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1. Introduction

In this paper, we investigate the relationship between equicontinuity and collections of functions which satisfy properties similar to equicontinuity. Historically, such collections have usually been introduced in an attempt to characterize compact sets of functions whose range does not admit a metric (or uniform) structure. We begin by reviewing some pertinent recent history. The reader is referred to [1, p. 382-383] for additional information.

In 1950, Gale [2] introduced a notion, later called property (G), which could serve to replace equicontinuity in Ascoli's Theorem for functions into T_3 spaces. He observed that this property is not implied by equicontinuity in the metric case, but left unanswered the question of whether property (G) implies equicontinuity.

In 1955, Kelley [4] defined the concept of even continuity and showed that it is strictly weaker than equicontinuity but still sufficiently strong for Ascoli's Theorem in a non-metric setting.

In 1969, Kaul [3] defined regularity, another concept stronger than even continuity, for which the Ascoli Theorem holds. Then in 1973, Yang [5] showed that Gale's property (G) implies regularity and, under certain restrictions on the family of functions, regularity implies equicontinuity.

Yang also gave an example of an equicontinuous collection that is not regular.

In Section 2, motivated by Kelley's definition of even continuity, we give a new characterization of equicontinuity in first-countable spaces, formulated in such a way that even continuity follows immediately. In Section 3, we characterize regularity in a similar manner and show that, when the domain is first countable, every regular collection is equicontinuous, with no restriction on the functions.

Throughout, X and Y will be Hausdorff topological spaces and every neighborhood (nhd) of a point will be open. We will denote by C(X,Y) the collection of all continuous functions from X to Y.

Since our intention is to examine the relationship between equicontinuity and various similar notions, we will assume the range space Y to be a metric space with metric d, unless specifically noted otherwise. We denote the ball of radius $\varepsilon > 0$ centered at $y \in Y$ by $B(y,\varepsilon)$ and the set $\cup\{B(a,\varepsilon)|a \in A\}$ by $B(A,\varepsilon)$.

2. Characterization of Equicontinuity

Recall that a collection $\mathcal{F} \subset C(X,Y)$ is said to be *equicontinuous at* $x_o \in X$ if for each $\varepsilon > 0$, there exists a nhd U of x_o such that for all $f \in \mathcal{F}$, $f(U) \subset B(f(x_o),\varepsilon)$.

Since Y is assumed to be a metric space, we can state the definition of even continuity as follows: $\mathcal{F} \subset C(X,Y)$ is *evenly continuous at* $x_o \in X$ if for each $y \in Y$ and for each $\varepsilon > 0$, there exists a nhd U of x_o and an $\varepsilon' > 0$ such that for all $f \in \mathcal{F}$, if $f(x_o) \in B(y,\varepsilon')$, then $f(U) \subset B(y,\varepsilon)$. Kelley [4, p. 237] showed that every equicontinuous collection is evenly continuous and whenever $\{f(x_o) | f \in \mathcal{F}\}$ has compact closure, the reverse implication holds as well.

It is evident that if, in the definition of even continuity, we replace the point y with an arbitrary set K and $B(y,\varepsilon)$ with $B(K,\varepsilon)$, a condition stronger than even continuity is obtained. It is less obvious that the resulting condition is implied by equicontinuity.

Proposition 1. *Let* $\mathcal{F} \subset C(X,Y)$ *be equicontinuous at* $x_o \in X$. *Then for each* $K \subset Y$ *and each* $\varepsilon > 0$, *there is a* nhd U *of* x_o *and an open* $W \subset Y$ *containing* K *such that for all* $f \in \mathcal{F}$, *if* $f(x_o) \in W$, *then* $f(U) \subset B(K,\varepsilon)$.

Proof. Let $K \subset Y$ and $\varepsilon > 0$ be given. Let $W = B(k,\varepsilon/2)$. Since $\mathcal{F}$ is equicontinuous, the subcollection $\{f \in \mathcal{F} | f(x_o) \in W\}$ is also. Hence there is a nhd U of x_o such that for all f in this subcollection, $f(U) \subset B(f(x_o),\varepsilon/2)$. Then using the triangle inequality we have that if $f(x_o) \in W$, then $f(U) \subset B(K,\varepsilon)$ as required.

We now show that, in first countable spaces, this condition characterizes equicontinuity, even if the condition holds only for closed sets K. The proof will employ the following technical lemma.

Lemma 2. *Let* X *and* Y *be sets*, $x_o \in X$, $\{U_n\}$ *a nested collection of subsets of* X, *each containing* x_o, *and* $\mathcal{B} = \{B_n\}$ *a collection of mutually disjoint subsets of* Y. *Let* $\{f_n\colon X \to Y\}$ *be a sequence of functions such that for* *each* $n \in N$, $f_n(x_o) \in B_n$ *and* $f_n(U_n) \not\subset B_n$. *Then there is a subsequence* $\{f_{n_i}\}$ *such that for each* $i \in N$, $f_{n_i}(U_{n_i}) \not\subseteq \cup\{B_{n_i} | i \in N\}$.

Proof. The proof will involve three cases: (1) some set $B \in \mathcal{B}$ meets infinitely many sets $f_n(U_n)$; or if each set B meets only finitely many of these sets, then either (2) infinitely many sets $f_n(U_n)$ meet only finitely many sets B, or (3) infinitely many sets $f_n(U_n)$ meet infinitely many sets B. The construction is different in each case.

Case (1). If there is a $B_{n_o} \in \mathcal{B}$ such that $f_n(U_n) \cap B_{n_o} \neq \emptyset$ for infinitely many indices $n_0, n_1, \cdots$, then the subsequence $\{f_{n_i} | i = 1,2\cdots\}$ suffices, because for each $i \in N$, $f_{n_i}(U_{n_i})$ meets B_{n_o} and B_{n_o} is disjoint from $\cup\{B_{n_i} | i \in N\}$.

If each $B \in \mathcal{B}$ meets only finitely many sets $f_n(U_n)$, then we partition N as $N_\alpha \cup N_\beta$, where $N_\alpha = \{n \in N | f_n(U_n)$ meets only finitely many $B \in \mathcal{B}\}$ and $N_\beta = \{n \in N | f_n(U_n)$ meets infinitely many $B \in \mathcal{B}\}$. The indices of the desired subsequence will be chosen exclusively from either N_α or N_β, depending on which of these is infinite. These are cases (2) and (3), respectively.

Case (2). N_α is infinite. For each $i \in N$, we will select a function so that $f_{n_i}(U_{n_i})$ is disjoint from all the sets B which correspond to previously selected functions. Formally, for each fixed index $n_\alpha \in N_\alpha$, we let $p_\alpha = \max\{n \in N_\alpha | f_{n_\alpha}(U_{n_\alpha}) \cap B_n \neq \emptyset\}$ and $q_\alpha = \max\{n \in N_\alpha | f_n(U_n) \cap$

$B_{n_\alpha} \neq \emptyset\}$. This construction forces each $f_{n_i}(U_{n_i})$ to be disjoint from each B_{n_j} for which $n_i \neq n_j$, since $n_i > n_j \Rightarrow n_i > q_j \Rightarrow f_{n_i}(U_{n_i}) \cap B_{n_j} = \emptyset$, while $n_j > n_i \Rightarrow n_j > p_i \Rightarrow f_{n_i}(U_{n_i}) \cap B_{n_j} = \emptyset$. Thus $\{f_{n_i}\}$ is the desired subsequence.

Case (3). N_β is infinite. For each $i \in N$, we will select a function so that $f_{n_i}(U_{n_i})$ meets at least one set B which corresponds to a function not in the subsequence. Formally, for each fixed index $n_\beta \in N_\beta$, we let $r_\beta = \min\{n \in N_\beta | n > n_\beta \text{ and } (U_{n_\beta}) \cap B_n \neq \emptyset\}$. Then to construct $\{f_{n_i}\}$, let $n_1 = \min\{n \in N_\beta\}$, and for $i \geq 1$, let $n_{i+1} = 1 + r_i$. Then for each i, $f_{n_i}(U_{n_i}) \cap B_{r_i} \neq \emptyset$, and by construction $r_i \neq n_j$ for any j, so $\{f_{n_i}\}$ is the desired subsequence.

Theorem 3. *Let* X *be a first countable space,* $x_o \in X$, *and* $\mathcal{F} \subset C(X,Y)$. *Then the following are equivalent:*

(i) $\mathcal{F}$ *is equicontinuous at* x_o.

(ii) *For each* $K \subset Y$ *and each* $\varepsilon > 0$, *there is a* nhd U *of* x_o *and an open* $W \subset Y$ *containing* K *such that for all* $f \in \mathcal{F}$, *if* $f(x_o) \in W$, *then* $f(U) \subset B(K,\varepsilon)$.

(iii) *For each closed* $K \subset Y$ *and each* $\varepsilon > 0$, *there is a* nhd U *of* x_o *and an open* $W \subset Y$ *containing* K *such that for all* $f \in \mathcal{F}$, *if* $f(x_o) \in W$, *then* $f(U) \subset B(K,\varepsilon)$.

Proof. (i) $\Rightarrow$ (ii). This is Proposition 1.

(ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (i). Suppose to the contrary that $\mathcal{F}$ is not equicontinuous at x_o. Then there exists $\varepsilon_o > 0$ and infinitely many functions $f_k \in \mathcal{F}$ such that

$f_k(U_k) \not\subset B(f_k(x_o),\varepsilon_o)$, where $\{U_k\}$ is a nested nhd base at x_o. By (iii), $\mathcal{F}$ is evenly continuous at x_o and hence so is $\{f_k\}$. But $\{f_k\}$ is obviously not equicontinuous at x_o, so the set $\{f_k(x_o)\}$ does not have compact closure. Therefore, without loss of generality we may suppose that $\{f_k(x_o)\}$ is an infinite set of distinct points. We will now contradict the hypothesis by extracting a subsequence of $\{f_k\}$ for which (iii) fails. The construction will depend on whether or not $\{f_k(x_o)\}$ is a totally bounded subset of Y.

If $\{f_k(x_o)\}$ is totally bounded, choose a tail of the Cauchy subsequence $\{f_{k_n}(x_o)\}$ and $\varepsilon < \varepsilon_o/2$ such that $\{f_{k_n}(x_o)\}$ lies in $B(f_{k_1}(x_o),\varepsilon)$. Let $K = \overline{\{f_{k_n}(x_o)\}}$ and note that $B(K,\varepsilon) \subset B(f_{k_1}(x_o),\varepsilon_o)$. Then for any open $W \subset Y$ containing K, we have that $f_{k_n}(x_o) \in W$ for each $n \in N$. But regardless of the choice of U, a nhd of x_o, $f_{k_n}(U) \not\subset B(K,\varepsilon)$, because U must contain some U_{k_n} and $f_{k_n}(U_{k_n}) \not\subset B(f_{k_1}(x_o),\varepsilon_o)$ for each $n \in N$. Thus condition (iii) fails to hold.

If $\{f_k(x_o)\}$ is not totally bounded, then some subsequence is not Cauchy. Hence, we may choose an infinite discrete closed subset of $\{f_k(x_o)\}$, call it $\{f_n(x_o)\}$, and a corresponding collection of disjoint balls $\{B_n\}$ of radius $\varepsilon < \varepsilon_o$, which separate the points of $\{f_n(x_o)\}$. Now the conditions of Lemma 2 are satisfied, so we may extract a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that for each $i \in N$, $f_{n_i}(U_{n_i}) \not\subset \cup\{B_{n_i} \mid i \in N\}$. Now let $K = \{f_{n_i}(x_o)\}$, which is closed in Y, and observe that $\cup\{B_{n_i} \mid i \in N\} = B(K,\varepsilon)$. Then for any open $W \subset Y$ containing K, we have that $f_{n_i}(x_o) \in W$ for each $i \in N$. But as before, regardless of the choice of U, a nhd of x_o, $f_{n_i}(U) \not\subset B(K,\varepsilon)$. Thus condition (iii) is again violated, and the proof is complete.

3. Regular Collections of Functions

A collection $\mathcal{F} \subset C(X,Y)$ is said to be *regular at* x_o if for each $\mathcal{G} \subset \mathcal{F}$ and each open $V \subset Y$, if $\overline{\{g(x_o) \mid g \in \mathcal{G}\}} \subset V$, then there exists a nhd U of x_o such that $g(U) \subset V$ for all $g \in \mathcal{G}$.

Kaul proved in [3] that if $\mathcal{F}$ is regular at x_o, then $\mathcal{F}$ is evenly continuous at x_o, and that they are equivalent whenever $\{f(x_o) \mid f \in \mathcal{F}\}$ has compact closure. His only assumption involved Y being a T_3 space. Yang noted in [5] that if $\{f(x_o) \mid f \in \mathcal{F}\}$ has compact closure and Y is a uniform space, then regularity and equicontinuity are equivalent.

We will take a different approach from that of Yang by showing that if X is first countable and Y is a metric space, then regularity implies equicontinuity with no restriction on the set $\{f(x_o) \mid f \in \mathcal{F}\}$.

We begin by stating a characterization of regularity which is analogous to the characterization of equicontinuity given in Theorem 3, except that here we do not require that X be first countable or that Y be a metric space. It is sufficient that Y be normal.

Theorem 4. *Let* $x_o \in X$, Y *be normal, and let* $\mathcal{F} \subset C(X,Y)$. *Then the following are equivalent:*

(i) $\mathcal{F}$ *is regular at* x_o.

(ii) *For each* $K \subset Y$ *and each open* $V \subset Y$ *containing* $\overline{K}$, *there is a* nhd U *of* x_o *and an open* $W \subset Y$ *containing* K *such that for all* $f \in \mathcal{F}$, *if* $f(x_o) \in W$, *then* $f(U) \subset V$.

(iii) *For each closed* $K \subset Y$ *and each open* $V \subset Y$ *containing* K, *there is a* nhd U *of* x_o *and an open* $W \subset Y$ *containing* K *such that for all* $f \in \mathcal{F}$, *if* $f(x_o) \in W$, *then* $f(U) \subset V$.

Proof. (i) $\Rightarrow$ (ii). Let K and V be given with $\overline{K} \subset V$. Let $W \subset Y$ be an open set with $\overline{K} \subset W \subset \overline{W} \subset V$ and define $\mathcal{G} = \{g \in \mathcal{F} | g(x_o) \in W\}$. Then $\overline{\{g(x_o) | g \in \mathcal{G}\}} \subset \overline{W} \subset V$, so by regularity, there is a nhd U of x_o such that $g(U) \subset V$ for all $g \in \mathcal{G}$. This shows that for all $f \in \mathcal{F}$, if $f(x_o) \in W$, then $f(U) \subset V$.

(ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (i). Let $V \subset Y$ be open and let $\mathcal{G} \subset \mathcal{F}$ be given such that $\overline{\{g(x_o) | g \in \mathcal{G}\}} \subset V$. Applying (iii) to the closed set $K = \overline{\{g(x_o)\}}$, there must be a nhd U of x_o and an open $W \subset Y$ containing K such that for all $f \in \mathcal{F}$, if $f(x_o) \in W$, then $f(U) \subset V$. In particular, $g(x_o) \in W$ for all $g \in \mathcal{G}$, and hence $g(U) \subset V$, as required.

Returning to the metric setting and noting that $B(K,\varepsilon)$ is an open set about K, we see by using Theorems 3 and 4 that every regular collection in a first countable space is necessarily equicontinuous.

Theorem 5. *Let* X *be a first countable space and* Y *a metric space. If* $\mathcal{F} \subset C(X,Y)$ *is regular, then* $\mathcal{F}$ *is equicontinuous.*

Example 2 in [5], which shows that an equicontinuous family need not be regular, is an unbounded collection of non-constant functions from R to R. Our next result shows that, in fact, no such collection on a first countable space can be regular.

Theorem 6. *Let* X *be a first countable space,* Y *a metric space and let* $\mathcal{F} \subset C(X,Y)$ *be regular at* x_o. *Then there is a bounded subset* $M \subset Y$ *and a* nhd U *of* x_o *such that for all* $f \in \mathcal{F}$, *if* $f(x_o) \notin M$, *then* f *is constant on* U.

Proof. Suppose to the contrary that for every bounded M and nhd U of x_o, there is an $f \in \mathcal{F}$ such that f is non-constant on U and $f(x_o) \notin M$. Let $\{U_n\}$ be a nested nhd base at x_o. By Theorem 5, $\mathcal{F}$ is equicontinuous at x_o, so we may assume without loss of generality that $f(U_1) \subset B(f(x_o),\frac{1}{2})$ for each $f \in \mathcal{F}$. Now let $f_0 \in \mathcal{F}$. We proceed by induction to construct a subcollection $\{f_i\} \subset \mathcal{F}$.

Pick $f_1 \in \mathcal{F} - \{f_0\}$ such that $f_1(x_o) \notin B(f_0(x_o),1)$ and such that f_1 is nonconstant on U_1. There must be such a function f_1 because $B(f_0(x_o),1)$ is bounded. Let $x_1 \in U_1$ with $d(f_1(x_1),f_1(x_o)) = \varepsilon_1 > 0$. Assuming $f_1,\cdots f_{n-1}$ and $x_1,\cdots x_{n-1}$ have been chosen, pick $f_n \in \mathcal{F} - \{f_0,\cdots f_{n-1}\}$ such that $f_n(x_o) \notin \cup\{B(f_i(x_o),1)|i = 1,\cdots n-1\}$ and such that f_n is nonconstant on U_n. Such a function exists for the same reason as before. Also as before, let $x_n \in U_n$ with $d(f_n(x_n),f_n(x_o)) = \varepsilon_n > 0$.

This gives us a collection $\{f_i\}$ for which $\{f_i(x_o)\}$ is closed. Letting $V = \cup\{B(f_i(x_o),\varepsilon_i/2)|i \in N\}$, the regularity of $\mathcal{F}$ at x_o assures us there is a basic nhd U_n of x_o such that $f_i(U_n) \subset V$ for each $i \in N$. But this is impossible because $f_n(x_n) \notin B(f_n(x_o), \varepsilon_n/2)$ by the choice of x_n, and if $i \neq n$, then $f_n(x_n) \notin B(f_i(x_o), \varepsilon_i/2)$ since $f_n(U_1) \subset B(f_n(x_o), \frac{1}{2})$. Hence $f_n(U_n) \not\subset V$, which is a contradiction, and the theorem is proved.

Now we are in a position to characterize the regular families of real-valued functions on a first countable space.

Theorem 7. *Let* X *be a first countable space and let* $\mathcal{F} \subset C(X,R)$. *Then the following are equivalent:*

(i) $\mathcal{F}$ *is regular at* x_o.

(ii) $\mathcal{F}$ *is equicontinuous at* x_o, *and there is a compact* $C \subset R$ *and a* nhd U *of* x_o *such that* f *is constant on* U *whenever* $f(x_o) \notin C$.

Proof. (i) $\Rightarrow$ (ii). This follows from Theorems 5 and 6.

(ii) $\Rightarrow$ (i). Let $\mathcal{G} \subset \mathcal{F}$ and let $V \subset R$ be an open set with $\overline{\{g(x_o) | g \in \mathcal{G}\}} \subset V$.
Let $\mathcal{G}_1 = \{g \in \mathcal{G} | g(x_o) \in C\}$ and $\mathcal{G}_2 = \mathcal{G} - \mathcal{G}_1$. Then $\mathcal{G}_1$ is equicontinuous and therefore regular since $\overline{\{g(x_o) | g \in \mathcal{G}\}}$ is compact. So let U_1 be a nhd of x_o for which $g(U_1) \subset V$ for all $g \in \mathcal{G}_1$, and let U_2 be a nhd of x_o on which each $g \in \mathcal{G}_2$ is constant. Then $U = U_1 \cap U_2$ is the required nhd of x_o.

Theorem 7 provides a method of partitioning a regular collection of real-valued functions into two subcollections, one equicontinuous at x_o, and the other consisting of functions which are constant on some fixed nhd of x_o.

We conclude by noting that some of the above theorems can be proved in a more general setting. Specifically, the assumption that the range space Y is a metric space in Theorem 3 can be relaxed to a uniform space, and to a normal uniform space in Theorems 5, 6, and 7. In each proof, we need only make the obvious modifications necessary in changing from a metric to a uniform space.

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