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1. Introduction

In 1955 A. D. Wallace [9] introduced the study of C-sets and investigated C-sets in semigroups. In this paper we investigate C-sets in Hausdorff continua (compact, connected Hausdorff spaces) and note some properties of C-sets pertaining to the study of mappings onto continua. If M is a continuum, a proper subset H of M is a *C-set* in M provided H is a subset of any subcontinuum of M which contains both a point in H and a point not in H . In Lemma 1 of [9, p. 639] Wallace observed that C-sets are connected and have no interior. Although C-sets do not have to be closed, it is not difficult to show that if K is a C-set which is not closed then \bar{K} is an indecomposable continuum. Moreover, if K is a C-set which is not closed the K is the union of some of the composants of \bar{K} . To see this suppose H is a subcontinuum of \bar{K} containing a point in K and a point not in K . Then H contains K and thus H contains \bar{K} , so each proper subcontinuum of \bar{K} which intersects K is a subset of K . Consequently, each component of \bar{K} which intersects K is a subset of K .

A continuum M is a *triod* provided it contains a subcontinuum C such that $M-C$ has at least three components. A continuum is said to be *atriodic* provided it contains no triod. The statement that the continuum M is *unicoherent*

means if A and B are continua whose union is M then $A \cap B$ is connected. In his doctoral dissertation at the University of Houston, Collins [2] introduced the class of IUC continua and proved [2, Theorem 6, p. 12] that atriodic continua have property IUC hereditarily. A continuum has *property IUC* provided every proper subcontinuum with interior is uni-coherent. Collins' result that atriodic continua have property IUC has been obtained independently by Mackowiak and Tymchatyn [7]. In this paper we generalize these results (Theorem 3).

The so-called "boundary bumping theorem" is used often in the proofs in this paper. For a proof of it in Hausdorff continua see [5, Theorem 2, p. 172].

2. A Characterization of C-Sets

The following theorem, although not stated in this manner, is essentially what Cook [1, Theorem 4, p. 243] and Read [8] (see also [6, 5.7, p. 111]) proved when they showed that a continuum is hereditarily indecomposable if and only if every mapping of a continuum onto it is confluent. The proof presented here differs only slightly and is included only for the sake of completeness.

Theorem 1. Suppose M is a Hausdorff continuum and H is a proper subcontinuum of M . Then H is a C-set in M if and only if for each mapping f of a continuum onto M every component of $f^{-1}(H)$ is thrown by f onto H .

Proof. Suppose M is a continuum, H is a subcontinuum of M which is a C-set and f is a mapping of a continuum X

onto M . Let K be a component of $f^{-1}(H)$ and G be a monotonic collection of subcontinua of X such that the common part of all the members of G is K and each member of G contains a point not in K . Then, if J is a continuum in G , $f[J]$ contains a point of H and a point not in H , so H is a subset of $f[J]$. Since X is a Hausdorff continuum, G is monotonic and K is the common part of all the members of G , $f[K] = \bigcap_{J \in G} f[J]$. Thus, $f[K] = H$.

On the other hand suppose H is not a C -set and C is a subcontinuum of M not containing H which contains a point of H and a point Q not in H . Let X be the continuum obtained by identifying $(Q,0)$ and $(Q,1)$ in $(M \times \{0\}) \cup (C \times \{1\})$ and f be the natural projection of X onto M . Then $f^{-1}(H)$ has two components one of which is not thrown onto H by f . This completes the proof of Theorem 1.

Remark. It is easy to show that a continuum is hereditarily indecomposable if and only if every proper subcontinuum of it is a C -set in it.

3. Atriodic Continua and C -Sets

In this section we often use the following property of atriodic continua: If M is a decomposable, atriodic continuum then M is the union of two continua A and B such that $A = \overline{A - (A \cap B)}$ and $B = \overline{B - (A \cap B)}$. For a proof of this see Collins [2] or [3]. It should be noted that in Collins' work he assumes that continua are metric but his arguments do not require changes for Hausdorff continua.

Lemma. If A and B are two continua which intersect such that (1) $A \cup B$ is atriodic, (2) $A = \overline{A - (A \cap B)}$ and $B = \overline{B - (A \cap B)}$, and (3) $A \cap B$ is the union of the two continua C_1 and C_2 , then A is irreducible from C_1 to C_2 .

Proof. Suppose P is a proper subcontinuum of A which intersects both C_1 and C_2 and let y be a point of A not in $P \cup C_1 \cup C_2$. That there is such a point y follows by the assumption that $A = \overline{A - (A \cap B)}$ for if P contains $A - (C_1 \cup C_2)$ then P contains A . There exist mutually exclusive open sets U_1 and U_2 containing C_1 and C_2 respectively such that $\overline{U_1}$ and $\overline{U_2}$ are mutually exclusive and neither contains y . Let B_1 and B_2 be the components of $B \cap U_1$ and $B \cap U_2$ containing C_1 and C_2 respectively. Then $A \cup (P \cup \overline{B_1}) \cup (P \cup \overline{B_2})$ is a triod.

The following theorem was proved independently by Maćkowiak and Tymchatyn [7, 13(2), p. 40]. In that paper they call C -sets which are continua terminal continua. This theorem is generalized in the next section of this paper.

Theorem 2. If M is an atriodic continuum then each proper subcontinuum of M which is not unicoherent is a C -set.

Proof. Suppose H is a proper subcontinuum of M such that H is not unicoherent and H is not a C -set. Then H is the union of two continua A and B such that $A \cap B$ is not connected and $A = \overline{A - (A \cap B)}$ and $B = \overline{B - (A \cap B)}$. Suppose K is a subcontinuum of M containing a point of $A \cup B$ and a

point not in $A \cup B$. Since $A \cap B$ is not connected and M is atriodic, $A \cap B$ is the union of two continua C_1 and C_2 . Suppose K contains a point of B . We now show that K contains A .

Suppose x is a point of $A - (A \cap B)$ which is not in K . There is an open set U containing x which does not contain a point of $B \cup K$. By the Lemma, A is irreducible from C_1 to C_2 so $A - (A \cap U)$ contains no continuum intersecting both C_1 and C_2 . Therefore, [5, Theorem 1, p. 168], $A - (A \cap U)$ is the union of two mutually exclusive closed point sets H_1 and H_2 containing C_1 and C_2 respectively. There exist mutually exclusive open sets U_1 and U_2 containing H_1 and H_2 respectively such that \bar{U}_1 and \bar{U}_2 are mutually exclusive and neither contains x . Let A_1 and A_2 denote the components of $A \cap U_1$ and $A \cap U_2$ containing C_1 and C_2 respectively. Then $(B \cup K) \cup (B \cup \bar{A}_1) \cup (B \cup \bar{A}_2)$ is a triod.

Now, since K contains A , by repeating the argument of the previous paragraph exchanging the roles of A and B , we obtain the K contains B . This will complete the proof.

4. HIUC Continua and C-Sets

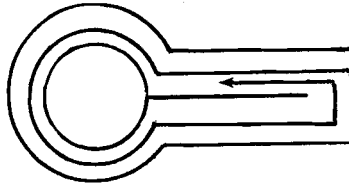
A continuum having property IUC hereditarily is said to have *property HIUC*.

Theorem 3. If M is a continuum with property HIUC then each proper subcontinuum of M which is not unicoherent is a C-set.

Proof. Suppose H is a non-unicoherent proper subcontinuum of M , and K is a subcontinuum of M containing a

point of H and a point not in H . Suppose x is a point of H which is not in K . Then there is an open set U containing x which contains no point of K . Then $H \cup K$ does not have property IUC since H is a non-unicoherent proper subcontinuum of $H \cup K$ which has interior in $H \cup K$.

Remark. It is easy to see from the example below that the hypothesis in Theorem 3 that M have property HIUC may not be weakened to M has property IUC for the circle is not a C -set in M .



5. Confluence and Weak Confluence

We conclude this paper with some consequences of Theorems 1, 2 and 3. First, we introduce some terminology which the author has found useful in discussing confluence and related properties.

Definitions. Suppose M is a continuum, H is a subcontinuum of M and f is a mapping of a continuum onto M . The statement that f is *confluent with respect to H* (respectively, *weakly confluent with respect to H*) means each (resp., some) component of $f^{-1}(H)$ is thrown by f onto H .

Thus, if f is a mapping of a continuum onto M then f is *confluent* (resp., *weakly confluent*) provided f is confluent (resp., weakly confluent) with respect to each

non-degenerate proper subcontinuum of M . Further, f is said to be *pseudo-confluent* provided f is weakly confluent with respect to each irreducible subcontinuum of M .

Theorem 1 may now be restated: A proper subcontinuum H of a continuum M is a C -set in M if and only if every mapping of a continuum onto M is confluent with respect to H .

The following theorems are immediate from Theorems 1 and 3.

Theorem 4. Suppose f is a mapping of a continuum onto the continuum M and M has property HIUC. If H is a non-unicoherent proper subcontinuum of M then f is confluent with respect to H .

Theorem 5. If f is a mapping of a continuum onto a continuum M having property HIUC then f is confluent (resp., weakly confluent) if and only if f is confluent (resp., weakly confluent) with respect to every unicoherent proper subcontinuum of M .

Corollary 1. Suppose f is a mapping of a continuum onto an atriodic continuum. Then f is pseudo-confluent if and only if f is weakly confluent.

Finally, we observe that Corollary 1 provides another proof of a theorem of Grispolakis and Tymchatyn [4, Theorem 5.3].

Corollary 2. Suppose M is an atriodic continuum. Then M is in Class W if and only if M is in Class P .

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