(\pi^{\leftarrow}(x))| = \aleph_0$ and $\beta f\colon \beta\kappa \to EX$ denotes the continuous extension of f.

(2) Let $g = \beta f|(\beta\kappa\backslash\kappa)$ and, for $x \in X$, $F_x = (\pi \circ g)^{\leftarrow}(x)$. Then $\{F_x\colon x \in X\}$ is a partition of $\beta\kappa\backslash\kappa$ with nonempty compact sets, and g is onto.

(3) Let $Y = \kappa \cup \{F_x\colon x \in X\}$ where $W \subseteq Y$ is defined to be open iff $F_x \in W$ implies $F_x \subseteq cl_{\beta\kappa}(W \cap \kappa)$. Now, Y is H-closed, has the simple extension topology, i.e., $Y = Y^+$, and $\chi(Y) \leq \chi(X)$.

(4) Modify the topology on Y to the strict extension topology (cf. [PV]), i.e., let $Y^{\#}$ be Y with the topology generated by $\{\hat{T}\colon T \subseteq \kappa\}$ where $\hat{T} = T \cup \{F_x\colon T \in \cap F_x\}$ (recall that $p \in F_x \subseteq \beta\kappa\backslash\kappa$ is an ultrafilter on κ implying that $\cap F_x$ is filter on κ). Now, $\{\hat{T}\colon T \subseteq \kappa\}$ is an open basis for $Y^{\#}$, and $Y^{\#}$ is an H-closed extension of κ. Also, for $T \subseteq \kappa$, it is easy to show that $\hat{T} = T \cup \{F_x\colon F_x \subseteq cl_{\beta\kappa}T\}$.

(5) Let $Z = Y^{\#}\backslash\kappa$, and define $\phi\colon Z \to X$ by $\phi(F_x) = x$.

Note that if $A \subseteq Z$, then $\phi(A) = \pi \circ g(\cup\{F_x: F_x \in A\})$. For $B \subseteq \beta\kappa\backslash\kappa$, recall that $(\pi \circ g)^{\#}(B) = \{x \in X: (\pi \circ g)^{\leftarrow}(x) \subseteq B\}$. Thus, for $T \subseteq \kappa$, it follows that $\phi(\hat{T}\backslash T) = \{x \in X: F_x \subseteq \mathrm{cl}_{\beta\kappa}T\}$ $= (\pi \circ g)^{\#}(\mathrm{cl}_{\beta\kappa}T\backslash\kappa)$.

Theorem 1.6. *$\phi: Z \to X$ is an open, θ-continuous bijection.*

Proof. Since ϕ is a bijection, it suffices to show ϕ is open and θ-continuous. Let $T \subseteq \kappa$. Since $\phi(\hat{T}\backslash T) =$ $(\pi \circ g)^{\#}(\mathrm{cl}_{\beta\kappa}T\backslash\kappa)$ and $\pi \circ g$ is closed, then ϕ is open as $\{\hat{T}\backslash T: T \subseteq \kappa\}$ is an open basis for Z. To show ϕ is θ-continuous, let U be an open neighborhood of $x \in X$. Then $\mathrm{cl}_{EX}\pi^{\leftarrow}(U)$ is clopen in EX and $\pi(\mathrm{cl}_{EX}\pi^{\leftarrow}(U)) = \mathrm{cl}\, U$. Then $W = g^{\leftarrow}(\mathrm{cl}_{EX}\pi^{\leftarrow}(U))$ is clopen in $\beta\kappa\backslash\kappa$ and $F_x \subseteq W$. So, for some $T \subseteq \kappa$, $F_x \subseteq \mathrm{cl}_{\beta\kappa}T\backslash T \subseteq W$. Now, $F_x \in \hat{T}\backslash T$ and $\phi(\hat{T}\backslash T) =$ $(\pi \circ g)\ (\mathrm{cl}_{\beta\kappa}T\backslash\kappa) \subseteq \pi \circ g(W) \subseteq \mathrm{cl}U$.

By Proposition 1 of Fedorcuk [F], Z_s and X_s are homeomorphic. Our next result extends part of 1.5 and is an interesting variation of the Alexandroff duplicate.

Corollary 1.7. *If* X *is an infinite* H-*closed space, then there is an* H-*closed extension* $h\kappa$ *of* κ *where* $\kappa = |X|$ *such that* $h\kappa\backslash\kappa$ *is homeomorphic to* X *and* $\chi(h\kappa) \leq \chi(X)$.

Proof. By 0.1 and 1.6, there is an H-closed extension $h\kappa$ of κ such that $h\kappa\backslash\kappa$ is homeomorphic to X and $Y^{+} \geq h\kappa \geq Y^{\#}$ where Y is described in (3). Also, by 0.1, $\chi(h\kappa) \leq \max\{\chi(Y^{\#}),$ $\chi(X)\}$. By (3), $\chi(Y^{+}) = \chi(Y) \leq \chi(X)$. It is easy to show that $\chi(Y^{+}) = \chi(Y^{\#})$ for any extension Y. Thus, $\chi(h\kappa) \leq \chi(X)$.

The result of 1.7 gives rise to the problem of identifying those Hausdorff spaces which are the remainders of H-closed extensions of discrete spaces. If X is a Hausdorff space with a coarser H-closed topology, then it follows by 0.1 and 1.7, that X is the remainder of an H-closed extension of some discrete space. J. Vermeer has communicated to the authors that the converse is true.

2. The Main Result

In [Po], Pol has given a simple proof that $|X| \leq 2^{\chi(X)}$ for a compact Hausdorff space X. In this section, we adapt Pol's proof to H-closed spaces with a dense set of isolated points. Then we use the results of the first section to extend to arbitrary H-closed spaces.

Theorem 2.1. *Let* X *be an* H-*closed extension of a discrete space* D. *Then* $|X| \leq 2^{\chi(X)}$.

Proof. Let $\kappa = \chi(X)$ and $\psi: \mathcal{P}(D) \to D$ be a choice function, i.e., for $E \in \mathcal{P}(D)\backslash\{\emptyset\}, \psi(E) \in E$. Recall that if $A \subseteq X$ with $|A| \leq 2^{\kappa}$ then $|cl_X A| \leq 2^{\kappa}$ since $\chi(X) \leq \kappa$ [J2]. We fix a neighborhood base $\{G(x,\alpha): \alpha < \kappa\}$ for each $x \in X$. For $F \subseteq X$ and $h: F \to \kappa$, let $G(F,h) = \cup_{x \in F} G(x,h(x))$.

We inductively define, for $\alpha < \kappa^+$, $A_\alpha \subseteq D$ with $|A_\alpha| \leq 2^{\kappa}$ as follows. Choose $d_0 \in D$ and let $A_0 = \{d_0\}$. Suppose for $\gamma < \alpha$ we have defined A_γ. Let $A_\alpha = \cup_{\gamma<\alpha} A_\gamma \cup \{\psi(D\backslash G(F,h)): F$ is a finite subset of $cl_X \cup_{\gamma<\alpha} A_\gamma$, $h: F \to \kappa$ and $D\backslash G(F,h) \neq \emptyset\}$. Since $|cl_X \cup_{\gamma<\alpha} A_\gamma| \leq 2^{\kappa}$, there are at most 2^{κ} pairs (F,h) where F is a finite subset of $cl_X \cup_{\gamma<\alpha} A_\gamma$ and $h: F \to \kappa$. Therefore $|A_\alpha| \leq 2^{\kappa}$. Let

$Y = cl_X \cup_{\alpha<\kappa^+} A_\alpha$ and observe that $Y = \cup_{\alpha<\kappa^+} cl_X A_\alpha$ since $\chi(X) \leq \kappa$ and $\{A_\alpha : \alpha < \kappa^+\}$ is a chain. Suppose that $X \backslash Y \neq \emptyset$. Since Y is closed and D is dense there is a $d \in D \backslash Y$. Define $h: Y \to \kappa$ such that $d \notin G(x,h(x))$ for $x \in Y$. Clearly $\{G(x,h(x)): x \in Y\}$ is a cover of Y and since Y is H-closed ($\cup A_\alpha$ is dense) there is a finite set $F \subseteq Y$ such that $G(F,h)$ is dense in Y. However there is an $\alpha < \kappa^+$ such that $F \subseteq cl_X A_\alpha$ which implies that $\psi(D \backslash G(F,h)) \in A_{\alpha+1}$. This contradicts that $G(F,h)$ is dense in Y. Therefore $Y = X$ and $|X| = |Y| \leq |cl_X \cup (A_\alpha : \alpha < \kappa^+)| \leq 2^\kappa$.

Definition 2.2 [G]. The H-pseudocharacter $\psi_H(X)$ of a space X is the smallest cardinal λ such that for each point $x \in X$ there is a system $\{U_\alpha : \alpha < \lambda\}$ of neighborhoods of x such that $\{x\} = \cap\{cl_X U_\alpha : \alpha < \lambda\}$.

If the H-pseudocharacter of an H-closed space X is λ then the semiregularization of X has character λ, in fact, $\psi_H(X) = \chi(X_s) \leq \chi(X)$.

Corollary 2.3. *Let* X *be an infinite* H-*closed space, then* $|X| \leq 2^{\psi_H(X)} \leq 2^{\chi(X)}$.

Proof. Let X be an H-closed space. The semiregularization of X, say Y, is H-closed and $\chi(Y) = \psi_H(X)$. By 1.5, there is an H-closed space Z which has a dense set of isolated points and $|Z| = |X|$, $\chi(Z) \leq \chi(Y)$. Therefore, by 2.1, $|X| = |Z| \leq 2^{\chi(Z)} \leq 2^{\psi_H(X)}$.

Since a zero set is a closed G_ω-set, by 1.5 and 2.1, we obtain the following corollary which is also immediate from a result by Arhangel'skii [A; Th. 4].

Corollary 2.4. *No compact Hausdorff space can be partitioned into more than* c *nonempty zero sets.*

Remark 2.5. Let X be a Hausdorff space. The *weak Lindelöf* number of X, denoted wL(X), is defined to be min{κ: each open cover of X has a subfamily of cardinality no greater than κ whose union is dense in X}. In [BGW], Bell, Ginsburg and Woods prove that for a normal space Z, $|Z| \leq 2^{\chi(Z)wL(Z)}$. They also give an example of a Hausdorff non-regular space Y such that $|Y| > 2^{\chi(Y)wL(Y)}$. They ask whether it is true that $|X| \leq 2^{\chi(X)wL(X)}$ when X is regular. Since a regular closed subset of X has the same weak Lindelöf number as X one can easily see how to generalize 2.1. That is, if X contains a dense set of isolated points then $|X| \leq 2^{\chi(X)wL(X)}$. Note that if X is H-closed wL(X) = ω.

3. First Countable

In this section we investigate in greater detail the possible cardinalities of first countable H-closed spaces. It is well known that if X is first countable and compact Hausdorff, then $|X|$ = c or $|X| \leq \aleph_0$. Since we now know that for H-closed spaces $|X| \leq 2^{\chi(X)}$, one would naturally wonder whether it is possible for a first countable H-closed space to have cardinality $\aleph_1$.

We have constructed in [DP] a first countable R-closed space of cardinality $\aleph_1$ suggesting that this may be possible for H-closed. We give a consistent example of a first countable H-closed space which is uncountable but of cardinality less than c. We also investigate a technique for constructing first countable H-closed spaces which to date has only given quasi-H-closed spaces.

By 1.5 (c) we know that there is a first countable H-closed space of size $\aleph_1$ if there is a compact space which can be partitioned into $\aleph_1$ zero sets. In the Cantor space, 2^ω, each closed set is a zero set. The following theorem was first proved by J. Baumgartner (unpublished) and rediscovered by A. W. Miller [Mi].

Theorem 3.1. *It is consistent that* $2^{\aleph_0} > \aleph_1$ *and that* 2^ω *is the* ω_1 *union of non-empty disjoint closed sets.*

To say that a statement P is consistent is to say that there is a model of ZFC in which P is true. The proof of Theorem 3.1 requires knowledge of forcing and is therefore omitted. Stern [St] and independently Kunen have shown that 3.1 holds in any random real extension of a model of CH.

In order to show absolutely that there is a first countable H-closed space of cardinality $\aleph_1$ it suffices to show that $\beta\omega_1\backslash\omega_1$ or $\beta\omega\backslash\omega$ can be partitioned into ω_1 zero sets of $\beta\omega_1$ or $\beta\omega$, respectively. We do not know if this can be done but we can show that each is the union of $\aleph_1$ zero sets of $\beta\omega_1$ or $\beta\omega$, respectively. Let θ_1 be the first countable weakly inaccessible cardinal and α_1 be the first measurable cardinal.

We show that for each $\alpha < \theta_1$ there is a T_1 quasi-H-closed first countable space of cardinality α and for each

$\alpha < \alpha_1$ there is a first countable quasi-H-closed T_1 space of cardinality 2^α. A space is *quasi*-H-*closed* if each open cover has a finite subcollection with dense union [PT].

3.2. To construct these spaces we introduce a method of constructing first countable spaces which seems particularly suited to quasi-H-closed spaces. Let N be the positive intergers and let I be any non-empty set. For any $D \subseteq N^I$, let, for each $i \in I$ and $n \in N$, $G(n,i) = \{d \in D: d(i) > n\} \cup \{i\}$. Let $X = D \cup I$ be given the following topology. Each point of D is isolated and a neighborhood base for $i \in I$ is $\{G(n,i) : n \in N\}$. With this topology X is a first countable T_1-space. If we wish X to have special properties we have to make careful selections of the set D. For the remainder of the paper when $X = D \cup I$ where $D \subset N^I$ we will assume that X is given the above topology.

Lemma 3.3. *Let* $D \subseteq N^I$ *and* $X = D \cup I$. *The space* X *is quasi*-H-*closed iff* D *is closed discrete in* N^I.

Proof. Suppose that X is quasi-H-closed. Let $h \in N^I$, we must find a finite $F \subset I$ such that $\{d \in D: d|_F = h|_F\}$ is finite. For a subset $F \subset I$ and $f,g \in N^I$, $f|_F \leq g|_F$ will mean $f(i) \leq g(i)$ for all $i \in F$. Since X is quasi-H-closed the open cover $D \cup \{G(h(i),i): i \in I\}$ has a finite subcollection with dense union. Therefore there is a finite $F \subset I$ such that $\cup_{i\in F}G(h(i),i)$ contains all but finitely many $d \in D$. By the definition of $G(h(i),i)$, $D\backslash\cup_{i\in F}G(h(i),i) = \{d \in D: d|_F \leq h|_F\}$. Therefore $\{d \in D: d|_F = h|_F\}$ is finite.

The converse is more difficult. Let $D \subset N^I$ be closed discrete. We will find, for $h \in N^I$, a finite set $F(h) \subseteq I$ such that $\{d \in D: d|_{F(h)} \leq h|_{F(h)}\}$ is finite. Let $h \in N^I$ and suppose that for each finite $F \subseteq I$, $D(F) = \{d \in D: d|_F \leq h|_F\}$ is infinite. There is then a free ultrafilter $p \in \beta D\backslash D$ such that $D(F) \in p$ for each finite $F \subseteq I$. Now, for each $i \in I$, let $g_i\colon \beta D \to \mathrm{cl}_{\mathbf{R}}\{\frac{1}{n}: n \in N\}$ be continuous and $g_i(d) = {}^1/d(i)$ for $d \in D$. Since $D(\{i\}) \in p$ and $d \in D(\{i\})$ implies $g_i(d) \geq {}^1/h(i)$, $g_i(p) \geq {}^1/h(i)$ for each $i \in I$. Therefore ${}^1/g_i(p) \in N$ for each $i \in I$. Let $f \in N^I$ be such that $f(i) = {}^1/g_i(p)$ for $i \in I$. We will show that f is in the closure of D. Let F be a finite subset of I. For each $i \in F$ we can choose $A_i \in p$ such that for $d \in A_i$ $g_i(d) = g_i(p)$. However this implies that for $d \in \cap_{i \in F} A_i \in p$, $f|_F = d|_F$. Since $\cap_{i\in F} A_i$ is infinite, we have contradicted that D is closed discrete in N^I. Note that this contradiction comes from the assumption that $g_i(p) \neq 0$ for each $i \in I$ only. (See Corollary 3.4). Therefore for each $h \in N^I$ there is a finite $F(h) \subseteq I$ such that $\{d \in D: d|_{F(h)} \leq h|_{F(h)}\}$ is finite. Now any open cover of X is refined by $D \cup \{G(h(i),i): i \in I\}$ for some $h \in N^I$. Therefore $\{d \in D: d|_{F(h)} \leq h|_{F(h)}\} \cup \cup_{i\in F} G(h(i),i)$ is a union of a finite subcollection containing D.

As pointed out in the proof of 3.3 we have the next result.

Corollary 3.4. *If* $D \subset N^I$ *is closed discrete, then* $\beta D\backslash D$ *is the union of* $|I|$ *zero sets of* βD.

We need the following results of Mycielski, Mrowka and Juhasz.

Theorem 3.5 [My]. *For* $\alpha < \theta_1$, N^α *has a closed discrete subset of cardinality* α.

Theorem 3.6 [Mr] *and* [J1]. *For* $\alpha < \alpha_1$, $N^{(2^\alpha)}$ *has a closed discrete subset of cardinality* α.

Remark 3.7. It is immediate that for $\alpha < \theta_1$ (resp. $\alpha < \alpha_1$), there is a first countable quasi-H-closed T_1 space of cardinality α (resp. 2^α). However a simple example of a first countable, quasi-H-closed T_1 space of arbitrary cardinality λ is a quotient of $(\omega+1) \times \lambda$ where $\omega+1$ is the one point compactification of ω and $\{n\} \times \lambda$ is identified to a point for each $n \in \omega$; note that this space has only a countable dense set of isolated points.

In order to construct $X = D \cup I$, for $D \subset N^I$, to be Hausdorff one would need, of course, that for $i \neq j \in I$ there is an $n \in N$ such that $G(n,i) \cap G(n,j) = \emptyset$. By 3.3 and 2.1, if D is closed discrete and $|D| > c$ this cannot be done. We do not know if this can be done for $|D| = \aleph_1$. However we give our own proof of 3.5 for $\alpha = \aleph_1$ in case it may be adaptable to yield a Hausdorff space by 3.3. Notice that by 3.4 and 3.5 $\beta\omega_1\backslash\omega_1$ can be covered by ω_1 zero sets.

A partially ordered set $(T,\leq)$ is a *tree*, if for each $x \in T$, the set $\{y \in T: y < x\}$ is well ordered by $<$. The order x, $0(x)$ is the order type of $\{y \in T: y < x\}$. A branch is a maximal linearly ordered subset of T and is an α-branch if it has length α. For $x \in T$ and $\beta \leq 0(x)$, $x|_\beta$ will be the unique $y \leq x$ with $0(y) = \beta$. The α-th level of T is the set $U_\alpha = \{x \in T: 0(x) = \alpha\}$ and $T|_\alpha = \cup_{\rho<\alpha} U_\rho$. The length of T is $\sup\{0(x) + 1: x \in T\}$. A tree T is an *Aronszajn tree* if each level of T is countable, length of T is ω_1 but T has no ω_1-branches. N. Arsonzajn showed that there are Aronszajn trees [Ku].

3.8. Let us first show that $\beta\omega\backslash\omega$ can be covered by an increasing ω_1 chain of zero sets. We are grateful to Eric van Douwen for this proof. Let $\mathcal{B}$ be a Boolean algebra of cardinality $\aleph_1$ which contains a dense Aronszajn tree T [Je]. That is, for each $0 \neq b \in \mathcal{B}$, there is a $t \in T\backslash\{0\}$ such that $t \leq b$. Let X be the Stone space of $\mathcal{B}$. Since the weight of X is $\aleph_1$, there is a continuous map from $\beta\omega\backslash\omega$ onto X [Pa]. For each $t \in T$, $[t]$ is the clopen subset of X consisting of all ultrafilters on $\mathcal{B}$ which contain t. For each $\alpha < \omega_1$, let $Z_\alpha = X\backslash\cup\{[t]: t \in T, 0(t) = \alpha\}$. Since T has no ω_1-branches $X = \cup_{\alpha<\omega_1} Z_\alpha$.

3.9. We now know that there is a first countable H-closed space of cardinality $\aleph_1$ iff there is a $D \subset N^{\omega_1}$ closed and discrete such that $X = D \cup \omega_1$ is Hausdorff. Let us therefore show how to find a closed discrete subset of N^{ω_1}. Let $\{\lambda_\alpha: \alpha < \omega_1\}$ be a strictly increasing indexing of the limit ordinals less than ω_1. For each $\alpha < \omega_1$ arbitrarily choose a countable closed discrete subset of $N^{[\lambda_\alpha,\lambda_{\alpha+1})}$, say $D(\alpha)$. Let T be an Aronszajn tree. For each $\alpha < \omega_1$ let $D(\alpha) = \{d_\alpha(t): t \in T|_{\alpha+1}\}$. For each $s \in T$

let $f_s \in N^{\omega_1}$ be defined by, for each $\alpha < 0(s)$, $f_s|_{[\lambda_\alpha,\lambda_{\alpha+1})} = d_\alpha(s|_\alpha)$ and for $0(s) < \alpha < \omega_1$ $f_s|_{[\lambda_\alpha,\lambda_{\alpha+1})} = d_\alpha(s)$. We will show that $D = \{f_s: s \in T\}$ is closed discrete in N^{ω_1}. Notice that $\{f_s|_{\lambda_{0(s)}}: s \in T\}$ ordered by inclusion is isomorphic to T. The essential idea now is that for each $\alpha < \omega_1$ everything branches into a closed discrete subset of $N^{[\lambda_\alpha,\lambda_{\alpha+1})}$. So the only possible accumulation point of D is an ω_1-branch. However there are none. Let $h \in N^{\omega_1}$ and let $\delta \leq \omega_1$ be the supremum of $\{\alpha < \omega_1: h|_{[\lambda_\alpha,\lambda_{\alpha+1})} = d_\alpha(t_\alpha) \in D(\alpha)\}$. If $\delta < \omega_1$ then $h|_{[\lambda_\delta,\lambda_{\delta+1})}$ is not a limit point of $D(\delta)$ and hence h is not a limit point of D. Therefore assume $\delta = \omega_1$. Since T contains no ω_1-branch there is a smallest $\alpha < \omega_1$ such that for some $\beta < \alpha$ $t_\beta \nleq t_\alpha$. Let β be the smallest $\beta < \alpha$ such that $t_\beta \nleq t_\alpha$. For $\gamma \in \{\alpha,\beta\}$ choose a finite subset $F_\gamma \subset [\lambda_\gamma,\lambda_{\gamma+1})$ such that for $d \in D(\gamma)\backslash\{d_\gamma(t_\gamma)\}, d|_{F_\gamma} \neq d_\gamma(t_\gamma)|_{F_\gamma}$. Now suppose $s \in T$ and $f_s|_{F_\beta \cup F_\alpha} = h|_{F_\beta \cup F_\alpha}$. By the definition of f_s, $s|_\beta = t_\beta$ and $s|_\alpha = t_\alpha$ (if $0(s) < \gamma$ then $s|_\gamma = s$). Since $t_\beta \nleq t_\alpha$, we must have that $0(s) < \alpha$ and $s = t_\alpha$. Therefore $|\{d \in D: d|_{F_\beta \cup F_\alpha} = h|_{F_\beta \cup F_\alpha}\}| \leq 1$ implying that D is closed and discrete.

4. Another Application

The technique developed in Section 3 lends itself very nicely to constructing first countable spaces with another covering property. Therefore we have included this section as an application of the technique. A space X is

κ-*Lindelöf* if each open cover of X has a subcover with cardinality less than κ. In [BG], Bell and Ginsburg show that it is independent with ZFC whether there is a first countable Lindelöf (= $\aleph_1$-Lindelöf) extension X of the discrete space ω_1 such that $|X\backslash\omega_1| \leq \omega$. We generalize this result to arbitrary cardinals κ by showing it is consistent that there is a first countable κ^+-Lindelöf extension X of κ^+ with $|X\backslash\kappa^+| \leq \kappa$. We also slightly improve their result by weakening the assumption.

A set $\mathcal{G} \subset N^\kappa$ is called *bounded* if there is an $f \in N^\kappa$ such that for each $g \in \mathcal{G}$ the set $\{\alpha \in \kappa: g(\alpha) > f(\alpha)\}$ is finite. A set $\mathcal{G} \subset N^\kappa$ is called *dominating* if for each $f \in N^\kappa$ there is a $g \in \mathcal{G}$ such that $\{\alpha \in \kappa: f(\alpha) > g(\alpha)\}$ is finite. Let $\underline{B(\kappa,\lambda)}$ denote the statement: any subset of N^κ with cardinality λ contains a bounded subset of cardinality λ. Let $\underline{D(\kappa,\lambda)}$ denote the statement: there is a dominating family $\mathcal{G} \subset N^\kappa$ of size λ. $B(\omega,\omega_1)$ (equivalent to the well-known $B(\omega_1)$) is a consequence of MAC and is independent with ZFC. We do not know the situation for $B(\kappa,\kappa^+)$ for $\kappa > \omega$. $D(\kappa,\kappa^+)$ follows trivially from GCH, and the statement $D(\omega,\omega_1)$ and $2^\omega > \omega_1$ is known to be consistent. We first show the following generalization of a result in [BG].

Theorem 4.1. $B(\kappa,\kappa^+)$ *implies there is no Hausdorff first countable* κ^+-*Lindelöf extension* X *of* κ^+ *with* $|X\backslash\kappa^+| \leq \kappa$.

Proof. Suppose that X is such an extension. Let for $i \in I = X\backslash\kappa^+, G(n,i)$ be a shrinking neighborhood base at i.

For $\alpha \in \kappa^+$, $d_\alpha \in N^I$ is defined such that $d_\alpha(i) = \min\{n: \alpha \notin G(n,i)\}$. If $|\{d_\alpha: \alpha < \kappa^+\}| < \kappa^+$ then there is a $d \in D = \{d_\alpha: \alpha < \kappa^+\}$ such that $|\{\alpha \in \kappa^+: d_\alpha = d\}| = \kappa^+$. However, for each $i \in I$, $G(d(i),i) \cap \{\alpha \in \kappa^+: d_\alpha = d\} = \emptyset$. This implies that $\{\alpha \in \kappa^+: d_\alpha = d\}$ is closed discrete in X contradicting that X is κ^+-Lindelöf. Now by $B(\kappa,\kappa^+)$ there is a subset $D_1 \subseteq D$ and an $h \in N^I$ such that $|D_1| = \kappa^+$ and, for each $d \in D_1$, $F_d = \{i \in I: d(i) > h(i)\}$ is finite. Since $|I| < |D_1|$, there is a finite set $F \subset I$ and $D_2 \subseteq D_1$ with $|D_2| = \kappa^+$ so that $F_d = F$ for each $d \in D_2$. Similarly, since there are only countably many functions from F to N, there is an $h_1: F \to N$ and a $D_3 \subseteq D_2$ with $|D_3| = \kappa^+$ such that $d|_F = h_1|_F$ for each $d \in D_3$. Finally, let $h_2 \in N^I$, be such that $h_2|_F = h_1|_F$ and $h_2|_{I\setminus F} = h|_{I\setminus F}$. Now for each $d \in D_3$ $d \leq h_2$ which also contradicts that X is κ^+-Lindelöf.

Now, we construct our first countable κ^+-Lindelöf extensions of κ^+. Recall that $\kappa^{\mathfrak{G}} = \Sigma(\kappa^\lambda: \lambda < \kappa)$.

Example 4.2. Assume $\kappa^{\mathfrak{G}} = \kappa$ and $D(\kappa,\kappa^+)$. We shall construct a first countable, zero-dimensional, Hausdorff κ^+-Lindelöf space X containing κ^+ as a dense and discrete subspace such that $|X\setminus\kappa^+| \leq \kappa$. We are going to choose an index set I with $|I| = \kappa$ and find $D \subseteq N^I$ so that $X = D \cup I$ will be first countable Hausdorff and κ^+-Lindelöf. The semi-regularization of X will be zero-dimensional. We shall need D to satisfy

(i) (κ^+-Lindelöf) for each $h \in N^I$, $|\{d \in D: d(i) \leq h(i)$ for all $i \in I\}| \leq \kappa$.

(ii) (Hausdorff) for $i \neq j$ there is an $n(i,j) \in N$ such that for $d \in D$ $d(i) > n(i,j)$ implies $d(j) \leq n(i,j)$.

Let $I = \cup_{\alpha<\kappa} N^{\alpha}$ (i.e. I is the set of all sequences of positive integers of length less than κ). Since $\kappa^{\delta} = \kappa$, $|I| = \kappa$. We will think of I as a tree ordered by inclusion; recall the definition of $O(i)$ from Section 3. Thus, for $i \in I$, $O(i)$ = domain of i and $O(i) < \kappa$. By $D(\kappa,\kappa^+)$, there is a $\mathcal{G}' \subset N^I$ so that for each $h \in N^I$ there is a $g \in \mathcal{G}'$ with $|\{i \in I: h(i) > g(i)\}| < \omega$ and $|\mathcal{G}'| = \kappa^+$. For each $n \in N$, define $\mathcal{G}(n) = \{g + n: g \in \mathcal{G}'\}$ (i.e. $(g+n)(i) = g(i)+n$). Let $\mathcal{G} = \cup_{n\in N}\mathcal{G}(n)$. It is easy to see that for any $h \in N^I$ there is a $g \in \mathcal{G}$ such that $g(i) \geq h(i)$ for all $i \in I$. Let $\mathcal{G} = \{g_\alpha: \alpha < \kappa^+\}$ be a well ordering of $\mathcal{G}$.

We will inductively choose, for $\xi < \kappa^+$, $d_\xi \in N^I$ so that each of the following hold.

(a) $\alpha < \xi$ implies $g_\alpha(i) < d_\xi(i)$ for some $i \in I$.

(b) If $d_\xi(i) = 1$, then $d_\xi(j) = 1$ for all $j \supseteq i$.

(c) If $i \subseteq j$ and $d_\xi(j) > 1$, then $j(O(i)) = d_\xi(i)$.

Suppose that $\lambda < \kappa^+$ and for $\xi < \lambda$ we have defined $d_\xi \in N^I$ satisfying (a)-(c). Since $|\lambda| \leq \kappa$, we can reorder $\{g_\alpha: \alpha < \lambda\}$ and $\{d_\xi: \xi < \lambda\}$ with order type $\leq\kappa$. Therefore without loss of generality and for ease of notation, suppose that $\lambda = \kappa$. We shall recursively define d_λ through the levels of I.

Define $d_\lambda(\emptyset) = g_0(\emptyset) + d_0(\emptyset) + 1$. Suppose for $\delta < \kappa$ and for $i \in \cup_{\alpha<\delta} N^{\alpha}$ we have defined $d_\lambda(i)$ satisfying (b) and (c) and such that for each $\alpha < \delta$ there is an $i \in N^{\alpha}$ with $d_\lambda(i) = g_\alpha(i) + d_\alpha(i) + 1$. It follows from (b) and (c) that

$\{i \in \bigcup_{\alpha<\delta} N^\alpha: d_\xi(i) > 1\}$ forms a chain and let $\ell = \bigcup\{i: d_\xi(i) > 1$ and $i \in \bigcup_{\alpha<\delta} N^\alpha\}$. For $j \in N^\delta$ such that for some $\alpha < \delta$, $d_\lambda(j|\alpha) = 1$ define $d_\lambda(j) = 1$ and (b) is satisfied. If δ is a limit then $\ell \in N^\delta$ and we define $d_\lambda(\ell) = g_\delta(\ell) + d_\delta(\ell) + 1$. If $\delta = \alpha + 1$ then we define $d_\lambda(j) = g_\delta(j) + d_\delta(j) + 1$ if $j \in N^\delta$ and $j(\alpha) = d_\lambda(\ell|\alpha)$, otherwise for $j \in N^\delta$ $d_\lambda(j) = 1$. This clearly satisfies (c).

Let us check that (a) holds for d_λ. If $\alpha < \lambda$ then there is an $i \in N^\alpha$ such that $d_\lambda(i) = g_\alpha(i) + d_\alpha(i) + 1$. This fact also ensures that for $\alpha < \lambda$ $d_\alpha \neq d_\lambda$.

Let $X = D \cup I$ be endowed with the topology described in 3.2. To show that X is κ^+-Lindelöf it suffices to show that any open set containing I contains all but κ many of the elements of D. This is equivalent to showing that, for $h \in N^I$, $|\{d \in D: d(i) \leq h(i)$ for all $i \in I\}| \leq \kappa$. However by the definition of $\mathcal{G}$ there is a $g_\alpha \in \mathcal{G}$ such that $g_\alpha \geq h$. Now, by construction, for each $\xi > \alpha$ there is an $i \in I$ such that $d_\xi(i) > g_\alpha(i) \geq h(i)$ by (a).

It is a little trickier to show that X is Hausdorff. Let $i \neq j \in I$, we must find $n \in N$ so that, for $d \in D$, $d(i) > n$ implies $d(j) \leq n$. First suppose that i and j are incomparable. Conditions (b) and (c) imply that for each $d \in D$ one of $d(i)$ and $d(j)$ is 1. Therefore assume that $i \subset j$ and that $i \in N^\alpha$. Note that, for $d \in D$, $d(j) > 1$ implies that $d(i) = j(\alpha)$ by (c). Therefore let $n = j(\alpha)$ and $d(i) > n$ implies $d(j) \leq n$. The case $j \subset i$ is identical by symmetry.

We will now take the semiregularization of X to make it zero-dimensional. Recall that for $n \in N$ and $i \in I$,

$G(n,i) = \{i\} \cup \{d \in D: d(i) > n\}$. We will show that cl $G(n,i)$ is clopen, for $i \in N^{\alpha}$ and $n \in N$, by showing the following two facts.

Fact 1. If $j \in$ cl $G(n,i)\backslash G(n,i)$, then $i \subset j$ and $j(0(i)) > n$. Recall that if i and j are incomparable, then $d(i) > n \geq 1$ implies that $d(j) = 1$ and $G(1,j) \cap G(n,i) = \emptyset$. Also if $j \subset i$, then by (c), $d(i) > n \geq 1$ implies that $d(j) = i(0(j))$ for $d \in D$. Therefore $G(i(0(j)),j) \cap G(n,i) = \emptyset$. Finally if $i \subset j$ and $j(0(i)) \leq n$, then $G(1,j) \cap G(n,i) = \emptyset$. Indeed if $d(j) > 1$, then $d(i) = j(0(i)) \leq n$ and therefore $d \notin G(n,i)$.

Fact 2. If $j \in$ cl $G(n,i)\backslash G(n,i)$, then $G(1,j) \subset$ cl $G(n,i)$. For if $d \in D$ and $d(j) > 1$, then $d(i) = j(0(i)) > n$ by Fact 1. Therefore, $d \in G(n,i)$. This completes the proof.

Remark. Bell and Ginsburg's [BG] construction for $\kappa = \omega$ required CH. We have slightly strengthened this result since $D(\omega,\omega_1)$ and $2^{\omega} > \omega_1$ is consistent.

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